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AN INVERSE PROBLEM FOR A LAYERED FILM ON A SUBSTRATE

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Abstract We consider a heat interaction of a layered film laying on a solid substrate with laser modulated with frequency ω. Under the assumption that the thickness of the film is much smaller than $1/\sqrt{\omega}$ while heat conductivity of the film is very high, we derive an approximate far field solution to the heat equation and demonstrate that the found asymptotics of the solution can be used for finding averaged parameters of the film.

Key words: inverse problem, heat equation, film, solid substrate.

AMS Mathematics Subject Classification: 35R30

1 Introduction

Let a layered film be on a homogeneous substrate which coincides with half-space $\mathbb{R}^3_+ := \{(x, y, z)|\, z > 0\}$. Suppose that the film consists of n layers $\Omega_j := \{(x, y, z)|\, z_j < z < z_{j-1}\}$, $j = 1, \dots, n$, where $0 = z_0 > z_1 > \dots > z_n = -h$, $h_j := z_{j-1} - z_j$ is the thickness of Ω_j and $h > 0$ is the total thickness of the film. We assume that $h << 1$. Let $T(x, y, z, t)$ be a temperature at point (x, y, z) at a time t. The heat conduction equation is

$$\gamma \frac{\partial T}{\partial t} - \operatorname{div}(\lambda \nabla T) = 0,$$

where $\gamma = c\rho$ and c, ρ, λ are the specific heat of the medium, its density and the thermal conductivity, respectively. Values of γ and λ in Ω_j and $\mathbb{R}^3_+$ are assumed to be positive constants and be γ_j, λ_j and γ_0, λ_0, respectively.

Assume that a heat source is a laser modulated with the frequency $\omega > 0$, which is applied on the boundary of the film at point (x, y, z_n). It corresponds to the boundary condition

$$\lambda_n \frac{\partial T}{\partial z} = I_0 \delta(x, y) \exp(\mathrm{i}\omega t), \quad z = z_n,$$

where I_0 is the intensity of the laser beam. Steady-state temperature can be represented in the form $T = u(r, z) \exp(\mathrm{i}\omega t)$, where $r = \sqrt{x^2 + y^2}$ and $u(r, z)$ is a solution to the equations

$$\operatorname{div}(\lambda \nabla u) - \mathrm{i}\omega \gamma_n u = 0, \tag{1}$$

satisfying to the boundary condition

$$\lambda_n \frac{\partial u}{\partial z} = I_0 \frac{\delta(r)}{2\pi r}, \quad z = z_n, \tag{2}$$

and the radiation condition at infinitely. Because parameters γ and λ are piecewise constant functions of z, equation (1) can be rewritten inside Ω_j and $\Omega_0 := \mathbb{R}^3_+$ in the form

$$\Delta u - \mathrm{i}\omega\mu_j u = 0, \quad j = 0, 1, \ldots, n, \tag{3}$$

where $\mu_j = \gamma_j/\lambda_j$ is the value that is inverse to the thermal diffusivity in Ω_j. It is also required that the following conditions must be satisfied on the boundary interfaces between Ω_{j-1} and Ω_j:

$$u|_{z=z_j-0} = u|_{z=z_j+0},\ \lambda_{j+1}\frac{\partial u}{\partial z}\bigg|_{z=z_j-0} = \lambda_j\frac{\partial u}{\partial z}\bigg|_{z=z_j+0},\ j = 0, 1, \ldots, n-1. \tag{4}$$

Let $F(r) := u(r, 0)$. In next section we derive an asymptotic formula for this function as $r \to \infty$, i.e., the far field asymptotics, under the condition that $\frac{1}{\gamma_j} << \omega << \frac{1}{\mu_j h^2}$. This formula contains parameters λ_0 and μ_0 of the substrate and the following integral parameters of the film: $a = \sum\limits_{j=1}^{n} h_j\lambda_j, b = \sum\limits_{j=1}^{n} h_j\gamma_j$. Values a/h and b/h are the averaged conductivity λ_f and specific heat γ_f of the film, respectively, and $a/b := 1/\mu_f$ is its averaged thermal diffusivity.

We consider the following inverse problem: knowing parameters of the substrate and the far field asymptotic on plane $z = 0$, find the parameters a and b of the film. This problem is important for applications.

In this paper we demonstrate that if $a\mu_0 - b = 0$, i.e., $a(\mu_f - \mu_0) = 0$, then far field asymptotics of $F(r)$ corresponds the field of the substrate. In this case having asymptotics of function $F(r)$ as $r \to \infty$, one can only conclude that $b - a\mu_0 \sim 0$. In opposite case, if $\mu_0 \neq \mu_f$, the far field asymptotics contains two terms, the first of these terms presents a surface heat wave [1] and the second one is related with a branch point determined only by parameters of the substrate. The first term depends on a and b while the second one depends on the combination $b - a\mu_0$ only. We show that under the condition $\mu_0 > \mu_f$ the first term dominates for sufficiently large frequencies, namely, if $\lambda_0 << 2a\sqrt{|\mu_0 - \mu_f|}\,\sqrt{\omega} << 2ab\omega/\lambda_0$. In this case one can uniquely find both parameters a and b. If the second term dominates (it is always if $\mu_0 < \mu_f$), then one can find only the combination $b - a\mu_0$, in general.

2 Far field asymptotics

Introduce the Bessel transform of function $u(r, z)$ with respect to r:

$$v(\xi, z) = \int\limits_0^\infty u(r, z) J_0(r\xi) r\, \mathrm{d}r.$$

Applying this transform to relations (1) – (4), we find that function $v(\xi, z)$ satisfy the equations and a boundary and interface conditions of the form

$$v'' - (\xi^2 + \mathrm{i}\omega\mu_j)v = 0, \quad z \in (z_j, z_{j-1}),\ j = 0, 1, \ldots, n, \tag{5}$$

$$\lambda_n v' = I_0/2\pi, \quad z = z_n, \tag{6}$$

$$v|_{z=z_j-0} = v|_{z=z_j+0},\ \lambda_{j+1}v'|_{z=z_j-0} = \lambda_j v'|_{z=z_j+0},\ j = 0, 1, \ldots, n-1, \tag{7}$$

where v' means derivative of v with respect to z and $z_{-1} := \infty$.

According to these relations, function v can be represented in the form

$$v(\xi, z) = \begin{cases} a_j e^{-\beta_j z} + b_j e^{\beta_j z}, & z \in (z_j, z_{j-1}), \quad j = 1, \dots, n, \\ a_0 e^{-\beta_0 z}, & z > 0, \end{cases} \tag{8}$$

where $\beta_j = \sqrt{\xi^2 + i\omega\mu_j}$, $j = 0, 1, \dots, n$. Here the such branch of the square root is taken that accepts positive values on real positive semi-axis. Constants a_j, b_j satisfy the relations

$$-\beta_n \lambda_n \left(a_n e^{-\beta_n z_n} - b_n e^{\beta_n z_n}\right) = I_0/2\pi, \tag{9}$$

$$\begin{aligned} a_j e^{-\beta_j z_{j-1}} + b_j e^{\beta_j z_{j-1}} &= a_{j-1} e^{-\beta_{j-1} z_{j-1}} + b_{j-1} e^{\beta_{j-1} z_{j-1}}, \\ \lambda_j \beta_j \left(a_j e^{-\beta_j z_{j-1}} - b_j e^{\beta_j z_{j-1}}\right) &= \lambda_{j-1}\beta_{j-1} \left(a_{j-1} e^{-\beta_{j-1} z_{j-1}} - b_{j-1} e^{\beta_{j-1} z_{j-1}}\right), \\ & \qquad j = 1, \dots, n, \end{aligned} \tag{10}$$

where $b_0 = 0$. Denote

$$c_j := a_j e^{-\beta_j z_j} + b_j e^{\beta_j z_j}, \quad d_j := \lambda_j \beta_j \left(a_j e^{-\beta_j z_j} - b_j e^{\beta_j z_j}\right).$$

Then

$$a_j = \left(\frac{c_j}{2} + \frac{d_j}{2\lambda_j \beta_j}\right) e^{\beta_j z_j}, \qquad b_j = \left(\frac{c_j}{2} - \frac{d_j}{2\lambda_j \beta_j}\right) e^{-\beta_j z_j}, \tag{11}$$

and relations (10) can be written as follows

$$\begin{aligned} & c_j \cosh(h_j \beta_j) - d_j \sinh(h_j \beta_j)/(\lambda_j \beta_j) = c_{j-1}, \\ & c_j \lambda_j \beta_j \sinh(h_j \beta_j) - d_j \cosh(h_j \beta_j) = -d_{j-1}, \\ & \qquad\qquad j = 1, \dots, n, \end{aligned}$$

Hence,

$$\begin{pmatrix} c_j \\ d_j \end{pmatrix} = S_j \begin{pmatrix} c_{j-1} \\ d_{j-1} \end{pmatrix}, \quad j = 1, \dots, n, \tag{12}$$

where the matrix S_j is determined by the formula

$$S_j = \cosh(h_j \beta_j) \begin{pmatrix} 1 & \tanh(h_j \beta_j)/(\lambda_j \beta_j) \\ \lambda_j \beta_j \tanh(h_j \beta_j) & 1 \end{pmatrix}. \tag{13}$$

Note that $c_0 = a_0, d_0 = a_0 \lambda_0 \beta_0$. Therefore,

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = a_0 S \begin{pmatrix} 1 \\ \lambda_0 \beta_0 \end{pmatrix}, \quad S = S_n S_{n-1} \dots S_1.$$

Denote elements of the matrix S by $s_{ik}(\xi), i, k = 1, 2$. Then relation (9) implies

$$a_0 = -\frac{I_0}{2\pi[s_{21}(\xi) + \lambda_0 \beta_0(\xi) s_{22}(\xi)]}. \tag{14}$$

The collection of formulae (8), (11) – (14) determines function $v(\xi, z)$ completely. Then function $u(r, z)$ is found as the inverse Bessel transform

$$u(r,z) = \int_0^\infty v(\xi, z) J_0(r\xi)\xi \, \mathrm{d}\xi. \tag{15}$$

Consider the function

$$f(\xi) := s_{21}(\xi) + \lambda_0 \beta_0(\xi) s_{22}(\xi),$$

which coincides with the denominator in formula (14). Suppose that

$$\lambda_j |\beta_j(\xi)| >> 1,$$

i.e., $\gamma_j \omega >> 1$ for $j = 1, 2, \ldots, n$. Then matrix S_j can be uniformly for all ξ approximated by the matrix

$$S_j = \cosh(h_j \beta_j) \begin{pmatrix} 1 & 0 \\ \lambda_j \beta_j \tanh(h_j \beta_j) & 1 \end{pmatrix}.$$

Hence,

$$S = \prod_{j=1}^n \cosh(h_j \beta_j) \begin{pmatrix} 1 & 0 \\ \sum_{j=1}^n \lambda_j \beta_j \tanh(h_j \beta_j) & 1 \end{pmatrix}$$

and

$$f(\xi) = \prod_{j=1}^n \cosh(h_j \beta_j) \left[\sum_{j=1}^n \lambda_j \beta_j \tanh(h_j \beta_j) + \lambda_0 \beta_0 \right].$$

Denote real and imaginary parts of β_j by χ_j, σ_j, respectively. Note that $\chi_j \geq \sqrt{\omega \mu_j / 2} > 0$, $0 < \sigma_j \leq \sqrt{\omega \mu_j / 2}$, and $\tanh(h_j \beta_j)$ is determined by the formula

$$\tanh(h_j \beta_j) = \frac{2 \sinh(h_j \chi_j) \cosh(h_j \chi_j) + \mathrm{i} \sin(2 h_j \sigma_j)}{2[\cosh^2(h_j \chi_j) + \sin^2(h_j \sigma_j)]}.$$

Assume that $h\sqrt{\omega \mu_j} << 1$, i.e., $h_j \sigma_j << 1$ for all $j = 1, \ldots, n$. Then the real and imaginary parts of $\tanh(h_j \beta_j)$ are positive. Therefore

$$\operatorname{Im} \left[\sum_{j=1}^n \lambda_j \beta_j \tanh(h_j \beta_j) + \lambda_0 \beta_0(\xi) \right] > 0$$

and $|f(\xi)| > 0$ for all $\xi \in \mathbb{R}$. Moreover, function $|f(\xi)|$ exponentially increases as $|\xi| \to \infty$. Hence, integral in (15) exists for every z and r.

Consider the asymptotics of the function

$$F(r) := u(r, 0) = \int_0^\infty v(\xi, 0) J_0(r\xi)\xi \, \mathrm{d}\xi = -\frac{I_0}{2\pi} \int_0^\infty \frac{J_0(r\xi)\xi \, \mathrm{d}\xi}{f(\xi)}. \tag{16}$$

as $r \to \infty$. Then only small neighborhood of the point $\xi = 0$ is important when the latter integral is computed. Therefore, one can use the following approximation for function $f(\xi)$:

$$f(\xi) = \sum_{j=1}^{n} h_j \lambda_j \beta_j^2 + \lambda_0 \beta_0(\xi), \tag{17}$$

which is valid for $\xi << 1/h$ and h is small by the assumption. Hence, $F(r)$ is well determined by the approximate formula

$$F(r) = -\frac{I_0}{2\pi} \int_0^\infty \frac{J_0(r\xi)\xi \, \mathrm{d}\xi}{\sum\limits_{j=1}^{n} h_j \lambda_j \beta_j^2(\xi) + \lambda_0 \beta_0(\xi)} . \tag{18}$$

as $r \to \infty$. To evaluate the last integral, use a deformation of the integration path in the complex plane $\zeta = \xi + \mathrm{i}\eta$. First of all consider the function

$$f(\zeta) := \sum_{j=1}^{n} h_j \lambda_j \beta_j^2(\zeta) + \lambda_0 \beta_0(\zeta) = a\zeta^2 + \mathrm{i}b\omega + \lambda_0 \sqrt{\zeta^2 + \mathrm{i}\omega\mu_0},$$

where

$$a = \sum_{j=1}^{n} h_j \lambda_j, \quad b = \sum_{j=1}^{n} h_j \gamma_j.$$

On real axis this function coincides with denominator of the integrand in formula (18). Function $f(\zeta)$ is analytic on two-sheeted Riemannian surface corresponding to square root $\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0}$. Define the upper sheet by the condition $\mathrm{Re}\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0} \geq 0$ and consider function $f(\zeta)$ on this sheet in the half-plane $\mathrm{Re}\,\zeta \geq 0$. In the fourth quadrant on the ζ-plane, there is a branch line which correspond to $\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0}$ and coincide with the part of the hyperbole $2\xi\eta + \mu_0\omega = 0$ on which $\xi^2 - \eta^2 < 0$. It is determined by the parametric equations $\zeta = \zeta(\tau) = \xi(\tau) + \mathrm{i}\eta(\tau)$, where

$$\xi(\tau) = \frac{\mu_0 \omega}{\sqrt{2\left(\tau + \sqrt{\tau^2 + (\mu_0\omega)^2}\right)}} , \; \eta(\tau) = -\frac{\sqrt{\tau + \sqrt{\tau^2 + (\mu_0\omega)^2}}}{\sqrt{2}}$$

and parameter $\tau \in [0, \infty)$. The branch point is $\zeta_b := (1 - \mathrm{i})\sqrt{\mu_0\omega/2}$. Consider two sides of the branch cat: $\Gamma_+ := \{\zeta(\tau) = \xi(\tau) + 0 + \mathrm{i}\eta(\tau) | \, \tau \in [0, \infty)\}$ and $\Gamma_- := \{\zeta(\tau) = \xi(\tau) - 0 + \mathrm{i}\eta(\tau) | \, \tau \in [0, \infty)\}$. Then $\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0} = -\mathrm{i}\sqrt{\tau}$ along Γ_+ and $\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0} = \mathrm{i}\sqrt{\tau}$ along Γ_-.

In the case $b = a\mu_0$, function $f(\zeta)$ can be represented in the form

$$f(\zeta) = \sqrt{\zeta^2 + \mathrm{i}\omega\mu_0}\left(a\sqrt{\zeta^2 + \mathrm{i}\omega\mu_0} + \lambda_0\right).$$

Hence, in half-plane $\mathrm{Re}\,\zeta \geq 0$ function $f(\zeta)$ vanishes only at the branch point. Assume now that $b \neq a\mu_0$, i.e., $\mu_f \neq \mu_0$, (recall that $\mu_f := b/a$). Then in that half-plane there

exists an unique point ζ_0 such that $f(\zeta_0)=0$ and ζ_0 belongs to the fourth quadrant. Indeed, the equation
$$a\zeta^2+\mathrm{i}b\omega+\lambda_0\sqrt{\zeta^2+\mathrm{i}\omega\mu_0}=0$$
is equivalent to the equation
$$a^2\zeta^4+(2\mathrm{i}ab\omega-\lambda_0^2)\zeta^2-b^2\omega^2-\mathrm{i}\lambda_0^2\mu_0\omega=0$$
under the condition that $\operatorname{Re}\zeta^2\le 0$. Hence,
$$\zeta_0=\sqrt{\lambda_0^2-2\mathrm{i}ab\omega-\lambda_0\sqrt{\lambda_0^2+\mathrm{i}\nu}}\Big/\Big(a\sqrt{2}\Big)$$
is the unique simple root of function $f(\zeta)$ in half-plane $\operatorname{Re}\zeta>0$. Here $\nu:=4a\omega(a\mu_0-b)=4a^2\omega(\mu_0-\mu_f)$. Check that the point $\zeta_0=\xi_0+\mathrm{i}\eta_0$ belongs to the fourth quadrant. Indeed, we have $\operatorname{Im}\sqrt{\lambda_0^2+\mathrm{i}\nu}>0$ if $\nu>0$. Hence, $\operatorname{Im}\zeta_0^2<0$ if $\mu_0\ge\mu_f$. However, it is still negative even if $\mu_0<\mu_f$ because in this case we can use the representation $-2ab\omega=-2a^2\mu_0\omega+\nu/2$ and find that $\operatorname{Im}\zeta_0^2=-\mu_0\omega+\nu^3A$, where $A=\Big(\lambda_0\sqrt{2}+\sqrt{\lambda_0^2+\sqrt{\lambda_0^4+\nu^2}}\Big)\Big(\lambda_0^2+\sqrt{\lambda_0^4+\nu^2}\Big)/(4a^2)$ and $\nu<0$. So, it is always $\operatorname{Im}\zeta_0^2<0$ and $\operatorname{Re}\zeta_0^2<0$. Hence, $\operatorname{Im}\zeta_0<0$. Thus $1/f(\zeta)$ is analytic function in the first quadrant and has simple pole at point ζ_0 in the forth quadrant if $\mu_f\neq\mu_0$.

Give a more detail location of ζ_0 in the fourth quadrant. Using the previous equality $\operatorname{Im}\zeta_0^2=-\mu_0\omega+\nu^3A$, we conclude that the sign of $\operatorname{Im}\zeta_0^2+\mu_0\omega=2\xi_0\eta_0+\mu_0\omega$ coincides with the sign of ν. We also have that $\operatorname{Re}\zeta_0^2=\xi_0^2-\eta_0^2<0$. These facts imply, that ζ_0 is located in the domain bounded by the branch line, the strait line $\xi=-\eta$, and axis η if $\mu_0>\mu_f$; and ζ_0 belongs to the domain located below the branch point ζ_b and bounded by the branch line and the diagonal $\xi=-\eta$ if $\mu_0<\mu_f$. Note that the branch point ζ_b belongs to the diagonal $\xi=-\eta$. If $4a\omega|a\mu_0-b|<<\lambda_0^2$, i.e., $4(h\lambda_f)^2|\mu_0-\mu_f|\omega<<\lambda_0^2$, then ζ_0 is close to ζ_b. If ω is high enough, namely, if it satisfies the inequalities $\lambda_0<<2a\sqrt{|\mu_0-\mu_f|}\sqrt{\omega}<<2ab\omega/\lambda_0$, then $\zeta_0^2\sim-\mathrm{i}\mu_f\omega$, hence ζ_0 is close to the diagonal $\xi=-\eta$. Moreover, $\operatorname{Im}\zeta_0>\operatorname{Im}\zeta_b$ if $\mu_0>\mu_f$ and $\operatorname{Im}\zeta_0<\operatorname{Im}\zeta_b$ if $\mu_0<\mu_f$.

We return now to calculation the asymptotics of $F(r)$ for large r. Bessel function J_0 can be expressed by the first and second Hankel functions $H_0^{(1)}$, $H_0^{(2)}$ as
$$J_0(r\zeta)=\frac{1}{2}\Big(H_0^{(1)}(r\zeta)+H_0^{(2)}(r\zeta)\Big).$$
Because function $H_0^{(1)}(r\zeta)$ exponentially decreases as $|\zeta|\to\infty$ and $\operatorname{Im}\zeta>0$, and the integrand is an analytic function in the first quadrant, one has
$$\int_0^\infty\frac{H_0^{(1)}(r\xi)\xi\,\mathrm{d}\xi}{f(\xi)}=-\int_0^\infty\frac{H_0^{(1)}(\mathrm{i}r\eta)\eta\,\mathrm{d}\eta}{f(\mathrm{i}\eta)}.$$
For calculating the integral of $H_0^{(2)}$, take the closed path of integration in the fourth quadrant which goes along real axis ξ from the origin to ∞, along the infinitely circular

arc in clockwise direction and then along the side of the branch cat $\Gamma_+ := \{\zeta(\tau) = \xi(\tau) + 0 + \mathrm{i}\eta(\tau) | \, \tau \in [0, \infty)\}$ from $\tau = \infty$ to $\tau = 0$ and then along the other side $\Gamma_- := \{\zeta(\tau) = \xi(\tau) - 0 + \mathrm{i}\eta(\tau) | \, \tau \in [0, \infty)\}$ from $\tau = 0$ to $\tau = \infty$ and along the imaginary axis η from $-\infty$ to the origin (see, e.g., the similar path in [2], p. 246). Then the integral along this closed path can be calculated by the residue theorem (if $\mu_f \neq \mu_0$) and the integral along the circular path vanishes because the function $H_0^{(2)}(r\zeta)$ exponentially decreases as $|\zeta| \to \infty$ and $\operatorname{Im}\zeta < 0$. Therefore,

$$\int\limits_0^\infty \frac{H_0^{(2)}(r\xi)\xi \, \mathrm{d}\xi}{f(\xi)} = -\int\limits_0^\infty \frac{H_0^{(2)}(-\mathrm{i}r\eta)\eta \, \mathrm{d}\eta}{f(-\mathrm{i}\eta)} - 2\pi\mathrm{i}\frac{H_0^{(2)}(r\zeta_0)\zeta_0}{f'(\zeta_0)} - \int\limits_\Gamma \frac{H_0^{(2)}(r\zeta)\zeta \, \mathrm{d}\zeta}{f(\zeta)} \, ,$$

where f' means the derivative of f with respect to ζ and $\Gamma = \Gamma_+ \bigcup \Gamma_-$. Because $f(-\mathrm{i}\eta) = f(\mathrm{i}\eta)$ and $H_0^{(2)}(-\mathrm{i}r\eta) = -H_0^{(1)}(\mathrm{i}r\eta)$, we find

$$F(r) = \frac{\mathrm{i}I_0 H_0^{(2)}(r\zeta_0)\zeta_0}{2f'(\zeta_0)} + \frac{I_0}{4\pi}\int\limits_\Gamma \frac{H_0^{(2)}(r\zeta)\zeta \, \mathrm{d}\zeta}{f(\zeta)} := F_1(r) + F_2(r), \tag{19}$$

Noting that

$$f'(\zeta_0) = 2a\zeta_0 + \frac{\lambda_0\zeta_0}{\sqrt{\zeta_0^2 + \mathrm{i}\mu_0\omega}} = \frac{\zeta_0[2a\sqrt{\zeta_0^2 + \mathrm{i}\mu_0\omega} + \lambda_0]}{\sqrt{\zeta_0^2 + \mathrm{i}\mu_0\omega}},$$

we find that

$$F_1(r) = \frac{\mathrm{i}I_0 H_0^{(2)}(r\zeta_0)\sqrt{\zeta_0^2 - \zeta_b^2}}{2\left[2a\sqrt{\zeta_0^2 - \zeta_b^2} + \lambda_0\right]} \, . \tag{20}$$

This function presents the heat surface wave which occurs as consequence of the layered medium (see also [1], [3], [4]). If $4a\omega|a\mu_0 - b| << \lambda_0^2$, i.e., $4(h\lambda_f)^2|\mu_0 - \mu_f|\omega << \lambda_0^2$, then ζ_0 is close to ζ_b and function $F_1(r)$ becomes small. If $a\mu_0 - b = 0$ then $F_1(r)$ vanishes and the asymptotics of $F_2(r)$ coincides with the asymptotics of the function

$$F_2^0(r) = \frac{I_0}{4\pi\lambda_0}\int\limits_\Gamma \frac{H_0^{(2)}(r\zeta)\zeta \, \mathrm{d}\zeta}{\sqrt{\zeta^2 + \mathrm{i}\mu_0\omega}} \, ,$$

which corresponds the field of the solid substrate.

Transform the integral over Γ to a more convenient for calculation form. Noting that

$$f(\zeta) = a\zeta^2(\tau) + \mathrm{i}\,(b\omega - \lambda_0\sqrt{\tau}), \quad \zeta(\tau) \in \Gamma_+,$$
$$f(\zeta) = a\zeta^2(\tau) + \mathrm{i}\,(b\omega + \lambda_0\sqrt{\tau}), \quad \zeta(\tau) \in \Gamma_-,$$

we find that

$$\begin{aligned} F_2(r) \quad &= \quad \frac{I_0}{4\pi}\int\limits_\Gamma \frac{H_0^{(2)}(r\zeta)\zeta \, \mathrm{d}\zeta}{f(\zeta)} = -\frac{I_0}{2\pi}\int\limits_0^\infty H_0^{(2)}(r\zeta(\tau)) \frac{\mathrm{i}\,\lambda_0\sqrt{\tau}\,\zeta(\tau)\,\zeta'(\tau)\,\mathrm{d}\tau}{[a\,\zeta^2(\tau) + \mathrm{i}\,b\omega]^2 + \lambda_0^2\tau} \\ &= -\frac{I_0}{2\pi}\int\limits_{\Gamma_+} H_0^{(2)}(r\zeta)\,\frac{\lambda_0\sqrt{\zeta^2 + \mathrm{i}\mu_0\omega}\,\zeta \ \mathrm{d}\zeta}{[a\,\zeta^2 + \mathrm{i}\,b\omega]^2 - \lambda_0^2(\zeta^2 + \mathrm{i}\mu_0\omega)} \, , \end{aligned} \tag{21}$$

where and $\zeta(\tau)$ determines the branch line and $\zeta'(\tau)$ stands for the derivative of ζ with respect to τ. In the last integral the integration direction along Γ_+ is taken towards to the branch point.

When r is large, we have the asymptotic expansion of the Hankel function such as

$$H_0^{(2)}(r) = \sqrt{\frac{2}{\pi r}}\, \mathrm{e}^{-\mathrm{i}(r-\pi/4)} \left[1 + \mathcal{O}(r^{-1})\right], \tag{22}$$

where $\mathcal{O}(r^{-1})$ means the first order term with respect to $1/r$. Taking into account formulae (20), (22), we find that

$$F_1(r) = \frac{1}{\sqrt{r}}\, \mathrm{e}^{-\mathrm{i}r\zeta_0} \left[p_0 + \mathcal{O}(r^{-1})\right], \quad r \to \infty, \tag{23}$$

where

$$p_0 = \frac{I_0(-1+\mathrm{i})\sqrt{\zeta_0^2 + \mathrm{i}\mu_0\omega}}{\sqrt{\pi\zeta_0}\,[2a\sqrt{\zeta_0^2+\mathrm{i}\mu_0\omega} + \lambda_0]}\,. \tag{24}$$

To determine the asymptotics of $F_2(r)$, consider δ-neighborhood of the branch point and inside its the segment $L := \{\zeta = \zeta_b - \mathrm{i}\tau|\, \tau \in (0,\delta)\}$ of the straight line along which the integrang exponentially decreases. For $\zeta \in L$ we have the following representation of the integrand:

$$-\frac{I_0}{2\pi} H_0^{(2)}(r\zeta)\, \frac{\lambda_0\sqrt{\zeta^2 + \mathrm{i}\mu_0\omega}\,\zeta}{[a\,\zeta^2 + \mathrm{i}\,b\omega]^2 - \lambda_0^2(\zeta^2 + \mathrm{i}\mu_0\omega)} \approx \frac{\mathrm{e}^{-\mathrm{i}r\zeta_b - r\tau}}{\sqrt{r}}\, \sqrt{\tau}\Big[g_0 + \mathcal{O}(\tau)\Big],$$

where

$$g_0 = \frac{I_0(1-\mathrm{i})\lambda_0\sqrt{\zeta_b}}{2\sqrt{\pi^3}\omega^2(b-a\mu_0)^2}. \tag{25}$$

Then the contribution from the neighborhood of the branch point ζ_b as $r \to \infty$ is as follows(see, e.g., [2], p. 228, where the similar calculations are made):

$$F_2(r) \approx g_0 \frac{\mathrm{e}^{-\mathrm{i}r\zeta_b}}{\sqrt{r}} \int\limits_0^\delta \mathrm{e}^{-r\tau}\, \sqrt{\tau}\, \mathrm{d}\tau = \frac{\mathrm{e}^{-\mathrm{i}r\zeta_b}}{r^2} \Big[q_0 + \mathcal{O}(r^{-1})\Big],$$

where $q_0 = g_0\sqrt{\pi}/2$.

Finally, we obtain the far field asymptotics of $F(r)$ as $r \to \infty$ of the form

$$F(r) = \frac{1}{\sqrt{r}}\, \mathrm{e}^{-\mathrm{i}r\zeta_0} \left[p_0 + \mathcal{O}(r^{-1})\right] + \frac{1}{r^2}\, \mathrm{e}^{-\mathrm{i}r\zeta_b} \left[q_0 + \mathcal{O}(r^{-1})\right], \quad b \neq a\mu_0. \tag{26}$$

Note that the latter formula holds good if $b - a\mu_0$ is not very close to zero. Recall also that formula (26) is derived under the condition that

$$\frac{1}{\gamma_j} << \omega << \frac{1}{\mu_j h^2} \qquad j = 1, 2, , \ldots, n.$$

We see that asymptotics of $F(r)$ consists of two terms. Depending on the location of ζ_0 and ζ_b, the first or the second term can dominate. The most interesting case for our goal is when the first term is leading. It is happened if and only if $\mu_0 > \mu_f$. It was shown above that in this case $0 < -\mathrm{Im}\zeta_0 < \mathrm{Im}\zeta_b$ for sufficiently large frequencies, hence, the first term in (26), which corresponds to the surface heat wave, dominates for large r.

If $\mu_f = \mu_0$ then the integrand in formula (21) has the following expansion for $\zeta \in L$:

$$-\frac{I_0}{2\pi} H_0^{(2)}(r\zeta)\, \frac{\lambda_0 \sqrt{\zeta^2 + \mathrm{i}\mu_0\omega}\, \zeta}{[a\,\zeta^2 + \mathrm{i}\, b\omega]^2 - \lambda_0^2(\zeta^2 + \mathrm{i}\mu_0\omega)} \approx \frac{\mathrm{e}^{-\mathrm{i}r\zeta_b - r\tau}}{\sqrt{r}}\, \frac{1}{\sqrt{\tau}} \Big[g^* + \mathcal{O}(\tau) \Big],$$

where

$$g^* = \frac{I_0(-1+\mathrm{i})\sqrt{\zeta_b}}{2\sqrt{\pi^3}\lambda_0}.$$

Therefore the far field asymptotics of $F(r)$ is found then by the formula

$$F(r) \approx \frac{q^*}{r}\, \mathrm{e}^{-\mathrm{i}r\zeta_b}, \quad r \to \infty, \quad q^* = g^*\sqrt{\pi}, \quad b = a\mu_0, \tag{27}$$

and coincides with the far field asymptotics for the solid substrate.

3 Recovering averaged parameters of the film

Now consider the problem of determining the parameters a and b via function $F(r)$ given for large r. We assume here that λ_0, μ_0 and ω, I_0 are known. Then the expression in right-hand side of (27) can be calculated. If the calculated function coincides with given function $F(r)$ enough good for large r, then we can conclude only that $a\mu_0 - b$ is close to zero. In opposite case, function $F(r)$ must be good approximated by the expression in the right-hand side of (26). Let us consider the function $\hat{F}(r) := r^2\, F(r) \exp(\mathrm{i}r\zeta_b)$. If this function is good approximated by a constant for large r, then this constant necessary coincides with q_0. It means also that in this case the second term in (26) is leading. Then we can find q_0 and $g_0 = 2q_0/\sqrt{\pi}$ and calculate the combination $b - a\mu_0$ using formula (25). In this case it is impossible to determine a and b separately, in general, since the first term can decrease very strongly as $r \to \infty$ and be lower a noise.

If function $\hat{F}(r)$ has no a finite limit as $r \to \infty$, it means that the first term in (26) dominates and function $F(r)$ must be good approximated by the formula

$$F(r) = \frac{p_0}{\sqrt{r}}\, \mathrm{e}^{-\mathrm{i}r\zeta_0}, \quad r \to \infty. \tag{28}$$

Then one can find $\zeta_0 = \xi_0 + \mathrm{i}\eta_0$ and, hence, calculate

$$a\zeta_0^2 + \mathrm{i}b\omega = -\lambda_0\sqrt{\zeta_0^2 + \mathrm{i}\mu_0\omega}.$$

This relation is uniquely determines a and b, because $\xi_0^2 - \eta_0^2 \neq 0$ and $2\xi_0\eta_0 + \mu_0\omega \neq 0$ (the latter was proved in the previous section).

To determine ζ_0 one can use the following algorithm. Take the function $\chi_1(r) := \log(|F(r)|\sqrt{r})$. According to formula (28), function $\chi_1(r)$ is a linear function with its slop equal to η_0. So, η_0 can be found by the least-squares method. Then the function $\chi_2(r) := F(r)\sqrt{r}\exp(-\eta_0 r)$ must be closed to the periodical function $p_0\exp(-\mathrm{i}r\xi_0)$, therefore one can easily find ξ_0.

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