EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 5, Issue 2 (2017) 66–79

MODIFICATION OF THE TIKHONOV METHOD UNDER SEPARATE RECONSTRUCTION OF COMPONENTS OF SOLUTION WITH VARIOUS PROPERTIES

Vasin V.V., Belyaev V.V.

Abstract In this paper a linear ill-posed is considered. Its solution is given in the form of a sum of two components: one contains breaks and the other is continuous, but admits breaks of derivative. For stable separate reconstruction of a solution, a modified Tikhonov method is applied. In this method, the stabilizer is chosen as a sum of two functionals with using total variation of function and its derivative, where every stabilizing functional depends on one component only. The convergence of the sum of the regularized components to a solution of the initial problem is proved. A scheme of finite-dimensional approximations of the regularized problem is investigated and the results of numerical experiments are presented.

Keywords: ill-posed problem, Tikhonov regularization, non-smooth solution, total variation

AMS Mathematics Subject Classification: 45Q05, 65J15, 65J20

1 Introduction

An ill-posed problem in the form of an operator equation

$$A(u) = f \tag{1.1}$$

is considered, where A is a linear operator acting on a pair of Banach spaces U, F; the right-hand side f is approximately given by the element $f_\delta : ||f - f_\delta|| \leq \delta$. Continuity of the operators A^{-1} is not assumed, therefore, equation (1.1) is an essentially ill-posed problem. In some applied ill-posed problems, often there is an additional information that along with a smooth background the sought for solution has various pecularities, for example, breaks and fractures. In fact, in this situation the problem of choosing the stabilizing functional arises for the Tikhonov regularization. Under reconstruction of a smooth solution, it is reasonable to use the stabilizer with a strong stabilizing effect, for example, the Sobolev norm. However, under finding a discontinuous solution, high precision is achieved with the stabilizer in the form of the total or classical variation [6, 13].

The idea of separate reconstructing solution components with various properties of smoothness had first appeared in the works devoted to processing noise signals (images) [2, 3]. For example, in [3], it is assumed that a solution of equation (1.1) is the sum the smooth component u_1 and discontinuous one u_2, and the stabilizing functional is constructed as $\Omega(u_1) + \Omega(u_2)$. Since the functional Ω_1 responding for the smooth component is taking the norm of space L_2, the total variation [4] is used for constructing the functional Ω_2. Numerical efficiency was demonstrated in [3], and the theoretical ground of this algorithm was given in [9].

An intermediate property between differentiability and discontinuity can be continuity of a solution with discontinuous derivative. In this case, it is necessary to investigate a solution representable in the form of the sum of three components [10]. For the third component, the stabilizing functional is taken as the norm of the Lipschitz space that guarantees the uniform convergence of the regularized approximations [10].

In the recently published work [11], the Tikhonov method was investigated for the three-component case; there the total variation of a function and its derivatives up to the second order were used as the stabilizing functionals. Note that the stabilizers based on using the classical variation of the n-th order derivatives of a function were applied in [7] for construction of the regularizing algorithm converged on variations.

In our work, the problem of constructing a regularizing algorithm (RA) for equation (1.1) is considered. It is presupposed that the solution of this equation can be presented as the sum of two components $u = u_1 + u_2$, one of which u_1 contains the breaks of the first kind, and the second component u_2 admits breaks of the derivative. To solve this problem, the Tikhonov regularization with the stabilizer in the form of the BV-norm of a function [1, 4] and its derivative [11] is used. It should be noted that in this case presence of a smooth (differentiable) component in a solution is not assumed. However, from the theoretical viewpoint more general three-component case with a added smooth component in the framework of our approach is similarly investigated [10, 11].

In Section 2, the componentwise convergence of the regularized solutions not only in the spaces L_p, W_p^1 is proved. But, in contrast to [11], the uniform convergence is additionally established for the first (discontinuous) component and the derivative of the second component on the segments, which does not contain the points of breaks. In Section 3, the finite-dimensional approximation of RA is investigated. In Section 4 the results of numerical experiments are given.

2 Convergence of regularized solutions

Let A be a linear bounded operator, acting from the space $L_p[a,b]\,(p \geq 1)$ into the space $L_2[c,d]$. It is assumed that the solution u of equation (1.1) can be presented as the sum of two component $u = u_1 + u_2$, where the functions u_1, $u_2^{(1)}$ admit the breaks of the first kind. It is evident that such presentation of the solution u as a sum of two components is ambiguous. Since the solution u is non-smooth, then, probably, for constructing RA, the most appropriate method is the Tikhonov algorithm with the stabilizer on the base of the total variation in the following form:

$$\min\{\frac{1}{2}||A(u_1+u_2)-f_\delta||_{L_2}^2+\alpha[||u_1||_{BV}+|u_2(a)|+||u_2^{(1)}||_{BV}] : u_1, u_2^{(1)} \in BV\} = \Phi_*, \quad (2.1)$$

where $u_2^{(1)}$ is the derivative of the first order for the function u_2, $||u||_{BV} = ||u||_{L_1} + J_a^b(u)$ is the norm the space $BV[a,b] = \{u : u \in L_1, J_a^b(u) < \infty\}$, $J_a^b(u)$ is the total variation of the function u defined by formula [4]

$$J_a^b(u) = \sup\{\int_a^b u(x)v^{(1)}(x)\,dx : |v(x)| \le 1, v \in C_0^1[a,b]\}.$$

Let us show that there exists such a pair of the components $(\hat{u}_1, \hat{u}_2)$, which gives the solution $\hat{u} = \hat{u}_1 + \hat{u}_2$ and delivers minimum in the stabilizer

$$\Omega(u_1, u_2) = ||u_1||_{BV} + |u_2(a)| + ||u_2^{(1)}||_{BV}.$$

Theorem 2.1. *Let A be linear continuous a operator acting from $L_p[a,b]$ into $L_2[c,d]$ and equation (1.1) have a unique solution $\hat{u} = A^{-1}(f)$. Then there exists, possibly, non-unique solution $(\hat{u}_1, \hat{u}_2)$, of the following problem:*

$$\min\{||u_1||_{BV} + |u_2(a)| + ||u_2^{(1)}||_{BV} : A(u_1+u_2) = f, u_1, u_2^{(1)} \in BV\} = \Psi_*. \tag{2.2}$$

Proof

Let (u_{1k}, u_{2k}) be a minimizing sequence in problem (2.2), then in view of boundedness of the sequences

$$||u_{1k}||_{BV} \le c_1,\ ||u_{2k}^{(1)}||_{BV} \le c_2,\ |u_{2k})(a)| \le c_3,$$

and compactness of embedding operator E of the space BV into L_p ([1], theorem 2.5) there exists subsequences strongly convergent in L_p . Without loss of generality, one can consider that they coincide with the original sequences, *i.e.*,

$$\lim_{k\to\infty} ||u_{1k} - \hat{u}_1||_{L_p} = 0,\ \lim_{k\to\infty} ||u_{2k}^{(1)} - \hat{v}_2||_{L_p} = 0,\ u_{2k} \to c_a.$$

It means that $\{u_{2k}\}$ is the Cauchy sequence in the space W_p^1. Therefore, there exists a function $\hat{u}_2 \in W_p^1$ such that $\hat{u}_2^{(1)} = \hat{v}_2, \hat{u}_2(a) = c_a$ and the following convergence holds

$$\lim_{k\to\infty} ||u_{2k} - \hat{u}_2||_{W_p^1} = 0.$$

From continuity of the operator A, it follows that the following relations are valid

$$0 \le ||A(\hat{u}_1 + \hat{u}_2) - f||_{L_2} = \lim_{k\to\infty} ||A(u_{1n} + u_{2n}) - f||_{L_2} = 0.$$

Besides, from weak lower semicontinuity of a norm and the total variation relatively the L_p-covergence, we have the following inequalities

$$||\hat{u}_1||_{BV} + |\hat{u}_2(a)| + ||\hat{u}_2^{(1)}||_{BV} \le \liminf_{k\to\infty}(||\hat{u}_{1n}||_{BV} + |\hat{u}_{2n}(a)| + ||\hat{u}_{2n}^{(1)}||_{BV}),$$

i.e., the pair $(\hat{u}_1, \hat{u}_2)$ attains minimum in (2.2).

Theorem 2.2. *Let the condition of* Theorem 2.1 *be fulfilled. Then for any $\alpha > 0$ there exists possible non-unique solution (u_1^α, u_2^α) of problem (2.1) and under connection of the parameters $\delta^2/\alpha \to 0$, $\alpha(\delta) \to 0$ as $\delta \to 0$, the following properties are valid:*
1) *$\{u_1^{\alpha(\delta)}\}$ is relatively compact in L_p, moreover, if $u_1^{\alpha(\delta_k)} \to \bar{u}_1$ as $\delta_k \to 0$ in L_p, then*

$$\lim_{\delta_k\to 0} J_a^b(u_1^{\alpha(\delta_k)}) = J_a^b(\bar{u}_1),\ u_1^{\alpha(\delta_k)} \rightrightarrows \bar{u}_1, \tag{2.3}$$

i.e., uniformly on every subsegment $[a', b'] \in [a, b]$ *non-containing break points of the function* u_1;
2) $\{u_2^{\alpha(\delta)}\}$ *is relatively compact in* W_p^1, *moreover, if* $u_2^{\alpha(\delta_k)} \to \bar{u}_2$ *as* $\delta_k \to 0$ *in* L_p, *then*

$$\lim_{\delta_k \to 0} J_a^b((u_2^{\alpha(\delta_k)})^{(1)}) = J_a^b((\bar{u}_1)^{(1)}),\ (u_2^{\alpha(\delta_k)})^{(1)} \rightrightarrows (\bar{u}_2)^{(1)}, \tag{2.4}$$

i.e., uniformly on every subsegment $[a', b'] \in [a, b]$ *non-containing break points of the function* $\bar{u}_1$;
3) *any solution* (u_1^α, u_2^α) *of problem* (2.1) *gives one and the same element* $u^\alpha = u_1^\alpha + u_2^\alpha$ *for all such pair* (u_1^α, u_2^α);
4) *if* $\bar{u}_1, \bar{u}_2$ *are the corresponding limit points of* $u_1^{\alpha(\delta_k)}, u_2^{\alpha(\delta_k)}$, *then the pair* $(\bar{u}_1, \bar{u}_2)$ *is a solution of* (2.2) *and, therefore,* $\bar{u} = \bar{u}_1 + \bar{u}_2 = A^{-1}f$.

Proof
Denote by Φ the objective functional in the minimization problem (2.1). Similarly to Theorem 2.1 for minimizing sequence (u_1^k, u_2^k), we have

$$\Phi(u_1^k, u_2^k) \to \Phi_*, k \to \infty$$

that implies boundedness of the sequences $\{u_1^k\}, \{(u_2^k)^{(1)}\}$ in the space BV. Due to the theorem on compact imbedding, we can set that

$$\lim_{k\to\infty} ||u_1^k - \bar{u}_1||_{L_p} = 0, \lim_{k\to\infty} ||u_2^k - \bar{u}_2||, u_2^k(a) \to c_a$$

for certain $\bar{u}_1 \in L_p, \bar{u}_2 \in W_p^1$. It implies the relation

$$\Phi_* \le \Phi(\bar{u}_1, \bar{u}_2) \le \liminf_{k\to\infty} \Phi(u_1^k, u_2^k) = \Phi_*,$$

from which it follows that $(\bar{u}_1, \bar{u}_2)$ is a solution of problem (2.1).

Rename $(\bar{u}_1, \bar{u}_2)$ by (u_1^α, u_2^α). Then for the solution of problem (2.2) and, therefore, for the solution of equation (1.1), the following inequality is true

$$\Phi(u_1^\alpha, u_2^\alpha) \le \Phi(\hat{u}_1, \hat{u}_2),$$

from which the following estimate holds:

$$||u_1^\alpha||_{BV} + |u_2^\alpha(a)| + ||(u_2)^{(1)}||_{BV} \le \frac{\delta^2}{\alpha} + ||\hat{u}_1||_{BV} + |\hat{u}_2(a)| + ||\hat{u}_2||_{BV}. \tag{2.5}$$

Due to the conditions on the parameter $\alpha(\delta)$ from (2.5), it follows that $\{u_1^{\alpha(\delta)}\}$, $\{u_2^{\alpha(\delta)}\}$ are bounded on the BV-norm and there exist the convergent subsequences

$$\lim_{k\to\infty} ||u_1^{\alpha(\delta_k)} - \breve{u}_1||_{L_p} = 0,\ \lim_{k\to\infty} ||u_2^{\alpha(\delta_k)} - \breve{u}_2||_{W_p^1} = 0$$

as $\delta_k \to 0$.

Taking into account (2.5), continuity of the operator A, and lower semicontinuity BV-norm relatively L_p-convergence, we obtain:

$$
\begin{gathered}
||A(\breve{u}_1+\breve{u}_2)-f|| \le \liminf_{k\to\infty}[||A(\breve{u}_1+\breve{u}_2)-f_{\delta_k}||+||f_{\delta_k}-f||] \\
\le 2^{1/2}\liminf_{k\to\infty}\{[\frac{1}{2}||A(u_1^{\alpha(\delta_k)}+u_2^{\alpha(\delta_k)})-f_{\delta_k}||^2+\alpha(\delta_k)(||u_1^{\alpha(\delta_k)}||_{BV} \\
+|u_2^{\alpha(\delta_k)}(a)|+||(u_2^{\alpha(\delta_k)})^{(1)}||_{BV})]^{1/2}+\delta_k\} \le 2^{1/2}\limsup_{k\to\infty}\{\frac{1}{2}[||A(\hat{u}_1+\hat{u}_2)-f_{\delta_k}||^2 \\
+\alpha(\delta_k)(||\hat{u}_1||_{BV}+|\hat{u}_2(a)|+||\hat{u}_2^{(1)}||_{BV})]^{1/2}+\delta_k\} = 2^{1/2}||A(\hat{u}_1+\hat{u}_2)-f||=0,
\end{gathered}
$$

i.e., the pair $(\breve{u}_1, \breve{u}_2)$ is a solution of problem (1.1). In view of (2.5) and the conditions on the parameter $\alpha(\delta_k)$, we have the following inequalities:

$$
\begin{gathered}
||\breve{u}_1||_{BV}+|\breve{u}_2(a)|+||\breve{u}_2^{(1)}||_{BV} \\
\le \liminf_{k\to\infty}(||u_1^{\alpha(\delta_k)}||_{BV}+|u_2^{\alpha(\delta_k)}(a)|+||(u_2^{\alpha(\delta_k)})^{(1)}||_{BV}) \\
+||\hat{u}_1||_{BV}+|\hat{u}_2(a)|+||\hat{u}_2^{(1)}||_{BV}.
\end{gathered} \tag{2.6}
$$

Therefore, the pair $(\breve{u}_1, \breve{u}_2)$ is also the solution of problem (2.2). Besides, from (2.6), we have

$$
\lim_{k\to\infty}||u_1^{\alpha(\delta_k)}|| = ||\hat{u}_1||_{BV},\ \lim_{k\to\infty}||(u_2^{\alpha(\delta_k)})^{(1)}||_{BV} = ||\hat{u}_2^{(1)}||_{BV},
$$

in particular,

$$
\lim_{k\to\infty} J_a^b(u_1^{\alpha(\delta_k)}) = J_a^b(\hat{u}_1),\ \lim_{k\to\infty} J_a^b(u_2^{\alpha(\delta_k)})^{(1)}) = J_a^b(\hat{u}_2^{(1)}),
$$

i.e., in (2.3) and (2.4) the first relations are satisfied.

Let us turn to investigation of the uniform convergence of components. Use the scheme of the proof from [12], where the one-component case for a some other stabilizing functional was considered. As it is known [1] (Theorem 1.17), for any function $\bar{u}$ there exists a sequence $\{\bar{u}_k\}$ such that

$$
\lim_{k\to\infty}||\bar{u}_k-\bar{u}||_{L_1}=0, \lim_{k\to\infty} J_a^b(\bar{u}_k) = J_a^b(\bar{u}). \tag{2.7}
$$

From the convergence of $\bar{u}_k$ in L_1 it follows that exists subsequence such that $\bar{u}_{k_i} \to \bar{u}(x)$ almost everywhere on the segment $[a,b]$. Let x_0 is a point of convergence of the sequence of $\bar{u}_{k_i}$. Then we have the estimate

$$
|\bar{u}_{k_i}(x)-\bar{u}_{k_i}(x_0)| \le V_a^b[\bar{u}_{k_i}] = J_a^b(\bar{u}_{k_i}) = \int_a^b |\bar{u}_{k_i}^{(1)}|dx \le const.
$$

According to the Helly theorem one can select from $\{\bar{u}_{k_i}\}$ a subsequence converging to a function $\tilde{u} \in V_a^b$ for every point $x \in [a,b]$. Without loss generality, one can let, that this subsequence already converges, *i.e.*, $\bar{u}_{k_i} \to \tilde{u}_{k_i}$ for any $x \in [a,b]$. Then $\bar{u}(x) = \tilde{u}(x)$ almost everywhere on $[a,b]$ and

$$
J_a^b(\bar{u}) = J_a^b)(\tilde{u}). \tag{2.8}
$$

Taking into account (2.7), (2.8) and the property upper semi-continuity of the classical variation relatively poitwise convergence, we obtain

$$V_a^b(\tilde{u}) \leq \liminf_{i\to\infty} V_a^b(\bar{u}_{k_i}) = \lim_{i\to\infty} J_a^b(\bar{u}_{k_i}) = J_a^b(\bar{u}) = J_a^b(\tilde{u}). \tag{2.9}$$

Because of [4, Property (d"), p. 29]

$$J_a^b(\bar{u}) = \inf\{V_a^b(g) : g(x) = \bar{u}(x) \ a.e. \ x \in [a,b]\}. \tag{2.10}$$

From (2.10) together with (2.9), we have

$$V_a^b(\tilde{u}) \leq J_a^b(\bar{u}) = J_a^b(\tilde{u}) = \inf\{V_a^b(g) : g(x) = \tilde{u}(x), \ a.e. \ x \in [a,b]\} \leq V_a^b(\tilde{u}). \tag{2.11}$$

Thus, from (2.11) it follows that for any function $\bar{u}$ there exists a function $\tilde{u}$ with the bounded classical variation $V_a^b(\tilde{)}u$ coinciding with the total variation $J_a^b(\bar{u})$; moreover, $\tilde{u}(x) = \bar{u}(x)$ almost on $[a,b]$. In the first part of the proof, it is established that there exists the solution (u_1^α, u_2^α) of the extremal problem (2.1) and the converging subsequences, correspondingly, in L_p and W_p^1

$$u_1^{\alpha(\delta_k)} \to \bar{u}_1, \quad u_2^{\alpha(\delta_k)} \to \bar{u}_2, \tag{2.12}$$

as $\alpha(\delta_k) \to 0, \delta_k^2/\alpha(\delta_k) \to 0, \delta_k \to 0$, where $\hat{u}_1 + \hat{u}_2 = A^{-1}f$. As it was proved that for any function $\bar{u} \in BV$ there exists an equivalent function $\tilde{u} \in V_a^b$ such that (2.11) is fulfilled, then it one can admit that for any α

$$J_a^b(u_1^\alpha) = V_a^b(u_1^\alpha), \quad J_a^b((u_2^\alpha)^{(1)}) = V_a^b(u_2^{\alpha(\delta_k)})^{(1)}).$$

In view of relation (2.12) one can set

$$u_1^{\alpha(\delta_k)}(x) \to \bar{u}_1(x), \quad u_2^{\alpha(\delta_k)}(x))^{(1)} \to (\bar{u}_2(x))^{(1)}$$

almost everywhere on $[a,b]$. Applying argumentation, which was above used for the sequence $\bar{u}_k$, property (2.11) and the Helly theorem, we can select from $\{u_1^{\alpha(\delta_k)}\}, \{u_2^{\alpha(\delta_k)}\}$ subsequences converging at every point $x \in [a,b]$. Without loss of generality, one can let that for any $x \in [a,b]$

$$u_1^{\alpha(\delta_k)}(x) \to \bar{u}_1(x), \quad (u_2^{\alpha(\delta_k)}(x))^{(1)} \to (\bar{u}_2(x))^{(1)}. \tag{2.13}$$

From the lower semicontinuity of the classical variation $V_a^b(u)$ and above established convergence the total variation for $\{u_1^{\alpha(\delta_k)}, (u_2^{\alpha(\delta_k)})^{(1)}\}$, we obtain

$$V_a^b(\bar{u}_1) \leq \liminf_{k\to\infty} V_a^b(u_1^{\alpha(\delta_k)}) \leq \liminf_{k\to\infty} .J_a^b(u_1^{\alpha(\delta_k)}) = J_a^b(\bar{u}_1), \tag{2.14}$$

This together with property (2.10) provides

$$V_a^b(\bar{u}_1) = J_a^b(\bar{u}_1). \tag{2.15}$$

Similarly the following equality is proved

$$V_a^b(\bar{u}_2^{(1)}) = J_a^b(\bar{u}_2^{(1)}). \tag{2.16}$$

On the basis of the result from the monograph [8] (Theorem 1 and Corollary 1, p. 258) and relations (2.13)-(2.16) it follows that for the sequences $\{u_1^{\alpha(\delta_k)}, (u_2^{\alpha(\delta_k)})^{(1)}\}$ the uniform convergence holds on every subsegment $[a', b'] \in [a,b]$ which does not contains points of breaks.

3 Finite-dimensional approximation of the regularization method

Let us define for simplicity, a uniform grid in the segment $[a,b]$ with the width h :

$$\Delta_n : a = x_0 < x_1 < ... < x_n = b, \quad x_i - x_{i-1} = h, \quad 1/h = n.$$

According to (2.1), let us consider the set of the of finite-dimensional problems

$$\min\{\frac{1}{2}||A(u_1+u_2 = f_\delta||^2_{L_2}+\alpha[||u_1||_{BV}+|u_2(a)|+||u_2^{(1)}||_{BV}] : u_1, u_2^{(1)} \in BV \cap U_n\} = \Phi_*, \tag{3.1}$$

where U_n is the subspace of the piecewise-linear functions constructed on the grid Δ_n.

Theorem 3.1. *Let the condition of* Theorem 2.2 *be fulfilled. Then problem* (3.1) *has a solution* $(\hat{u}_{1n}, \hat{u}_{2n})$*, for which the following properties hold:*
1) $\{\hat{u}_{1n}\}$ *is relatively compact in* L_p *and all its limit points are the first components for the solution of problem* (2.1)*;*
2) $\{\hat{u}_{2n}\}$ *is relatively compact in* W^1_p *and all its limit points are the second components for the solution of problem* (2.1)*.*

Proof
Let (u^m_{1n}, u^m_{2n}), be a minimizing sequence in problem (3.1) for fixed $\alpha > 0$ and n. Then it is evident that the following components are uniformly bounded on m :

$$||u^m_{1n}||_{BV} \leq c_1, \quad ||(u^m_{2n})^{(1)}|| \leq c_2, \quad |u^m_{2n}| \leq c_3.$$

From compactness of the embedding operator from BV into L_p, it follows that there exist converged subsequences

$$\lim_{k\to\infty} ||u^{m_k}_{1n} - \hat{u}_{1n}||_{L_k}, \quad \lim_{k\to\infty} ||(u^{m_k}_{2n})^{(1)} - \hat{v}_{2n}||$$

and $u^{m_k}_{2n} \to c_2$. From this it follows that $\{u^{m_k}_{2n}\}$ is the Cauchy sequence in W^1_p. Hence, there exists $\hat{u}_{2n}$ such that

$$\hat{v}_{2n} = \hat{u}^{(1)}_{2n} \quad \lim_{k\to\infty} ||u^{m_k}_{2n} - \hat{u}_{2n}||_{W^1_p} = 0.$$

and $\hat{u}_{1n}, \hat{u}^{(1)}_{2n} \in BV \cap U_n$. Taking into account the properties of the operator and BV-norm, we arrive to the relations:

$$\Phi^n_* \leq \Phi(\hat{u}_{1n}, \hat{u}_{2n}) \leq \liminf_{k\to\infty} \Phi(u^{m_k}_{1n}, u^{m_k}_{2n}) \leq \limsup_{k\to\infty} \Phi(u^{m_k}_{1n}, u^{m_k}_{2n}) = \Phi^n_*.$$

It means that the pair $(\hat{u}_{1n}, \hat{u}_{2n})$ is a solution of problem (3.1).

On the basis of Theorem 1.17 from [4] (see, also, relation (2.7)) there exist the functions $u^\epsilon_1, u^\epsilon_2 \in C^\infty$ such that

$$\Phi(u^\epsilon_1, u^\epsilon_2) \leq \Phi_* + \epsilon.$$

Let us denote by p_n the projector that acts as $p_n(u) = u_n$, where u_n is piecewise linear function constructed on the grid Δ_n. Then, in the view of the uniform convergence of the piecewise linear approximation to a smooth function, we come to the following relations:

$$\Phi_* \leq \liminf_{n\to\infty} \Phi_*^n \leq \liminf_{n\to\infty} \Phi(p_n u_1^\epsilon, p_n u_2^\epsilon)$$
$$\leq \limsup_{n\to\infty}\{\frac{1}{2}||A(p_n u_1^\epsilon + p_n u_2^\epsilon) - f_\delta||^2\}$$
$$+\alpha[||p_n u_1^\epsilon||_{BV} + |p_n u_2^\epsilon(a)| + ||(p_n u_2^\epsilon)^{(1)}||]\} = \Phi(p_n u_1^\epsilon, p_n u_2^\epsilon) \leq \Phi_* + \epsilon$$

for arbitrary small $\epsilon > 0$, *i.e.*,

$$\lim_{n\to\infty} \Phi_*^n = \Phi_*. \tag{3.2}$$

From (3.2) it follows that solutions of the finite-dimensional problem (3.1) are bounded

$$||\hat{u}_{1n}|| \leq \bar{c}_1, \quad |\hat{u}_{2n}(a)| \leq \bar{c}_2, \quad ||\hat{u}_{2n}^{(1)}|| \leq \bar{c}_3.$$

As in the proof of solvability of problem (3.1) one can select subsequences converging in L_p

$$\hat{u}_{1n_k} \to \bar{u}_1, \quad \hat{u}_{2n_k} \to \bar{u}_2, \quad \hat{u}_{2n_k}^{(1)} \to \bar{v}_2, k \to \infty,$$

where $\bar{v}_2 = \bar{u}_2^{(1)}$. It implies the following relations

$$\Phi_* \leq \Phi(\bar{u}_1, \bar{u}_2) \leq \liminf_{k\to\infty} \Phi(\hat{u}_{1n_k}, \hat{u}_{2n_k}) = \lim_{k\to\infty} \Phi_*^n = \Phi_*.$$

Thus, all limit points $(\bar{u}_1, \bar{u}_2)$ of the solutions $(\hat{u}_{1n}, \hat{u}_{2n})$ of the finite-dimensional problem (3.1) are the solutions of the infinite-dimensional problem (2.1). Here for the first component the strong convergence in L_p and for the second component in W_p^1 hold.

Remark 3.1 After piecewise linear approximation of problem (3.1), the stabilizing functional takes more simple and suitable for computation form, in particular,

$$\begin{aligned}
J_a^b(u_{1n}) &= \sup\{\textstyle\int_a^b u_{1n} v^{(1)}(x)\,dx : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \sup\{-\textstyle\int_a^b u_{1n}^{(1)}(x) v(x)\,dx : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \sup\{-\textstyle\sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{u_{1n}^i - u_{1n}^{i-1}}{h} v(x)\,dx : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&\textstyle\sum_{i=1}^n |u_{1n}^i - u_{1n}^{i-1}|,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
J_a^b(u_{2n}^{(1)}) &= \sup\{\textstyle\int_a^b u_{2n}(x) v^{(1)}(x)\,dx : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \sup\{\textstyle\sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{u_{2n}^i - u_{2n}^{i-1}}{h} v^{(1)}(x)\,dx : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \sup\{\textstyle\sum_{i=1}^n \frac{u_{2n}^i - u_{2n}^{i-1}}{h} (v_i - v_{i-1}) : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \sup\{\textstyle\sum_{i=1}^n \frac{(u_{2n}^i - u_{2n}^{i-1}) - (u_{2n}^{i+1} - u_{2n}^i)}{h} v_i : \quad |v(x)| \leq 1, v \in C_0^1\}\\
&= \textstyle\sum_{i=1}^n h|\frac{u_{2n}^{i+1} - 2u_{2n}^i + u_{2n}^{i-1}}{h^2}|,
\end{aligned} \tag{3.4}$$

$$\int_a^b |u_{1n}(x)|\,dx = \sum_{i=1}^n \frac{h}{2}|u_{1n}^i + u_{1n}^{i-1}|, \int_a^b |u_{2n}^{(1)}(x)|\,dx = \sum_{i=1}^n h|\frac{u_{2n}^i - u_{2n}^{i-1}}{h}|.$$

4 Newton method and numerical experiments

For solving the minimization problem (3.1) and finding the regularized components, the Newton method is used. For this in the stabilizer the non-differentiable BV-norms are preliminary approximated by the following smooth functionals (cf. [1], [13]):

$$||u^\beta||_{L_1} = \int_a^d \sqrt{u^2(x)+\beta^2}dx, \quad J_a^b(u^\beta) = \int_a^b \sqrt{u^{(1)}(x))^2+\beta^2},$$

and $||u^{(1)}||_{L_1}, J_a^b(u^{(1)})$ are replaced by the functionals

$$||(u^\beta)^{(1)}||_{L_1} = \int_a^b \sqrt{u^{(1)}(x))^2+\beta^2}\,dx, \quad J_a^b((u^\beta)^{(1)}) = \int_a^b \sqrt{(u^{(2)}(x))^2+\beta^2}\,dx,$$

where β is a small parameter and $u^{(2)}(x) = d^2u(x)/dx^2$.

Further, for the smooth BV- norms the discrete approximation is applied, *i.e.*, the derivatives are replaced by difference relations as in (3.3), (3.4), and the integrals are changed by sums on the basis of the quadrature formula of rectangles. Introduce the following notations:

$$\psi(t) = \sqrt{t+\beta^2}, \quad u_i = u(x_i).$$

After the discrete approximation the functionals $||u^\beta||_{L_1}, J_a^b(u^\beta)$ have the form:

$$||u^\beta||_{L_1} \sim \sum_{i=0}^{n} \psi(u_i^2)h, \quad J_a^b(u^\beta) \sim \sum_{i=1}^{n} \psi((< D_i^{(1)}, u >)^2)h,$$

where $D_i^1 = (0, ..., 0, -1/h, 1/h, 0, ..., 0) \in R^{n+1}$, the element $1/h$ has the i-th position. Using these relations, we obtain the formulae for the gradients:

$$\text{grad}\,(\sum_{i=0}^{n} \psi(u_i^2)h) = L_1(u)u, \quad L_1 = \text{diag}_1\,(\psi^{(1)}(u)),$$

$$\text{grad}\,(\sum_{i=1}^{n} \psi(< D_i^1, u >^2)h) = L_2(u)u,$$

where $L_2(u) = D_1^T \text{diag}_2\,(\psi^1(u))D_1$, D_1 is the matrix $n \times (n+1)$, the i-th row of which is $D_i^{(1)}$, $\text{diag}_2(\psi^{(1)}(u))$ is the matrix $n \times n$ with i-th diagonal element is equal to $\psi^{(1)}(< D_i^{(1)}, u >^2)$.

Since $||u^{\beta(1)}||_{L_1} = J_a^b(u^\beta)$, the gradient of the discrete variant of this norm has the form

$$\text{grad}\,(\sum_{i=1}^{n} \psi((< D_i^1, u >)^2)h) = L_3(u)u,$$

where $L_3(u) = L_2(u)$. Discrete approximation of $J_a^b(u^{\beta(1)})$ brings to the relation

$$J_a^b(u^{\beta(1)}) \sim \sum_{i=1}^{n-1} h\,\psi(< D_i^2, u >^2),$$

where $D_i^2 = (0, ..., 0, 1/h^2, -2/h^2, 1/h^2, 0, ..., 0) \in R^{n+1}, i = 1, 2, ..., n-1$. The relation for the gradient takes the form

$$\text{grad}\,(\sum_{i=1}^{n-1} h\psi(< D_i^2, u >^2)) = L_4(u)u,$$

where $L_4(u) = h\, D_2^T \text{diag}_4\,(\psi^{(1)}(u))D_2, D_2$ is the matrix of dimension $(n-1)\times(n-1)$, the i-th row of which is D_i^2, $\text{diag}_4\,(\psi^{(1)}(u))$ is the matrix of dimension $(n-1)\times(n-1)$, the i-th element which is equal to $\psi^{(1)}(< D_i^2, u >^2)$.

The functional $|u^\beta(u)| = \psi((u_0)^2)$, therefore,

$$\text{grad}\,(|u^\beta(a)|) = L_5(u)u, \quad L_5(u) = \text{diag}_5(\psi^{(1)}(u));$$

here $\text{diag}_5\,(\psi^{(1)})$ is the matrix of dimension $(n+1\times)(n+1)$, in which the element with the index $(0,0)$ is equal $\psi^{(1)}(u_0)$ and the rest ones are equal to zero. For the discrepancy functional, we have

$$\text{grad}\,(\frac{1}{2}||Au - f_\delta||_{L_2}^2) = A^*(Au - f_\delta)).$$

After discrete approximation, A, A^* are replaced by the matrices A_n, A_n^* and f_δ is changed by a vector of the corresponding dimension. Let us retain the notation $\Phi(u_1, u_2)$ for the objective functional also and in the discrete variant. Then with taking into account of the obtained formulae for the gradients, the minimization problem is reduced to the following system of equations:

$$\text{grad}\,(\Phi(u_1,u_2)) = \begin{pmatrix} A_n^T(A_n(u_1+u_2) - f_n) + \alpha_1(L_1(u_1) + L_2(u_1)) = 0 \\ A_n^T(A_n(u_1+u_2) - f_n) + \alpha_2(L_3(u_2)u_2) + L_4(u_2)u_2 + L_5(u_2)u_2) = 0. \end{pmatrix}$$

Introduce the notation $B(u_1, u_2) = \text{grad}\,(\Phi(u_1, u_2))$. For formation of the Newton method, it is necessary to calculate the derivative of the operator $B(u_1, u_2)$. It should be noted that each of gradients of the functionals entering in the stabilizer, has the form $F_i(u) = L_i(u)u$. Therefore, for its derivative the following formula holds:

$$(L_i(u)u)^{(1)} = L_i(u) + L_i^1(u)u. \tag{4.1}$$

Numerical experiments show that without essential loss of precision the second term in (4.1) can be neglected. After this simplification, the derivative of the operator $B(u_1, u_2)$ adopts the form

$$B^{(1)}(u_1,u_2) = \begin{pmatrix} A^TA + \alpha_1(L_1(u_1) + L_2(u_1)) & A^TA \\ A^TA & A^TA + \alpha_2(L_3(u_2) + L_4(u_2) + L_5(u_2)) \end{pmatrix}$$

and the Newton method can be written as

$$(u_1^{k+1}, u_2^{k+1})^T = (u_1^k, u_2^k)^T - [B^{(1)}(u_1^k, u_2^k)]^{-1}\text{grad}\,(\Phi(u_1^k, u_2^k)), \tag{4.2}$$

where u_1, u_2 are vectors of dimension $n+1$.

For testing the Newton method, iterative process (4.2) is applied to the integral equation, which arises under continuation of a gravitational field on the depth H [5]

$$A\,u \equiv \frac{1}{\pi}\int_{-1}^{1} \frac{H}{(x-s)^2 + H^2} u(s)ds = f(x).$$

The operator A is approximated by the matrix A_n with component $\{a_{ij}\}$, where

$$a_{ij} = \frac{Hh}{(x_i - s_j)^2 + H^2}, \ h = 2/101, \ n = 101, \ i, j = 0, 1, ..., n, \ H = 0.3.$$

In the first experiment, the Newton method is used to reconstruct the solution u, which contains only one peculiarity: either a break of the solution, or a break of the solution derivative. As the stabilizing functional either $||u_1||_{BV}, (u_2 = 0)$, or $||u_2||_{BV}, (u_1 = 0)$ in problem (2.1) are used. As the model solutions the following functions are taken:

$$u_1(x) = \begin{cases} 0, & \text{if } -1 \le x < -0.8; \\ 3, & \text{if } -0.8 \le x \le -0.5; \\ 0, & \text{if } -0.5 < x < 0; \\ 2, & \text{if } 0 \le x \le 0.25; \\ 1, & \text{if } 0.25 < x \le 0.75; \\ 0, & \text{if } 0.75 < x \le 1. \end{cases}$$

$$u_2(x) = \begin{cases} 0, & \text{if } -1 \le x < -0.5; \\ 4x + 2, & \text{if } -0.5 \le x < -0.25; \\ -2x + 0.5, & \text{if } -0.25 \le x < 0; \\ 2x + 0.5, & \text{if } 0 \le x < 0.25; \\ -4x + 2, & \text{if } 0.25 \le x < 0.5; \\ 0, & \text{if } 0.5 \le x \le 1. \end{cases}$$

In Fig. 1 the exact $u_1(x)$ (solid line) and reconstructed u_1^N (dotted line) solutions are presented. Here, the parameter of regularization is $\alpha = 0.5\,10^{-5}$, the number of iterations $N = 50$, the relative error of the right-hand side is $\Delta = 0.012$, the relative error of the solution is u_1^N is $\bar{\Delta} = 0.227$.

Fig. 2 shows the results obtained under reconstruction of the model solution $u_2(x)$ for the same error level of the right-hand side. Here $\alpha = 0.5\,10^{-5}, N = 50$. The relative error of the numerical solution $\Delta = 0.101$. In the second experiment the model solution contains both peculiarities: a break and a fracture (break of derivative). The model solution has the form $u(x) = u_1(x) + u_2(x)$, where

$$u_1(x) = \begin{cases} 0, & \text{if } -1 \le x < 0.25; \\ 1, & \text{if } 0.25 \le x \le 0.75; \\ 0, & \text{if } 0.75 < x \le 1. \end{cases}$$

$$u_2(x) = \begin{cases} 0, & \text{if } -1 \le x < -0.8; \\ \frac{10}{3}x + \frac{8}{3}, & \text{if } -0.8 \le x < -0.5; \\ -\frac{10}{3}x - \frac{2}{3}, & \text{if } -0.5 \le x < -0.2; \\ 0, & \text{if } -0.2 \le x \le 1. \end{cases}$$

Fig. 3 contains the exact and numerical solutions for such model. In this case $\alpha = 0.5\,10^{-5}, \ N = 50, \ \Delta = 0.003, \ \bar{\Delta} = 0.22$. In contrast to the first experiment, here we have more smoothed the numerical solution.

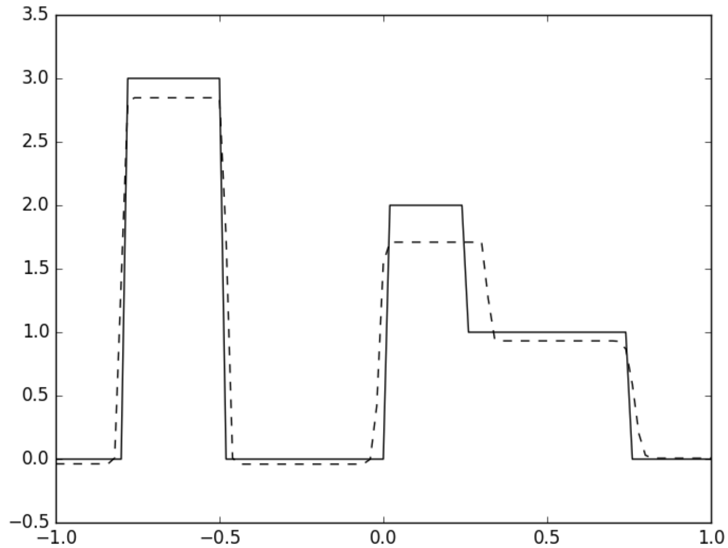

Figure 1: The exact component u_1 (solid line) and reconstructed one u_1^N (dotted line).

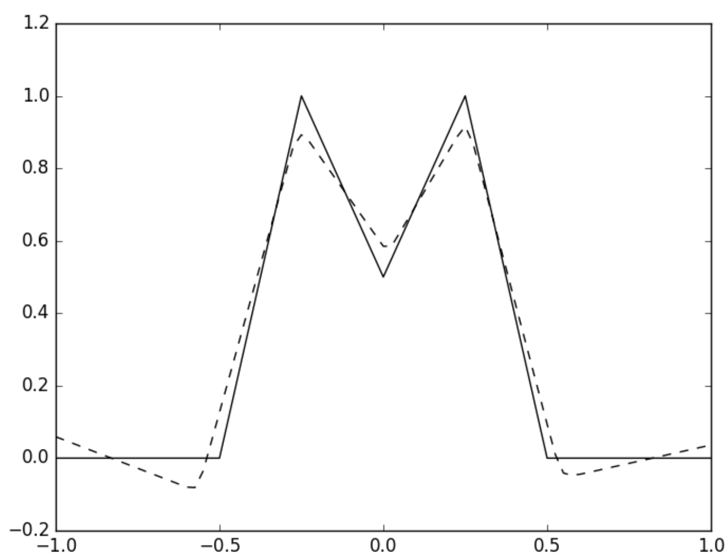

Figure 2: The exact component u_2 (solid line) and reconstructed one u_2^N (dotted line).

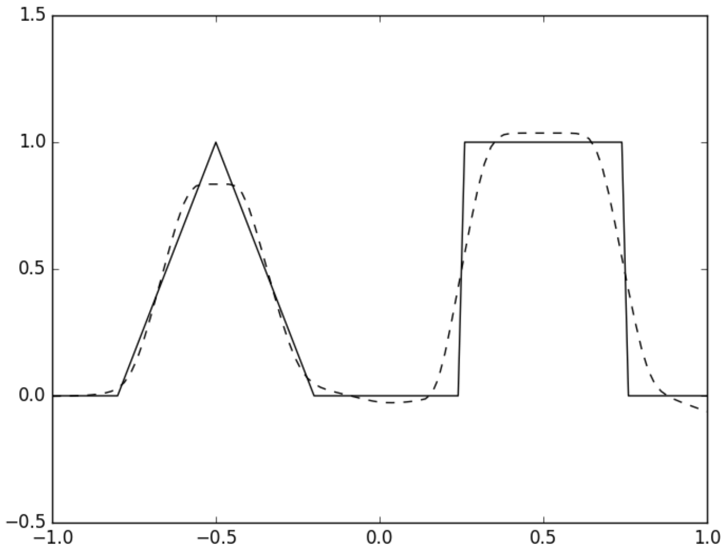

Figure 3: The exact model solution $u = u_1 + u_2$ (solid line) and reconstructed one $u^N = u_1^N + u_2^N$ (dotted line).

Conclusion. From the theoretical view point for the modified Tikhonov method, the L_p-convergence and the piecewise uniform convergence was proved for the regularized components responding to approximation of the discontinuous component of the solution. Also, the W_p-convergence for the second component and the piecewise uniform convergence for its derivative were ascertained. The numerical results obtained with using the Newton method show that in the case when there is only one type of peculiarity the constructing RA provides the quite good results with preservation of the subtle structure of a solution. In finding a solution with both peculiarities the quality of a solution is some worse. In this case there is oversmoothing the approximate solution. Probably, here it is necessary to choose more carefully the control parameters.

Acknowledgement

The work was supported by Russian Foundation for Basic Research (projects no 15-01-00629).

References

[1] R. Acar and C.R. Vogel, *Analysis of bounded variation penalty methods for ill-posed problems*, Inverse Problems 10 (1994),1217-1229.

[2] E.J. Gandes, J. Romberg and T. Tao, *Stable signal recovery from incomplete and inaccurate measurments*, Pure Appl. Math. 59 (2006), 1207-1223.

[3] A. Gholami and S.M. Hosseini, *A balanced combination of Tikhonov and total variation regularization for reconstruction of piecewise- smooth signal*, Signal Processing 93 (2013), 1945-1960.

[4] E. Giusti, *Minimal Surfaces Functions of Bounded Variation, Monogr. Math. 80*, Birkhäuser, Basel, 1984.

[5] M.M. Lavrentiev, V.G. Romanov, S.P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis* (in Russian), Nauka, Moscow, 1980.

[6] A.S. Leonov, *Piecewise-uniform regularization of ill-posed problems with discontinuous solutions* (in Russion), Zh. Vychisl. Mat. Mat. Fiz. 22 (1982), 516-531.

[7] A.S. Leonov, *On the total variation convergence of regularizing algorithms for ill-posed prpblems* (in Russian) Zh. Vychisl. Mat. Mat. Fiz. 47 (2007), 767-783.

[8] A.N. Tikhonov, A.S. Leonov, A.G. Yagola, *Nonlinear Ill-Posed Problems*, Chapman and Hall, London, v. 1, 2, 1998.

[9] V.V. Vasin, *Reconstruction of smooth and discontinuous components of solutions to linear ill-posed problems*, Dokl. Math. 87 (2013), 23-25.

[10] V.V. Vasin, *Approximation of solution with singulasrities of various type for linear ill-posed problems*, Dokl. Math. 89 (2014), 30-33.

[11] V.V. Vasin, *Regularization of ill-posed problems by using stabilizers in the form of the total variation of a function and its derivatives*, J. Inverse Ill-Posed Probl. 24 (2016), 149-158.

[12] V.V. Vasin, M.A. Korotkii, *Tikhonov regularization with nondiferentiable stabilizing functionals*, J. Inverse Ill-Posed Probl. 15 (2007), 853-865.

[13] C.R. Vogel, *Computational Methods for Inverse Problems, Front. Appl. Math. 23*, SIAM, Philadelphia, 2002.

Vladimir Vasin,
N. N. Krasovskii Institute of Mathematics and Mechanics UB RAS,
S. Kovalevskay Street, 16, Ekaterinburg 620990, Russia
Ural Federal University,
Lenin Avenue, 51, Ekaterinburg 620000, Russia,
Email: `vasin@imm.uran.ru`,

Vladimir Belyaev,
N. N. Krasovskii Institute of Mathematics and Mechanics UB RAS,
S. Kovalevskaya street, 16, Ekaterinburg, 620990, Russia,
Ural Federal University,
Lenin Avenue, 51, Ekaterinburg, 620000, Russia,
Email: `beliaev_vv@mail.ru`

Received 03.04.2017, Accepted 21.04.2017