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INTEGRALS OF SPHERICAL HARMONICS WITH FOURIER EXPONENTS IN MULTIDIMENSIONS

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Abstract We consider integrals of spherical harmonics with Fourier exponents on the sphere $\mathbb{S}^n$, $n \geq 1$. Such transforms arise in the framework of the theory of weighted Radon transforms and vector diffraction in electromagnetic fields theory. We give analytic formulas for these integrals, which are exact up to multiplicative constants. These constants depend on choice of basis on the sphere. In addition, we find these constants explicitly for the class of harmonics arising in the framework of the theory of weighted Radon transforms. We also suggest formulas for finding these constants for the general case.

Key words: Fourier transform, spherical harmonics, weighted Radon transforms

AMS Mathematics Subject Classification: 33C55, 42B10

1 Introduction

We consider the integrals

$$I_k^m(p,\rho) \stackrel{def}{=} \int\limits_{\mathbb{S}^n} Y_k^m(\theta) e^{i\rho(p,\theta)}\, d\theta,\ p \in \mathbb{S}^n,\ \rho \geq 0,\ n \geq 1, \tag{1}$$

where $\{Y_k^m \,|\, k \in \mathbb{N} \cup \{0\},\ m = \overline{1, a_{k,n+1}}\}$ is orthonormal basis of spherical harmonics on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ (see e.g. [8], [10]), $a_{k,n}$ is defined as follows:

$$a_{k,n+1} = \binom{n+k}{k} + \binom{n+k-2}{k-2},\ a_{0,n} = 1,\ a_{1,n} = n, \tag{2}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k!)},\ n,k \in \mathbb{N} \cup \{0\}. \tag{3}$$

We recall that spherical harmonics $\{Y_k^m\}$ are eigenfunctions of the spherical Laplacian $\Delta_{\mathbb{S}^n}$ and the following identity holds (see e.g. [8], [10]):

$$\Delta_{\mathbb{S}^n} Y_k^m = -k(n+k-1)Y_k^m,\ m = \overline{1, a_{k,n+1}}, \tag{4}$$

where $a_{k,n}$ is defined in (2).

Integrals I_k^m arise, in particular, in connection with iterative inversions of the weighted Radon transforms in dimension $d = n + 1 = 2$; see [7]. In addition, an exact (up to multiplicative coefficient depending on k) analytic formula for (1) was given in [7] for $d = n + 1 = 2$, $k = 2j$, $j \in \mathbb{N} \cup \{0\}$.

We recall that weighted Radon transform operator R_W is defined (in dimension $d = n+1$) as follows (see e.g. [6], [7]):

$$R_W f(s,\theta) \overset{def}{=} \int\limits_{x\theta=s} W(x,\theta) f(x)\, dx,\ s\in\mathbb{R},\ \theta\in\mathbb{S}^n, \tag{5}$$

where W is the weight function on $\mathbb{R}^{n+1}\times\mathbb{S}^n$, f is a test-function.

The present work is strongly motivated by the fact that integrals I_k^m also arise in the theory of weighted Radon transforms defined by (5) for higher dimensions $d = n+1 \geq 3$. This issue will be presented in detail in the subsequent work [4].

On the other hand, in [9] integrals I_k^m were considered for the case of $n = 2$ in connection with vector diffraction in electromagnetic theory and exact analytic formulas were given for this case.

In addition, for the case of dimension $n = 2$ more general forms of integrals I_k^m were considered in the recent work [1]. In particular, the results of [1] coincide with the results of the present work for the case of dimension $n = 2$.

In the present work we prove that

$$I_k^m(p,\rho) = c(m,k,n) Y_k^m(p) \rho^{(1-n)/2} J_{k+\frac{n-1}{2}}(\rho), \tag{6}$$

where $J_r(\cdot)$ is the r-th Bessel function of the first kind, $c(m,k,n)$ is a constant which depends on indexes m,k of spherical harmonic Y_k^m and on dimension n; see Theorem 1 in Section 2.

This result is new for the case of $d = n+1 = 2$ for odd k and for $d = n+1 > 3$ in general.

In the framework of applications to the theory of weighted Radon transforms, integrals I_k^m arise for the case of even $k = 2j$, $j\in\mathbb{N}\cup\{0\}$; see formula (7) of [7] for $n = 1$ and subsequent work [4] for $n \geq 2$. For $k = 2j$ we find explicitly the constants $c(m,2j,n)$ arising in (6); see Theorem 2.1 in Section 2.

It is interesting to note that the constants $c(m,2j,n)$ are expressed via the eigenvalues of the Minkowski-Funk transform $\mathcal{M}$ on $\mathbb{S}^n$, where operator $\mathcal{M}$ is defined as follows (see e.g. [3], [5]):

$$\mathcal{M}[f](p) = \int\limits_{\mathbb{S}^n,\,(\theta p)=0} f(\theta)\, d\theta,\ p\in\mathbb{S}^n, \tag{7}$$

where f is an even test-function on $\mathbb{S}^n$; see Section 2 for details .

In Section 3 we give proofs of Theorem 2.1.

2 Main results

Theorem 2.1. *Let $I_k^i(p,\rho)$ be defined by (1). Then:*

(i) The following formulas hold:

$$I_k^m(p,\rho) = c(m,k,n) Y_k^m(p) \rho^{(1-n)/2} J_{k+\frac{n-1}{2}}(\rho), \tag{8}$$

$$I_k^m(p,-\rho) = (-1)^k I_k^m(p,\rho), \tag{9}$$

$$p\in\mathbb{S}^n,\ \rho\in\mathbb{R}_+ = [0,+\infty),$$

where $J_r(\rho)$ is the standard r-th Bessel function of the first kind, $c(m,k,n)$ depends only on integers m, k, n for fixed orthonormal basis $\{Y_k^m\}$.

(ii) In addition, for $k = 2j$, $j \in \mathbb{N} \cup \{0\}$, the following formulas hold:

$$c(m,2j,n) = \frac{2^{(n-1)/2}\pi\Gamma\left(j+\frac{n}{2}\right)\lambda_{j,n}}{\Gamma\left(j+\frac{1}{2}\right)}, \tag{10}$$

$$\lambda_{j,n} = 2(-1)^j\left[\sqrt{\pi}\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+1)}\right]^{n-1}, \tag{11}$$

where $c(m,k,n)$ are the constants arising in (8), $\Gamma(\cdot)$ is the Gamma function, $\lambda_{j,n}$ are the eigenvalues of the Minkowski-Funk operator $\mathcal{M}$ defined in (7).

In the case of $n=1$, $k=2j$ formulas (8)-(9) arise in formula (7) of [7]. In the case of $n=2$ formulas (8)-(9) and constants $c(m,k,n)$ for general k were given in formula (1) of [9].

We didn't success to find in literature the explicit values for the eigenvalues $\lambda_{j,n}$ of operator $\mathcal{M}$ defined in (14) for $n>2$.

In particular, formulas (8)-(11) are very essential for inversion of weighted Radon transforms; see [7] for $n=1$ and the subsequent work [4] for $n \geq 2$.

3 Proofs

3.1 Proof of formula (8)

From the Funk-Hecke Theorem (see e.g. [8], Chapter 2, Theorem 2.39) it follows that:

$$\int\limits_{\mathbb{S}^n} Y_k^m(\theta)e^{i\rho(p\theta)}\,d\theta = Y_k^m(p)c_k^m(\rho),\ m=\overline{1,a_{k,n+1}}. \tag{12}$$

Formula (8) follows from (12), the following differential equation:

$$\frac{1}{\rho^n}\frac{d}{d\rho}\left(\rho^n\frac{dc_k^m}{d\rho}\right)+\left(1-\frac{\mu_{k,n}}{\rho^2}\right)c_k^m=0,\ c_k^m(0)=0 \text{ for } k\geq 1, \tag{13}$$

$$\mu_{k,n} = k(n+k-1), \tag{14}$$

where function $c_k^m(\rho)$ arises in the right hand-side of (12), and from the fact that the solution of equation (13) with indicated boundary condition is given by the formula:

$$c_m^k(\rho) = c(m,k,n)\rho^{(1-n)/2}J_{k+\frac{n-1}{2}}(\rho), \tag{15}$$

where $c(m,k,n)$ is some constant depending on integers m,k,n; $J_r(\rho)$ is the r-th Bessel function of the first kind (see e.g. [11]).

Formulas (12), (15) imply formula (8).

It remains to prove that formulas (13)-(15) hold. First, we prove that formulas (13), (14) hold.

We recall that Laplacian Δ in $\mathbb{R}^{n+1}$ in spherical coordinates is given by the formula:

$$\Delta u = \frac{1}{\rho^n}\left(\rho^n \frac{d}{d\rho}u\right) + \frac{1}{\rho^2}\Delta_{\mathbb{S}^n}u,\ \rho \in (0,+\infty), \tag{16}$$

where u is a test function.

From formulas (1), (4), (12), (16) it follows that

$$\Delta I_k^m(p,\rho) = Y_k^m(p)\left(\frac{1}{\rho^n}\frac{d}{d\rho}\left(\rho^n \frac{dc_k^m}{d\rho}\right) - \frac{\mu_{k,n}c_k^m}{\rho^2}\right), \tag{17}$$

where $\mu_{k,n}$ is defined in (14).

On the other hand,

$$\begin{aligned}\Delta I_k^m(p,\rho) &= \int_{\mathbb{S}^n} Y_k^m(\theta)\Delta e^{i\rho(p\theta)}\, d\theta \\ &= -\int_{\mathbb{S}^n} Y_k^m(\theta)\,|\theta|^2\, e^{i\rho(p\theta)}\, d\theta = -I_k^m(p,\rho),\ p\in\mathbb{S}^n,\ \rho\geq 0. \end{aligned}\tag{18}$$

Formulas (13), (14) follow from (1), (17), (18). In particular, the boundary condition in (13) follows from orthogonality of $\{Y_k^m\}$ on $\mathbb{S}^n$ and the following formulas:

$$I_k^m(0,p) = \int_{\mathbb{S}^n} Y_k^m(\theta)\, d\theta = \begin{cases} 0,\ k\geq 1, \\ \mathrm{vol}(\mathbb{S}^n)c,\ k=0\end{cases}, \tag{19}$$

$$Y_0^1(p) = c \neq 0,\ p\in\mathbb{S}^n, \tag{20}$$

where I_k^m is defined in (1), $\mathrm{vol}(\mathbb{S}^n)$ denotes the standard Euclidean volume of $\mathbb{S}^n$.

Next, formula (15) is proved as follows.

We use the following notation for fixed k, m:

$$y(t) = c_k^m(\rho), t = \rho \geq 0. \tag{21}$$

Using differential equation (13) and the notations from (21) we obtain:

$$ty''(t) + ny'(t) + \left(t - \frac{\mu_{k,n}}{t}\right)y(t) = 0,\ y(0) = 0,\ \text{ for } k\geq 1. \tag{22}$$

In order to solve (22) we make the following change of variables:

$$y(t) = t^{(1-n)/2}Z(t),\ t\geq 0. \tag{23}$$

Formula (23) implies the following expressions for $y'(t)$, $y''(t)$ arising in (22):

$$y'(t) = \frac{1-n}{2}t^{-(1+n)/2}Z(t) + t^{(1-n)/2}Z'(t), \tag{24}$$

$$y''(t) = \frac{(n^2-1)}{4}t^{-(1+n)/2}\frac{Z(t)}{t} + (1-n)t^{-(1+n)/2}Z'(t) + t^{(1-n)/2}Z''(t),\ t\geq 0, \tag{25}$$

where $Z(t)$ is defined in (23).

Using formulas (22), (24), (25) we obtain:

$$tZ''(t) + Z'(t) + \left(t - \frac{\left(k + \frac{n-1}{2}\right)^2}{t}\right) Z(t) = 0,\ t \geq 0. \tag{26}$$

Differential equation (26) for unknown function $Z(t)$ is known as Bessel differential equation of the first kind with parameter $k + (n-1)/2 \in \mathbb{R}$ (see e.g. [11]). The complete solution of (26) is given by the following formula:

$$Z(t) = C_1 J_{k+\frac{n-1}{2}}(t) + C_2 Y_{k+\frac{n-1}{2}}(t),\ t \geq 0, \tag{27}$$

where $J_r(t), Y_r(t)$ are r-th Bessel functions of the first and second kind, respectively, C_1, C_2 are some constants; see e.g. [11]. Boundary condition in (22) implies that

$$Z(t) = C_1 J_{k+\frac{n-1}{2}}(t),\ t \geq 0. \tag{28}$$

Formulas (21), (23), (28) imply that (15) is the complete solution of (22).

Formula (8) is proved.

3.2 Proof of formula (9)

Formula (9) follows from definition (1) and the following property of the spherical harmonic Y_k^m:

$$Y_k^m(-\theta) = (-1)^k Y_k^m(\theta),\ \theta \in \mathbb{S}^n. \tag{29}$$

Property (29) reflects the fact that $Y_k^m(\theta) = Y_k^m(\theta_1, \ldots, \theta_{n+1})$ is a homogeneous polynomial of degree k restricted to $\mathbb{S}^n$.

Formula (9) in Theorem 2.1 is proved.

3.3 Proof of formulas (10), (11)

Formula (10) follows from orthonormality of $\{Y_k^m\}$ (in the sense of $L^2(\mathbb{S}^n)$), from formulas (1), (8), (9), the following formula:

$$\mathcal{M}[Y_{2k}^m] = \lambda_{k,n} Y_{2k}^m \tag{30}$$

and from the following identities:

$$\int_{\mathbb{S}^n} \overline{Y_{2k}^m}(p)\,dp \int_{-\infty}^{+\infty} I_{2k}^m(p,\rho)\,d\rho = 2c(m,2k,n)\int_{\mathbb{S}^n} |Y_{2k}^m(p)|^2 dp \int_0^{+\infty} J_{2k+\frac{n-1}{2}}\rho^{(1-n)/2}d\rho \tag{31}$$

$$= 2c(m,2k,n)\int_0^{+\infty} J_{2k+\frac{n-1}{2}}\rho^{(1-n)/2}d\rho$$

$$= c(m,2k,n)\frac{2^{(3-n)/2}\Gamma(\frac{1}{2}+k)}{\Gamma(k+\frac{n}{2})},$$

$$\int_{\mathbb{S}^n} \overline{Y_{2k}^m}(p) \int_{-\infty}^{+\infty} I_{2k}^m(p,\rho)\,d\rho = 2\pi \int_{\mathbb{S}^n} \overline{Y_{2k}^m}(p)\,dp \int_{\mathbb{S}^n} Y_{2k}^m(\theta)\delta(p\theta)\,d\theta \tag{32}$$

$$= 2\pi \int_{\mathbb{S}^n} \overline{Y_{2k}^m}(p)\mathcal{M}[Y_{2k}^m](p)\,dp$$

$$= 2\pi\lambda_{k,n}\int_{\mathbb{S}^n} |Y_{2k}^m|^2(p)\,dp = 2\pi\lambda_{k,n}, \tag{33}$$

where $c(m,2k,n)$ arises in (8), $\delta = \delta(s)$ is 1D Dirac delta function, $\overline{Y_k^m}$ is the complex conjugate of Y_k^m, $\mathcal{M}[Y_k^m]$ is defined in (7), $\lambda_{k,n}$ is given in (11), $\Gamma(\cdot)$ is the Gamma function.

Formula (30) reflects the known property of the Funk-Minkowski transform $\mathcal{M}$ that the eigenvalue $\lambda_{k,n}$ of operator $\mathcal{M}[\cdot]$ defined in (7) corresponds to the eigensubspace of harmonic polynomials of degree $2k$ on $\mathbb{R}^{n+1}$ restricted to $\mathbb{S}^n$ (see e.g. [5], Chapter 6, p. 24). Note that, in [5] it was proved that formula (30) holds also for all harmonic polynomials in $\mathbb{R}^3$ (i.e. $n = 2$) restricted to $\mathbb{S}^2$, however these considerations admit a straightforward generalization to the case of arbitrary dimension $n \geq 1$.

Formulas (31), (32) imply formula (10).

Now, it remains to find the explicit value for $\lambda_{k,n}$ in formula (30). We obtain it according to [5] (Chapter 2, page 24), where the case of dimension $n = 2$ was considered.

In particular, formula (30) holds for any homogeneous harmonic polynomial P_{2k} of degree $2k$ in $\mathbb{R}^{n+1}$, where P_{2k} is restricted to the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$; see [5] (Chapter 2).

We consider the following harmonic polynomial in $\mathbb{R}^{n+1}$:

$$P_{2k}(x) = P_{2k}(x_1,\ldots,x_{n+1}) = (x_n + ix_{n+1})^{2k},\ x \in \mathbb{R}^{n+1}. \tag{34}$$

From formula (34), aforementioned results of [5] (Chapter 6) and their generalizations to the case of arbitrary dimension $n \geq 1$ it follows that P_{2k} being restricted to sphere $\mathbb{S}^n$ is an eigenfunction of operator $\mathcal{M}$ which corresponds to the eigenvalue $\lambda_{k,n}$.

We consider the spherical coordinates in $\mathbb{R}^{n+1}$ given by the following formulas:

$$\begin{aligned}
x_1 &= \cos(\theta_n),\\
x_2 &= \sin(\theta_n)\cos(\theta_{n-1}),\\
&\cdots\\
x_n &= \sin(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_2)\cos(\phi),\\
x_{n+1} &= \sin(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_2)\sin(\phi),\\
\theta_n,\, \theta_{n-1}, &\cdots, \theta_2 \in [0,\pi],\ \phi \in [0, 2\pi).
\end{aligned} \tag{35}$$

Formulas (34), (35) imply that polynomial P_{2k} being restricted to $\mathbb{S}^n$ may be rewritten as follows:

$$P_{2k}|_{\mathbb{S}^n} = P_{2k}(\theta_n,\, \theta_{n-1}, \ldots, \theta_1, \phi) = e^{i2k\phi}\prod_{i=2}^{n}\sin^{2k}(\theta_i), \tag{36}$$

where $(\theta_n, \ldots, \theta_1, \phi)$ are the coordinates on $\mathbb{S}^n$ according to (35).

From formulas in (34), (35), (36) it follows that:

$$P_{2k}|_{\mathbb{S}^n} = P_{2k}(\pi/2, \ldots, \pi/2, 0) = 1. \tag{37}$$

From formulas (30), (37), (36) we obtain:

$$\begin{aligned}
\lambda_{k,n} = \mathcal{M}[P_{2k}](\pi/2, \ldots, \pi/2, 0) &= 2(-1)^k \prod_{i=2}^{n}\int_0^{\pi}\sin^{2k}(\theta_i)\, d\theta_i\\
&= 2(-1)^k\left[\sqrt{\pi}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\right]^{n-1},
\end{aligned} \tag{38}$$

which implies (11).

Formulas (10), (11) are proved.

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