EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 4, Number 4 (2016) 60–73

METHODS FOR SOLVING NONLINEAR ILL-POSED PROBLEMS BASED ON THE TIKHONOV-LAVRENTIEV REGULARIZATION AND ITERATIVE APPROXIMATION

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Abstract A problem of constructing a stable approximate solution for a nonlinear irregular operator equation is investigated. For approximating solutions of the equations regularized by the Tikhonov-Lavrentiev methods, the Levenberg-Marquardt and Newton type processes are used, and the linear convergence rate and the Fejér property are proved. An asymptotic stopping rule of iterations is formulated that guarantees the regularizing properties of iterations and error estimate. Analogous questions are briefly discussed for the gradient methods.

Key words: ill-posed problem, Tikhonov-Lavrentiev regularization, Levenberg-Marquardt and Newton type methods, stopping rule, error estimate

AMS Mathematics Subject Classification: 45Q05, 65J15, 65J20

1 Introduction

The inverse problem in the form of the operator equation is considered

$$A(u) = f, \tag{1.1}$$

where A is a nonlinear differentiable operator acting on a pair of Hilbert spaces U, F. Continuity of the operators A^{-1} and $A'(u)^{-1}$ is not assumed, therefore, equation (1.1) is an essentially ill-posed (irregular) problem. Discontinuity of the operator $A'(u)^{-1}$ does not allow us to use the traditional Newton, Gauss-Newton, or Levenberg-Marquardt processes for approximating a solution of equation (1.1). So, in such a case, it is necessary to pass either to regularized analogues of the mentioned methods on the basis of the iterative regularization principle or to apply these iterative processes not to (1.1), but to the Tikhonov regularized equation

$$A'(u)^*(A(u) - f_\delta) + \alpha(u - u^0) = 0 \tag{1.2}$$

for an any differetiable operator A or to the Lavrentiev regularized equation

$$A(u) + \alpha(u - u^0) - f_\delta = 0 \tag{1.3}$$

for a monotone operator A, where u^0 is a trial solution (initial approximation), $||f - f_\delta|| \leq \delta$. Solvability of equations (1.2) and (1.3) is assumed. Certain sufficient conditions for solvability of (1.2) and (1.3) are discussed in Remark 5.2

For approximate solving the regularized equation (1.2), we propose the modified Levenberg-Marquardt method (MLM)

$$u^{k+1} = u^k - \gamma[A'(u^0)^* A'(u^0) + \bar{\alpha} I]^{-1} L_\alpha(u^k) \equiv T(u^k), \tag{1.4}$$

and, for solving the regularized equation (1.3), the following modified Newton method

$$u^{k+1} = u^k - \gamma[A'(u^0) + \bar{\alpha} I]^{-1} M_\alpha(u^k) \equiv T(u^k), \tag{1.5}$$

where

$$L_\alpha(u) = (A'(u)^*(A(u) - f_\delta) + \alpha(u - u^0), \quad M_\alpha(u) = A(u) + \alpha(u - u^0) - f_\delta,$$

$\alpha, \bar{\alpha}$ are positive regularization parameters, $\bar{\alpha} \geq \alpha$. So, we have two-stage algorithm: on the first stage the regularizing schemes (1.2) or (1.3) are used, and on the second stage the iterative processes (1.4) or (1.5) are applied. Having convergence theorems or error estimates for regularization methods (1.2), (1.3) and the iterative approximation in the form (1.4), (1.5), we can construct a one-stage regularizing algorithm (RA) for equation (1.1).

Let us present a brief survey on the Levenberg-Marquardt and the Newton type methods. It is necessary to note that for solving equation (1.1) by the Levenberg–Marquardt method (LM)

$$u^{k+1} = u^k - [A'(u^k)^* A'(u^k) + \alpha I]^{-1} A'(u^k)^*(A(u^k) - f_\delta) \tag{1.6}$$

one must impose rather heavy structural conditions on the operator A and on the choice of the regularizing parameter [3, 6, 7, 9, 23]. In [6, 7] (see also [9]), it was suggested the following rule for choosing the parameter in the method (1.6), namely, the parameter $\alpha = \alpha_k$ would be such that the relation holds

$$||f_\delta - A(u^k) - A'(u^k)(u^{k+1}(\alpha) - u^k)|| = q||f_\delta - A(u^k)||, \tag{1.7}$$

where $0 < q < 1$. Under this, existence of a unique solution $\alpha = \alpha_k$ for equation (1.7) is guaranteed if the following inequality holds:

$$||f_\delta - A(u^k) - A'(u^k)(z - u^k)|| < q||f_\delta - A(u^k)||,$$

where z is a solution for equation (1.1) of the minimal norm.

The strong convergence of method (1.6) to a solution of equation (1.1) was proved under the condition that the operator A satisfies the property

$$||A(u) - A(\tilde{u}) - A'(u)(u - \tilde{u})|| \leqslant C||A(u) - A(\tilde{u})|| \tag{1.8}$$

in some neighborhood $S_r(u^0) = \{u : ||u - u^0|| \leqslant r\}$ of a probe solution u^0 [7, 9]. In paper [23], two variants of the LM were considered. The first one has the form

$$u^{k+1} = u^k - \beta[A'(u^k)^* A'(u^k) + \alpha I]^{-1} A'(u^k)^*(A(u^k) - f) \equiv T(u^k), \tag{1.9}$$

i.e., in method (1.6), a positive parameter β is introduced. In the second variant (as an analogy to the modified Newton method), the derivative in the inverse operator is calculated at a fixed point u^0, namely,

$$u^{k+1} = u^k - \beta[A'(u^0)^* A'(u^0) + \alpha I]^{-1} A'(u^k)^*(A(u^k) - f) \equiv T_0(u^k). \tag{1.10}$$

It was established that operators T and T^0 are pseudo-contracting if the local condition

$$||A(u) - f||^2 \leqslant \varkappa \langle B^{-1}(u) S(u), u - z\rangle, \tag{1.11}$$

is satisfied, where $S(u) = A'(u)^*(A(u) - f), B^{-1}(u) = (A'(u)^* A'(u) + \alpha I)^{-1}$ for method (1.9) and the condition

$$||A(u) - f||^2 \leqslant \varkappa \langle S(u), u - z\rangle \tag{1.12}$$

for method (1.10) under $\beta < \beta_0(\varkappa)$ in some neighborhood $O_\rho(z)$ of a solution for equation (1.1). This property of operators T and T^0 implies a week convergence of iterations to a solution of this equation for exact input data.

Thus, for the strong or weak convergence of the LM to a solution of equation (1.1), rather heavy conditions (of the forms (1.8) or (1.11), or (1.12)) must be satisfied for the operator A. By these reasons, the passage from equation (1.1) to the regularized equation (1.2) is fruitful since the operator L_α, which is the gradient of the Tikhonov functional

$$\Phi(u) = \frac{1}{2}(||A(u) - f_\delta||^2 + \alpha||u - u^*||^2),$$

has (under some conditions) the better structural properties than A and A'^*A. For example, if relations (1.17) and (1.18) hold, then the operator S_α has the property of strong monotonicity [10, 11], but, otherwise, for A an A'^*A this property does not hold. It is interesting to note that for the linear operator A not only the regularized operator but and the regularized solutions u_α from (1.2) and (1.3) posses improved property of approximation than the initial guess u^0, as it follows from the next lemma.

Lemma 1.1. *Let z be a solution of equation* (1.2) *and $z \neq u^0$. Then for the exact right-hand side $f = f_\delta$ and for any $\alpha > 0$, the strong inequality holds*

$$||u_\alpha - z|| < ||u^0 - z|| \tag{1.13}$$

i.e., the regularized solution is closer to the solution of (1.1) *than any initial guess.*

Proof

For the Tikhonov regularization, we have the evident relation

$$||u_\alpha - z|| = (A^*A + \alpha I)^{-1}(A^* f + \alpha u^0) - z = \alpha(A^*A + \alpha I)^{-1}(u^0 - z). \tag{1.14}$$

Define $B = \alpha(A^*A + \alpha I)^{-1}$ and show that for the operator B the inequality

$$||Bw||^2 \leq ||w||^2 - ||w - Bw||^2 \tag{1.15}$$

is fulfilled for any $w \in U$. Inequality (1.15) is equivalent to the following one:

$$||Bw||^2 \leq < w, Bw >,$$

which is true because $||B|| \leq 1$ and $||Bw||^2 \leq ||B^{1/2}||^2||B^{1/2}w||^2 =< w, Bw >$. As $w = u^0 - z \neq 0$, relations (1.14) and (1.15) imply (1.13) since

$$||u_\alpha - z||^2 = ||B(u^0 - z)||^2 \leq ||u^0 - z||^2 - ||u^0 - z - B(u^0 - z)||^2 < ||u^0 - z||^2.$$

These properties allows us to provide the strong convergence of the iterative processes for equation (1.2). Together with the Tikhonov regularization, this gives an opportunity to build a regularizing algorithm for the initial problem (1.1), in particular, for some classes of inverse problems in geophysics [22, 4, 23, 24]. It should be noted that Lemma 1.1 is also valid for the Lavrentiev regularization (1.3).

In the author's work [15], the iterative ML–M process is investigated in the form

$$u_\alpha^{k+1} = u_\alpha^k - \gamma[A'(u_\alpha^k)^*A'(u_\alpha^k) + \bar{\alpha}I]^{-1}L_\alpha(u_\alpha^k) \tag{1.16}$$

and its modified variant in the form (1.4) for approximation of the solution u_α of the regularized equation (1.2). It was proved there that under the following conditions on the operator

$$||A'(u)|| \leqslant N_1, \quad ||A'(u) - A'(v)|| \leqslant N_2||u - v||, \tag{1.17}$$

$\gamma \leqslant \gamma(\alpha, \bar{\alpha})$ and the sourcewise represented solution

$$z - u^* = A'(z)^*v, \quad ||v|| < 1/N_2 \tag{1.18}$$

for iterations u_α^k of processes (1.5), (1.16) the strong convergence to the solution u_α of equation (1.2) holds. Namely, the following theorems hold [15].

Theorem 1.1. *Let conditions (1.17) and (1.18) be satisfied, and, moreover,* $\delta \leqslant \alpha/(12N_2)$, $\bar{\alpha} \geqslant 3N_1^2$, $\alpha < (1-N_2||v||)N_1^2$, $r = \alpha/(12N_1N_2)$. *Then under* $\gamma < \alpha/(2\bar{\alpha})$ *for any initial guess* $u_\alpha^0 \in S_\alpha(u_\alpha)$, *the iterative process (1.16) converges to the regularized solution* u_α *and the strong Fejér property holds*

$$||u_\alpha^{k+1} - u_\alpha||^2 \leqslant ||u_\alpha^k - u_\alpha||^2 - \nu||u_\alpha^{k+1} - u_\alpha^k||^2. \tag{1.19}$$

where $\nu = \alpha/(2\bar{\alpha}\gamma) - 1$.

Theorem 1.2. *Let conditions of Theorem 1.1 be satisfied. Then under* $\gamma < 2\alpha/(3\bar{\alpha})$ *for any initial guess* $u_\alpha^0 \in S_r(u_\alpha)$, *the iterative process* (1.4) *converges to the regularized solution* u_α *and the strong Fejér property holds under* $\nu = 2\alpha/(\gamma 3\bar{\alpha}) - 1$. *Moreover, under special dependence* $\alpha(\delta)$ *beginning from some* $k \geqslant k_0$, *all iteration points* u_α^k *belong to some neighborhood* $O_r(z)$ *of a radius* $r = c\delta^{1/2}$ *of the solution* z.

Properties of iteration of processes (1.4) and (1.16) established in Theorems 1.1 and 1.2 are consequences of the strong Fejér property of the step operators in these methods. Note that (see [20, 21]) the class of operators with such a property is closure with respect to operations of multiplication and convex summation. This allows one:
1) to build new classes of hybrid iterative processes for solving the initial problem (1.1);

2) to construct the step operator of the main process in the form of superposition of operators responsible for approximation of solutions of some sub-problems that compile the initial problem;

3) in an economical way, to take into account additional *a priori* constraints on the desired solution in the resolving iterative process.

In Section 2 of this paper, it is proved that if the premises of Theorem 1.2 are satisfied, then for process (1.4) not only the strong Fejér property is valid but, also, an error estimate holds.

As it is known, for approximate solving equation (1.1) with the monotone operator the following iteratively regularized Newton method

$$u^{k+1} = u^k - (A'(u^k) + \alpha_k I)^{-1}(A(u^k) + \alpha_k(u^k - u^0) - f_\delta) \tag{1.20}$$

was proposed and justified [1, 2]. In the author's work [17], it was shown for process (1.5) that if conditions (1.17) are equitable and $A'(0)$ is a self-adjoint non-negative operator, then for an appropriate value of parameter γ, the strong Fejér property (1.19) and the strong convergence of iterations hold. In Section 3, it is proved that for the same conditions process (1.5) has the linear rate of the convergence. Section 4 is devoted to the error estimate for the two-stage methods (1.2)–(1.4) and (1.3)–(1.5).

It should be noted, that along with MLM (1.4) and MNM (1.5) one can use the method of gradient type. In particular, for approximation of the solution of equation (1.2), the regularized steepest descent method [16]

$$u^{k+1} = u^k - \gamma \frac{||L_\alpha(u^k)||^2}{||A'(u^k)L_\alpha(u^k)|^2 + \alpha||L_\alpha(u^k)||^2} L_\alpha(u^k) \tag{1.21}$$

or its a modified version [18] (when the operator $A'(u^k)$ is replaced by $A(u^0)$) can be applied. For approximate solving equation (1.3) (as $\alpha = \epsilon$) with a monotone operator, the nonlinear variant of α-processes [19] in the form

$$u^{k+1} = u^k - \gamma \frac{< (A'(u^0) + \epsilon I)^\alpha M_\epsilon(u^k), M_\epsilon(u^k) >}{< (A'(u^0) + \epsilon I)^{\alpha+1} M_\epsilon(u^k), M_\epsilon(u^k) >} M_\epsilon(u^k) \tag{1.22}$$

can be used. If in method (1.21) $\alpha = 0$, we come to the steepest descent method investigated in work [13]. If in process (1.22) A is a linear operator and $\epsilon = 0$, we obtain the classical α-processes [12] for solving well posed linear equations. The concluding Section 5 contains a brief survey concerning the theorems of the converges and error estimates for processes (1.21), (1.22).

2 Modified Levenberg-Marquardt method

Theorem 2.1. *Let conditions* (1.17), (1.18) *be satisfied and equation* (1.2) *have a solution u_α. Let for the parameters $\alpha, \bar{\alpha}, N_1, N_2, r$ and the initial guess u^0 the following relations be fulfilled:*

$$\delta \le \alpha/12N_2, \quad r = \alpha/(12N_1N_2), \quad ||v|| \ge 1/24N_2, \quad \bar{\alpha} \ge 3N_1^2.$$

Then for any initial guess $u^0 \in S_r(u_\alpha)$:
1) under the condition $\gamma < 4\alpha/27\bar{\alpha}$, *the iterations* u^k *generated by process* (1.4) *converge to the regularized solution* u_α *and for the iterations the strong Fejér property* (1.19) *holds;*
2) under $\gamma < 2\alpha/27\bar{\alpha}$ *the following error estimate is valid:*

$$||u^k - u_\alpha|| \le q^k r, \quad q = \sqrt{1 - \frac{\alpha^2}{81\bar{\alpha}^2.}} \tag{2.1}$$

Proof
Define

$$B(u^0) = A'(u^o)^* A'(u^0) + \alpha I,$$
$$F_0(u) = B^{-1}(u^0)[A'(u)^*(A(u) - f_\delta) + \alpha(u - u^0)].$$

First, let us prove certain properties of the operator $F_0(u)$ for u from the ball $S_r(u_\alpha)$, using a technique from [15]. We have the following relations:

$$\begin{gathered} < F_0(u), u - u_\alpha >=< F_0(u) - F_0(u_\alpha) > \\ = \alpha < B^{-1}(u^0)(u - u_\alpha), u - u_\alpha > \\ + < B^{-1}(u^0)[A'(u)^*(A(u) - f_\delta) - A'(u_\alpha)^*(A(u_\alpha) - f_\delta), u - u_\alpha > \\ = \alpha < B^{-1}(u^0)(u - u_\alpha), u - u_\alpha > \\ +B^{-1}(u^0)[A'(u)^*(A(u) - A(u_\alpha))], u - u_\alpha > \\ + < B^{-1}(u^0)(A'(u)^* - A'(u_\alpha)^*)(A(u_\alpha) - f_\delta), u - u_\alpha > . \end{gathered}$$

Now estimate each term below. From spectral decomposition of a self-adjoint operator it follows

$$\alpha < B^{-1}(u^0)(u - u_\alpha), u - u_\alpha \ge \alpha/(N_1^2 + \bar{\alpha}). \tag{2.2}$$

Using the Lagrange formula for operators, we obtain

$$\begin{gathered} < B^{-1}(u^0)[A'(u)^*(A(u) - A(u_\alpha))], u - u_\alpha > \\ =< B^{-1}(u)(B(u^0) - B(u))B^{-1}(u^0)A'(u)^*(A(u) - A(u_\alpha), u - u_\alpha > \\ + < B^{-1}(u)A'(u)^* \int_0^1 [A'(u_\alpha + t(u - u_\alpha))(u - u_\alpha) - A'(u^0)(u - u)]\, dt, u - u_\alpha > \\ + < B^1(u)A'(u)^*A'(u)(u - u_\alpha), u - u_\alpha > \\ \ge -[\tfrac{2N_1^3 N_2||u-u^0||}{\bar{\alpha}^2} + \tfrac{N_1 N_2||u-u_\alpha||}{\bar{\alpha}}]||u - u_\alpha||^2. \end{gathered} \tag{2.3}$$

As it is known [5], condition (1.18) implies the following estimates for the solution u_α of equation (1.2):

$$||A(u_\alpha) - f_\delta|| \le \delta + 2\alpha||v||,$$

therefore,

$$\begin{gathered} < B^{-1}(u^0)(A'(u)^* - A'(u_\alpha)^*)(A(u_\alpha) - f_\delta), u - u > \\ \ge -\frac{N_2(\delta + 2\alpha||u - u_\alpha||^2)}{\bar{\alpha}}. \end{gathered} \tag{2.4}$$

Combining (2.2)–(2.4), taking into account the inequality $||u-u^0|| \leq ||u-u_\alpha||+||u_\alpha - u^0|| \leq 2r$ and the conditions on the parameters, we arrive to the final estimate

$$< F_0(u), u-u_\alpha > \geq \frac{\alpha}{6\bar{\alpha}}||u-u_\alpha||^2. \quad (2.5)$$

Besides, the evident inequalities hold

$$\begin{aligned} ||B^{-1}(u^0)L_\alpha|| &= ||B^{-1}(u^0)L_\alpha(u) - B^{-1}(0)L_\alpha(u_\alpha)|| \\ &\leq ||B^{-1}(u^0)(u-u_\alpha)|| + ||B^{-1}(u^0)A'(u)^*(A(u)-A(u_\alpha))|| \\ &+||(A'(u)^* - A'(u_\alpha))(A(u_\alpha)-f_\delta)|| \\ &\leq [\tfrac{\alpha}{\bar{\alpha}} + \tfrac{N_1^2}{\bar{\alpha}} + \tfrac{N_2(\delta+2\alpha)||v||}{\bar{\alpha}}]||u-u_\alpha|| \leq \tfrac{3}{2}||u-u_\alpha||. \end{aligned} \quad (2.6)$$

Joining estimates (2.5)–(2.6), we obtain the relation

$$||F_0(u)||^2 \leq \frac{27\bar{\alpha}}{2\alpha} < F_0(u), u-u_\alpha > . \quad (2.7)$$

The strong Fejér property for an operator means that the inequality

$$||T(u)-u_\alpha||^2 \leq ||u-u_\alpha||^2 - \nu||u-T(u)||^2 \quad (2.8)$$

is fulfilled for certain ν and $Fix(T) = \{u_\alpha\}$. Relation (2.8) for the step operator T of process (1.4) is equivalent to the following one:

$$-2/(1+\nu)\gamma) < u-u_\alpha, F_0(u) > +||F_0(u)||^2 \leq 0. \quad (2.9)$$

Comparing (2.7)–(2.9), we come to the conclusion that for $\gamma < 4\alpha/27\bar{\alpha}$ relation (2.8) is true for $\nu = 4/27\gamma - 1$. Substituting $u = u^k$, into (2.8), we obtain the inequality of the form (1.19).

From (2.5) and (2.7), we have have the following relations

$$\begin{aligned} ||u^{k+1}-u_\alpha||^2 &= ||u^k - u_\alpha - \gamma F_0(u^k)||^2 \\ &= ||u^k-u_\alpha||^2 - 2\gamma < F_0(u^k), u^k - u_\alpha > +\gamma^2||F_0(u^k)||^2 \\ &\leq (1 - 2\gamma\frac{\alpha}{6\bar{\alpha}} + \gamma^2\frac{9}{4})||u^k-u_\alpha||^2 \end{aligned}$$

The condition

$$\phi(\gamma) = (1 - 2\gamma\frac{\alpha}{6\bar{\alpha}} + \gamma^2\frac{9}{4}) < 1$$

implies the inequality $\gamma < 4\alpha/27\bar{\alpha}$ that guarantees the convergence of process (1.4). The minimum condition of the function $\phi(\gamma)$ gives the value $\gamma_{opt} = 2\alpha/27\bar{\alpha}$, for which (2.1) is fulfilled.

Assume that there is *a priori* information that u_α belongs to Q, where Q is a convex closed subset of a Hilbert space. Let P_Q be a strong Fejér mapping with a constant ν_1 and the fixed point $Fix(P_Q) = Q$. Form the iterative process

$$u^{k+1} = P_Q(T(u^k)), \quad u^0 \in S_r(u_\alpha) \cap Q, \quad (2.10)$$

where T is the step operator in method (1.4).
Theorem 2.2. *Let the conditions of theorem 2.1 be fulfilled, but inequalities (1.17) be valid only for $u, v \in Q$. Then the sequence $\{u^k\}$ generated by process (1.16) strongly converges to u_α, i.e.,*

$$\lim_{k\to\infty} ||u^k - u_\alpha|| = 0. \tag{2.11}$$

Proof
Taking into account the Fejér property of mappings, we have the following relations:

$$||P_Q(T(u)) - u_\alpha||^2 \le ||T(u) - u_\alpha||^2 - \nu_1||T(u) - P_Q(T(u))||^2$$
$$\le ||u - u_\alpha||^2 - \nu||u - T(u)||^2 - \nu_1||T(u) - P_Q(T(u))||^2.$$

Substituting $u = u^k$ in the last relation, we have

$$-||u^{k+1} - u_\alpha||^2 \le ||u^k - u_\alpha||^2 - \nu||u^k - T(u^k)||^2$$
$$\nu_1||T(u_k) - P_Q(T(u^k))||^2.$$

It implies boundedness of the sequence $\{u^k\}$ and the relation

$$\lim_{k\to\infty} ||u^k - T(u^k)|| = \gamma||F_0(u^k)|| = 0. \tag{2.12}$$

The convergence of iteration (2.11) follows from (2.5) and (2.12).
Remark 2.1. Instead of conditions (1.17), where $u \in S_r(u_\alpha)$, it is sufficient to require only the second inequality. Actually, from the following inequalities

$$||A'(u) - A'(u^0)|| \le N_2||u - u^0||, \quad ||u - u^0|| \le ||u - u_\alpha|| + ||u_\alpha - u^0||,$$

we obtain

$$||A'(u)|| \le |||A'(u^0)|| + 2N_2 r = N_1.$$

3 Modified Newton method

Prove that the MNM has the linear rate of the convergence.

Theorem 3.1. *Let $A'(u^0)$ be a non-negatively defined self-adjoint operator and conditions (1.17) be fulfilled. Let for the parameters the following conditions be valid:*

$$||u^0 - u_\alpha|| \le r,\ 0 < \alpha \le \bar{\alpha},\ r = \alpha/6N_2,\ \bar{\alpha} \ge 3N_1. \tag{3.1}$$

Then:
1) for $\gamma < \alpha\bar{\alpha}/(\alpha + N_1)^2$ process (1.5) converges strongly to the solution u_α and for iterations the strong Fejér property (1.19) is true;
2) for $\gamma_{opt} = \alpha\bar{\alpha}/2(\alpha + N_1)^2$ the following error estimate is valid:

$$||u^k - u_\alpha|| \le q^k r, \quad q = \sqrt{1 - \frac{\alpha^2}{4(\alpha + N_1)^2.}} \tag{3.2}$$

Proof
Introduce the notations

$$B_0 = (A'(u^0) + \bar{\alpha} I), \quad F(u) = B_0^{-1}(A(u) + \alpha(u - u^0) - f_\delta).$$

Under conditions (3.1) on the parameters, we have the following estimate:

$$\begin{gathered} < F(u), u - u_\alpha > = < F(u) - F(u_\alpha), u - u_\alpha > \\ = \alpha < B_0^{-1}(u - u_\alpha), u - u_\alpha > + < B_0^{-1}(A(u) - A(u_\alpha), u - u_\alpha > \\ = \alpha < B_0^{-1}(u - u_\alpha), u - u_\alpha > \\ + < B_0^{-1} \int_0^1 A'(u_\alpha + t(u - u_\alpha))(u - u_\alpha) dt, u - u_\alpha > \\ = \alpha < B_0^{-1} A'(u^0)(u - u_\alpha), u - u_\alpha > + < \int_0^1 [A'(u_\alpha + t(u - u_\alpha)) \\ - A'(u^0)](u - u_\alpha) dt, u - u_\alpha > + \alpha < B_0^{-1}(u - u_\alpha), u - u_\alpha > \\ \geq \tfrac{\alpha}{\bar{\alpha} + N_1} ||u - u_\alpha||^2 - \tfrac{N_2}{2\bar{\alpha}}(||u - u_\alpha|| + 2||u^0 - u_\alpha||)||u - u_\alpha||^2 \\ \geq \tfrac{\alpha}{2\bar{\alpha}} ||u - u_\alpha||^2. \end{gathered} \tag{3.3}$$

Taking into account the evident estimate

$$\begin{gathered} ||F(u)|| = ||B_0^{-1}[(A(u) + \alpha(u - u^0) - f_\delta) - (A(u_\alpha) + \alpha(u_\alpha - u^0) - f_\delta)]|| \\ \leq \alpha ||B_0^{-1}(u - u_\alpha)|| + ||B_0^{-1}(A(u) - A(u_\alpha))|| \leq \tfrac{(\alpha + N_1)}{\bar{\alpha}} ||u - u_\alpha|| \end{gathered} \tag{3.4}$$

and (3.3), we come to the relation

$$||F(u)||^2 \leq 2(\alpha + N_1)^2 / (\alpha \bar{\alpha}) < F(u), u - u_\alpha > . \tag{3.5}$$

The strong Fejér property of the step operator of method (1.5) means that for certain $\nu > 0$ the following inequality is valid:

$$||F(u)||^2 \leq \frac{2}{\gamma(1 + \nu)} < F(u), u - u_\alpha > . \tag{3.6}$$

From (3.5) and (3.6) it follows that this property holds for $\gamma < \alpha\bar{\alpha}/(\alpha + N_1)2$. Combining (3.3) and (3.4), we find that

$$||u^{k+1} - u_\alpha||2 \leq (1 - 2\gamma \frac{\alpha}{2\bar{\alpha}} + \gamma^2 \frac{(\alpha + N_1)^2}{\bar{\alpha}^2})||u - u_\alpha||2.$$

If for the function $\psi(\gamma) = 1 - 2\gamma\frac{\alpha}{2\bar{\alpha}} + \gamma^2 \frac{(\alpha + N_1)^2}{\bar{\alpha}^2}$ the relation $\psi(\gamma) < 1$ is true, then we have $\gamma < \alpha\bar{\alpha}/(\alpha + N_1)^2$, for which the convergence of iterations to u_α holds. The minimal value of this function is attained for $\gamma_{opt} = \alpha\bar{\alpha}/2(\alpha + N_1)^2$ that implies estimate (3.2).

4 Error estimate of the two-stage methods

To estimate the error of the two-stage method consisting of the Tikhonov regularization (1.2) and MLM (1.4), we should to have an error estimate for the solution u_α of equation (1.2). For this, let us mention one assertion from book [5]

Theorem 4.1. *Let conditions* (1.18) *be fulfilled. Then for the solution* u_α *of equation* (1.2) *the following relations are equitable*

$$||A(u_\alpha) - f_\delta|| \le \delta + 2\alpha||v||, \quad ||u_\alpha - z|| \le \frac{\delta + \alpha||v||}{\sqrt{\alpha(1 - N_2||v||)}}. \tag{4.1}$$

Put $\alpha(\delta) = \delta/||v||$, then from (4.1) we have

$$||u_{\alpha(\delta)} - z|| \le M\sqrt{\delta}, \quad M = 2\sqrt{||v||}/\sqrt{1 - N_2||v||}. \tag{4.2}$$

Prove that estimate (4.2) is order optimal. Actually, for a linear operator A on the class of correctness $z - u^0 \in M_R = BS_R(0)$, the linear optimal method R^{opt} is below estimated by the module of continuity $\omega(\delta, R)$ of the inverse operator A^{-1} (see [8], Lemma 4.2.3)

$$\omega(\delta, R) \le \sup\{||R^{opt}(f_\delta) - z|| : \quad ||A(z) - f_\delta|| \le \delta,\ z - u^0 \in M_R\}. \tag{4.3}$$

Here, B is a linear bounded operator and the function $\omega(\delta, R)$ is calculated by the formula (see [8], theorem 4.9.1)

$$\omega(\tau, R) = R\sqrt{g(\tau^2/R^2)}, \tag{4.4}$$

where $B_1 = g(C_1)$, $B_1 = B^*B$, $C_1 = C^*C$, $C = AB$. If $B = A^*$, $R = ||v||$ (see (1.18)), then $B_1 = AA^*$, $C_1 = (AA^*)^2$, therefore, $\omega(\tau, ||v||) = \sqrt{\tau}||v||$. On the other hand, on the class M_R and for $\alpha(\delta) = \delta/||v||$, the Tikhonov regularization method has the following upper bound (see [8]):

$$||u_{\alpha(\delta)} - z|| \le C(||v||, ||A||)\omega(\delta, ||v||). \tag{4.5}$$

For (4.3)–(4.5) it follows that estimate (4.2) is order optimal for the linear operator A. It means that for the nonlinear operator A this estimate is order optimal, too.

Now let us address to the two-stage algorithm (1.2), (1.4). From (2.1), (4.2) we have the following inequalities:

$$||u^k - z|| \le ||u^k - u_{\alpha(\delta)}|| + ||u_{\alpha(\delta)} - z|| \le q^k r + M\sqrt{\delta}. \tag{4.6}$$

Equating the terms in in the right-hand side of relations (4.6), we obtain the number of iterations $k(\delta) = ln(M\sqrt{\delta}/r)/lnq$, for which from (4.6) the order optimal estimate of error holds

$$||u^{k(\delta} - z|| \le 2M\sqrt{\delta}. \tag{4.7}$$

To obtain the error estimate of the two-stage algorithm (1.3), (1.5), we use the result of work [14], according to which, if the solution of (1.1) is source representable in the form

$$z - u^0 = A'(z)v$$

and for the derivative of the monotone operator A the Lipschitz condition with constant N_2 holds, then

$$||u_\alpha - z|| \le \frac{\delta}{\alpha} + k_0\alpha, \quad k_0 = (1 + \frac{N_2}{2}||v||)||v||. \tag{4.8}$$

Minimizing the right-hand side of inequality (4.8) on the parameter α, we find $\alpha = \sqrt{\delta/k_0}$ and the estimate

$$||u_{\alpha(\delta)} - z|| \le 2\sqrt{\delta k_0}. \tag{4.9}$$

For the linear self-adjoint non-negatively defined operator A, estimate (4.9) is order optimal (see [8], Corollary 4.7.2, p. 173), hence, for the nonlinear monotone operator A this property is valid, too.

Combining (3.2) with (4.9) yields the inequalities

$$||u^k - z|| \le ||u^k - u_{\alpha(\delta)}|| + ||u_{\alpha(\delta)} - z|| \le rq^k(\delta) + 2\sqrt{\delta k_0}. \tag{4.10}$$

Equating the terms on the right-hand side of (4.10), we find the iteration number and the order-optimal estimate

$$k(\delta) = [ln(\frac{2}{r}\sqrt{k_0\delta})/lnq(\delta)], \quad ||u^k - z|| \le 4\sqrt{\delta k_0}.$$

5 Gradient type methods

The regularized version of the steepest method is constructed by the following way:

$$u^{k+1} = u^k - \beta(A'(u)^*(A(u) - f_\delta) + \alpha(u - u^0). \tag{5.1}$$

Linearizing the operator A at the point u^k

$$A'(u^k)u = G, \quad G = f - A(u^k) + A'(u^k)u^k,$$

we arrive to the linear equation

$$A(u) \simeq A(u^k) + A'(u^k)(u - u^k).$$

The value of the parameter β in (5.1) is found from the minimum condition of the regularized discrepancy

$$\min_\beta \{||A'(u^k)(u^k - \beta L_\alpha(u^k)) - G||^2 + \alpha||u^k - \beta L_(u^k) - u^0||^2\}.$$

Using the necessary condition of the extremum, we obtain the value of the sought for parameter $\beta = \beta(u^k)$

$$\beta(u^k) = \frac{||L_\alpha(u^k)||^2}{||A'(u^k)L_\alpha(u^k)|^2 + \alpha||L_\alpha(u^k)||^2}.$$

Theorem 5.1. *Let conditions* (1.17), (1.18) *be fulfilled and let for the parameters the following conditions be valid:*

$$\delta < \alpha/6N_2, ||v|| \le 1/12N_2, r = \alpha/(12N_1N_2).$$

Then for any initial guess $u^0 \in S_r(u_\alpha)$ and $\gamma < \alpha^2/M^2$, $M = N_1^2 + (4/3)\alpha$ process (1.21) *converges strongly to the solution u_α of equation* (1.2) *and the strong Fejér condition* (1.19) *holds.*

Proof is given in [16].

For solving equation (1.3) one can apply process (1.22).

Theorem 5.2. *Let A be a monotone operator and $A'(u^0)$ be a self-adjoint operator such that*

$$||A'(u^0)|| \leq N_0, \quad ||A(u) - A(v)|| \leq N||u - v||, \quad u^0 \in S_r(u_\alpha).$$

Then:
1) for $\gamma < 2\epsilon^3/(N_0 + \epsilon)(N + \epsilon)^2$ the iterations generated by process (1.22) converge strongly to the solution of equation (1.3) as $\alpha = \epsilon$;
2) for $\gamma < \epsilon^3/(N_0 + \epsilon)(N + \epsilon)^2$ the following estimate is valid:

$$||u^k - u_\epsilon|| \leq q^k||u^0 - u_\epsilon||, \quad q(\epsilon) = \sqrt{1 - \frac{\epsilon^4}{(N_0 + \epsilon)^2(N + \epsilon)^2}}.$$

This theorem was announced in work [19].

Remark 5.1. Using technique from the previous sections, it is possible to obtain an error estimate for the two-stage algorithms (1.2), (1.21) and (1.3), (1.22).

Remark 5.2. In all theorems of Sections 1-5 the existence of a solution for equations (1.2), (1.3) is assumed . There are certain sufficient conditions, under which solvability holds.

1. If a nonlinear differentiable operator A is (weakly) closed, then equation (1.2) has a solution possible non-unique (see [8], theorem 3.3.1).
2. If an operator A is monotone and satisfies the Lipshitz condition, then equation (1.3) has the unique solution because the operator $A(u) + \alpha(u - u^0)$ is uniformly monotone.

Acknowledgement

The work is supported by Russian Foundation for Basic Research (projects no 15-01-00629, 16-51-50064, 15-01-05984).

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Received 11.10.2016, Accepted 01.11.2016