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SOME METHODS FOR SOLVING OF 3D INVERSE PROBLEM OF MAGNETOMETRY

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Abstract Recovery of magnetic target parameters from magnetic sensor measurements has attracted wide interests and found many practical applications. However, difficulties present in identifying the magnetization due to the complications of magnetization distributions over investigated object, errors and noises of measurement data, degrade the accuracy and quality of the restored parameters. In this paper we consider a modern model for the mentioned problem (magnetic inversion based on both total magnetic intensity data and full tensor gradient magnetic data) and some method of its solving. This method involves taking into account the round-off errors, accumulation of which could significantly influences the restored solution in the case of using model with full tensor gradient magnetic data. Tikhonov regularization has been applied in solving the inversion problem with the modified generalized discrepancy principle (that include information about accumulated round-off errors) for the choosing regularization parameter.

Key words: magnetometry, full tensor magnetic gradient, inverse problem, Tikhonov regularization, generalized discrepancy principle, round-off errors.

AMS Mathematics Subject Classification: 45B05, 45Q05, 65R32, 65F22.

1 Introduction

Recovery of magnetization parameters from magnetic sensor measurements has attracted wide interests and found many practical applications. The inverse problem under investigation is to discover the magnetization of some object with magnetic field measurements made outside the ship.

One of the practical interests to the inverse problem is to recover the magnetization parameters of some object with a minimum number of sensor measurements [1, 2, 3, 4]. The magnetization of some object can be separated into two types of components. The first type is the induced magnetization, which is generated by the object immersing in the earth magnetic field. The induced magnetization can be predicted with numerical computation. The second type is the permanent magnetization. Its strength and direction depend on the magnetic history of mechanical and thermal constraints, magnetostriction, e.t.c. Because we have no *a priori* knowledge to this magnetic history, the conventional approach is to use magnetic sensor measurements to determine it with an assumption of uniform magnetization. In this paper we consider the inverse problem of restoring total magnetization. However, difficulties present in identifying the total magnetization due to the complications of magnetization distributions over the investigated object, errors and noises of measurement data, degrade the accuracy and quality of the parameter identification.

At this article we consider both traditional magnetic data model (based on total magnetic intensity) and modern magnetic data model (based on full tensor gradient magnetic data). The modern model has been considered in detail at work [5] (the following statements based on the information from this work). Using this model for solving real world practical applications became possible with the development of a high temperature superconducting quantum interference device operating in liquid nitrogen, based on which a novel rotating magnetic gradiometer system has been designed. This system allows to measure components of the gradient tensor. Gradient measurements are relatively insensitive to orientation because gradients arise largely from anomalous sources, and the background geomagnetic gradient is low. This contrasts with the field vector, which is dominated by the background field from Earth's core. Gradient measurements also provide valuable additional information, compared to conventional total-field measurements, when the field is undersampled. Many discussions are given on the on the advantages of magnetic gradient tensor surveys as compared to the conventional total magnetic intensity surveys.

But for numerical investigation the fact that geomagnetic gradient is low leads to the situation when round-off errors could significantly influence the result of calculations [6]. So, we try to apply the method [7] which is able to solve this problem via generalization of the "generalized discrepancy principle" [8, 9] for choosing regularization parameter in Tikhonov regularization procedure.

2 3D model of the full tensor magnetic gradient data

The equation describing magnetic field $\boldsymbol{B}_{field\ dipole}$ of dipole sources $\boldsymbol{m}$ is defined as

$$\boldsymbol{B}_{field\ dipole} = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3}\right),$$

where $\boldsymbol{m} = m_x\boldsymbol{i} + m_y\boldsymbol{j} + m_z\boldsymbol{k}$, $\boldsymbol{r} = (x - x_s)\boldsymbol{i} + (y - y_s)\boldsymbol{j} + (z - z_s)\boldsymbol{k}$, $r = \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}$ is a distance between point (x_s, y_s, z_s), which corresponds to allocation of the triaxial sensor that measures magnetic field $\boldsymbol{B}_{field\ dipole}$, and point (x, y, z) of dipole source $\boldsymbol{m}$, μ_0 is a permeability in vacuum.

Transforming $\boldsymbol{B}_{field\ dipole}$ into following form

$$\boldsymbol{B}_{field\ dipole} = B_{x\ dipole}\boldsymbol{i} + B_{y\ dipole}\boldsymbol{j} + B_{z\ dipole}\boldsymbol{k} = \frac{\mu_0}{4\pi}\left(\left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})(x-x_s)}{r^5} - \frac{m_x}{r^3}\right)\boldsymbol{i} + \right.$$
$$\left. + \left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})(y-y_s)}{r^5} - \frac{m_y}{r^3}\right)\boldsymbol{j} + \left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})(z-z_s)}{r^5} - \frac{m_z}{r^3}\right)\boldsymbol{k}\right)$$

and redefining the variables as $i = x, y, z$ and $p = (p_x, p_y, p_z) = (x_s, y_s, z_s)$, we have following representation for components of vector $\boldsymbol{B}_{field\ dipole}$:

$$B_{i\ dipole} = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})(i-p_i)}{r^5} - \frac{m_i}{r^3}\right).$$

Taking derivative of $B_{i\,dipole}$ with respect to spatial variable $i = x, y, z$ and $j = x, y, z \neq i$, we have the diagonal elements and non-diagonal elements of tensor matrix $\mathbf{B}_{tensor}$:

$$B_{ii} = \frac{\mu_0}{4\pi}\left(\frac{6m_i(i-p_i)}{r^5} + \frac{3(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{15(\boldsymbol{m}\cdot\boldsymbol{r})(i-p_i)(i-p_i)}{r^7}\right),$$
$$B_{ij} = \frac{\mu_0}{4\pi}\left(\frac{3m_i(j-p_j)}{r^5} + \frac{3m_j(i-p_i)}{r^5} - \frac{15(\boldsymbol{m}\cdot\boldsymbol{r})(i-p_i)(j-p_j)}{r^7}\right).$$

Note, that we define full tensor magnetic gradient $\mathbf{B}_{tensor}$, which unlike to magnetic induction $\boldsymbol{B}_{field\,dipole}$ (that has only 3 components) has 9 components and can be written in the following matrix form:

$$\mathbf{B}_{tensor} = [B_{ij}] = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{yx} & B_{yy} & B_{yz} \\ B_{zx} & B_{zy} & B_{zz} \end{bmatrix},$$

where $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$, $\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x}$, $\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y}$ and $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$. So, actually, we have only 5 different components of the tensor matrix.

Thus, for the whole object, for volume V of which we want to restore the magnetic moment density $\boldsymbol{M}$ ($\boldsymbol{M} = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k}$), we have the following 3D Fredholm integral equations of the 1^{st} kind:

$$\boldsymbol{B}_{field\,dipole} = \frac{\mu_0}{4\pi}\iiint\limits_V \left(\frac{3(\boldsymbol{M}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{M}}{r^3}\right)dv,$$
$$B_{ii} = \frac{\mu_0}{4\pi}\iiint\limits_V \left(\frac{6m_i(i-p_i)}{r^5} + \frac{3(\boldsymbol{M}\cdot\boldsymbol{r})}{r^5} - \frac{15(\boldsymbol{M}\cdot\boldsymbol{r})(i-p_i)(i-p_i)}{r^7}\right)dv,$$
$$B_{ij} = \frac{\mu_0}{4\pi}\iiint\limits_V \left(\frac{3m_i(j-p_j)}{r^5} + \frac{3m_j(i-p_i)}{r^5} - \frac{15(\boldsymbol{M}\cdot\boldsymbol{r})(i-p_i)(j-p_j)}{r^7}\right)dv,$$

which can be rewritten as the following system of two 3D Fredholm integral equations of the 1^{st} kind:

$$\begin{cases} \boldsymbol{B}_{field\,dipole}(x_s, y_s, z_s) = \dfrac{\mu_0}{4\pi}\displaystyle\iiint\limits_V \mathbf{K}_1(x-x_s, y-y_s, z-z_s)\boldsymbol{M}(x,y,z)dv, \\ \boldsymbol{B}_{tensor\,dipole}(x_s, y_s, z_s) = \dfrac{\mu_0}{4\pi}\displaystyle\iiint\limits_V \mathbf{K}_2(x-x_s, y-y_s, z-z_s)\boldsymbol{M}(x,y,z)dv, \end{cases} \tag{1}$$

where $\boldsymbol{B}_{field\,dipole} = [B_x\ B_y\ B_z]^T$ and $\boldsymbol{B}_{tensor\,dipole} = [B_{xx}\ B_{xy}\ B_{xz}\ B_{yz}\ B_{zz}]^T$. Kernels $\mathbf{K}_1$ and $\mathbf{K}_2$ of these integral equations can be written as

$$\mathbf{K}_1(x-x_s, y-y_s, z-z_s) = \frac{1}{r^5}\begin{bmatrix} 3(x-x_s)^2 - r^2 & 3(x-x_s)(y-y_s) & 3(x-x_s)(z-z_s) \\ 3(y-y_s)(x-x_s) & 3(y-y_s)^2 - r^2 & 3(y-y_s)(z-z_s) \\ 3(z-z_s)(x-x_s) & 3(z-z_s)(y-y_s) & 3(z-z_s)^2 - r^2 \end{bmatrix}$$

and

$$\mathbf{K}_2(x-x_s, y-y_s, z-z_s) = \frac{3}{r^7}\times$$
$$\times\begin{bmatrix} (x-x_s)[3r^2-5(x-x_s)^2] & (y-y_s)[r^2-5(x-x_s)^2] & (z-z_s)[r^2-5(x-x_s)^2] \\ (y-y_s)[r^2-5(x-x_s)^2] & (x-x_s)[r^2-5(y-y_s)^2] & -5(x-x_s)(y-y_s)(z-z_s) \\ (z-z_s)[r^2-5(x-x_s)^2] & -5(x-x_s)(y-y_s)(z-z_s) & (x-x_s)[r^2-5(z-z_s)^2] \\ -5(x-x_s)(y-y_s)(z-z_s) & (z-z_s)[r^2-5(y-y_s)^2] & (y-y_s)[r^2-5(z-z_s)^2] \\ (x-x_s)[r^2-5(z-z_s)^2] & (y-y_s)[r^2-5(z-z_s)^2] & (z-z_s)[3r^2-5(z-z_s)^2] \end{bmatrix}.$$

If we take into account that $V \subset P = \{(x,y,z):\ L_x \leqslant x \leqslant R_x,\ L_y \leqslant y \leqslant R_y,\ L_z \leqslant z \leqslant R_z\}$ and the system of sensor planes is restricted by rectangular parallelepiped $Q = \{(x_s,y_s,z_s) \equiv (s,t,r):\ L_s \leqslant s \leqslant R_s,\ L_t \leqslant t \leqslant R_t,\ L_r \leqslant r \leqslant R_r\}$, we can rewrite the system (1) in the following operator form

$$\mathbf{A}\boldsymbol{M} = \frac{\mu_0}{4\pi}\int\limits_{L_x}^{R_x}\int\limits_{L_y}^{R_y}\int\limits_{L_z}^{R_z}\mathbf{K}(s,t,r,x,y,z)\boldsymbol{M}(x,y,z)dxdydz = \boldsymbol{B}(s,t,r), \qquad (2)$$

where $\boldsymbol{B}(s,t,r)$ and $\boldsymbol{M}(x,y,z)$ are vector–functions: $\boldsymbol{B} = [B_x\ B_y\ B_z\ B_{xx}\ B_{xy}\ B_{xz}\ B_{yz}\ B_{zz}]^T$ and $\boldsymbol{M} = [M_x\ M_y\ M_z]^T$, kernel $\mathbf{K}(s,t,r,x,y,z)$ is a matrix-function: $\mathbf{K} = [\mathbf{K}_1\ \mathbf{K}_2]^T$ ($\mathbf{K} = [\mathbf{K}_1]^T$ in the case of total magnetic intensity model without using full tensor magnetic gradient data).

Then we will be assume that $\boldsymbol{M} \in W_2^2(P)$, $\boldsymbol{B} \in L_2(Q)$, and operator $\mathbf{A}$ with kernel $\mathbf{K}$ is continuous and unique. Norms of the right-hand side of equation (2) and the solution are introduces as follows: $\|\boldsymbol{B}\|_{L_2} = \sqrt{\|B_x\|_{L_2}^2 + \|B_y\|_{L_2}^2 + \|B_z\|_{L_2}^2 + \|B_{xx}\|_{L_2}^2 + \|B_{xy}\|_{L_2}^2 + \|B_{xz}\|_{L_2}^2 + \|B_{yz}\|_{L_2}^2 + \|B_{zz}\|_{L_2}^2}$, $\|\boldsymbol{M}\|_{W_2^2} = \sqrt{\|M_x\|_{W_2^2}^2 + \|M_y\|_{W_2^2}^2 + \|M_z\|_{W_2^2}^2}$. Suppose that instead of accurately known $\bar{\boldsymbol{B}}$ and $\mathbf{A}$ their approximate values $\boldsymbol{B}_\delta$ and $\mathbf{A}_h$ are known, such that $\|\boldsymbol{B}_\delta - \bar{\boldsymbol{B}}\|_{L_2} \leqslant \delta$, $\|\mathbf{A} - \mathbf{A}_h\|_{W_2^2 \to L_2} \leqslant h$. So, the inverse problem is ill-posed and it is necessary to build a regularizing algorithm for its solving. We use the algorithm based on minimization of the Tikhonov functional [9]

$$F^\alpha[\boldsymbol{M}] = \|\mathbf{A}_h\boldsymbol{M} - \boldsymbol{B}_\delta\|_{L_2}^2 + \alpha\|\boldsymbol{M}\|_{W_2^2}^2. \qquad (3)$$

For any $\alpha > 0$ an unique extremal of the Tikhonov functional $\boldsymbol{M}_\eta^\alpha$, $\eta = \{\delta, h\}$, which implements minimum of $F^\alpha[\boldsymbol{M}]$, exists. To select the regularization parameter the generalized discrepancy principle can be used [8, 9]. When we choose the parameter $\alpha = \alpha(\eta)$ accordingly to the generalized discrepancy principle

$$\rho(\alpha) = \|\mathbf{A}_h\boldsymbol{M}_\eta^\alpha - \boldsymbol{B}_\delta\|_{L_2}^2 - \left(\delta + h\|\boldsymbol{M}_\eta^\alpha\|_{W_2^2}\right)^2 = 0 \qquad (4)$$

$\boldsymbol{M}_\eta^\alpha$ tends to exact solution as $\eta \to 0$ in W_2^2. The minimal element of the Tikhonov functional for fixed $\alpha > 0$ can be found by the application of the conjugate gradient method.

3 Structure of the algorithm

For numerical minimization of functional (3) we used algorithms which were described in details at works [10, 11], including some recommendations of its effective parallelization. Thereby, in this section we describe some new approach that has not been mentioned at [10, 11], but was introduced only at [6].

After discretization an approximate solution M, which realizes the minimum of functional (3), can be found as a solution of the system

$$(A_h^T A_h + \alpha R^T R)\, M = A_h^T B_\delta, \tag{5}$$

where R — finite-difference approximation of the operator $\mathbf{R}$: $\|\boldsymbol{M}\|_{W_2^2} = \|\mathbf{R}\boldsymbol{M}\|_{L_2}$, dimensions of matrix A: $(N_A \times N)$, matrix R: $(N_R \times N)$, vector M: $(N \times 1)$.

For numerical solving of system (5) we use the conjugate gradient method in the form that was proposed at [7].

Let $M^{(s)}$ — minimizing sequence, $p^{(s)}$, $q^{(s)}$ — auxiliary vectors, $p^{(0)} = 0$, $M^{(1)}$ — any arbitrary point. Then formulae of the conjugate gradient method for searching of solution $M^{(N)}$ of system (5) can be rewritten as follow:

$$\begin{aligned}
&r^{(s)} = \begin{cases} A_h^T\big(A_h\, M^{(s)} - B_\delta\big) + \alpha\, R^T\big(R\, M^{(s)}\big), & \text{if} \quad s = 1, \\ r^{(s-1)} - q^{(s-1)}/\big(p^{(s-1)}, q^{(s-1)}\big), & \text{if} \quad s \geqslant 2, \end{cases}\\
&p^{(s)} = p^{(s-1)} + \frac{r^{(s)}}{\big(r^{(s)}, r^{(s)}\big)},\\
&q^{(s)} = A_h^T\big(A_h\, p^{(s)}\big) + \alpha\, R^T\big(R\, p^{(s)}\big)\,,\\
&M^{(s+1)} = M^{(s)} - \frac{p^{(s)}}{\big(p^{(s)}, q^{(s)}\big)}.
\end{aligned}$$

Now, we have to remind that the conjugate gradient method can theoretically be viewed as a direct method, because it produce the exact solution after finite number of iterations (in the absence of round-off errors), which is no larger than size N of matrix $(A_h^T A_h + \alpha R^T R)$ of system (5). However, the conjugate gradient method is unstable with respect to even small perturbation, and exact solution is never obtained. Different situations are probable. The first one is when pretty soon we come to a neighborhood of the minimum, but because of round-off errors discrepancy $r^{(s)}$ is not reduced further, subsequent iterations are made with no sense. So it is very important to develop a criterion for earlier termination of the iterative process. The second one is that due to the round-off errors most directions are not in practice conjugate, and iterative process may stop far enough away from the exact minimum. Fortunatelly, the conjugate gradient method can be used as an iterative method as it provides monotonically improving approximations to the exact solution, which may reach the required tolerance after a relatively small (compared to the problem size) number of iterations. The method that is able to help us to take into account the round-off errors is called as «unreliability of discrepancy criterion» and was proposed by Kalitkin and Kuzmina at [7].

Let us consider the component-wise calculation of the residual on the sst iteration:

$$r_n^{(s)} = \sum_{k=1}^{N_A} A_{kn} \big(A\, M^{(s)}\big)_k + \alpha \sum_{k=1}^{N_R} R_{kn} \big(R\, M^{(s)}\big)_k - \big(A^T B_\delta\big)_n. \tag{6}$$

In the calculations of right-side (6) of this equation the round-off errors of the multiplications are negligible. The round-off error of additions can be taken into account by statistical rules:

$$\big(\sigma^{(s)}\big)^2 = \sum_{n=1}^{N} \Bigg(\big(A^T B_\delta\big)_n^2 + \sum_{k=1}^{N_A} \Big(\big(A_{kn}\big)^2 \big(A\, z^{(s)}\big)_k^2 + \big(A_{kn}\big)^2 \big(M^{(s)}\big)_n^2 + \big(A_{kn}\big)^2 \big(B_\delta\big)_k^2 \Big) + \\ + \alpha \sum_{k=1}^{N_R} \Big(\big(R_{kn}\big)^2 \big(R\, M^{(s)}\big)_k^2 + \big(R_{kn}\big)^2 \big(M^{(s)}\big)_n^2 \Big) \Bigg).$$

Therefore, the discrepancy is calculated reliably if the following condition is true

$$\|r^{(s)}\|^2 \gg \Delta^2 \big(\sigma^{(s)}\big)^2, \quad \|r^{(s)}\|^2 = \sum_{n=1}^{N} \big(r_n^{(s)}\big)^2,$$

where Δ — round-off error of a single calculation (e.g., $\Delta = 10^{-16}$ for calculations with «double preision» and $\Delta = 10^{-32}$ calculations with «quad precision»).

Considering also the accumulation of errors and mistakes by steps we can come to the following stoping criteria of the iterative process [7]. The iterative process have to be interrupted on iteration with number $N_{optimal}$, when the following condition is true

$$\Delta^2 \sum_{s=1} \left(\frac{\big(\sigma^{(s)}\big)^2}{\|r^{(s)}\|^2} \right) > 1.$$

And, finally, we can take into account the rounding error while chousing the regularization parameter via generalized discrepancy principle (4) [8, 9]:

$$\rho(\alpha) = \|A_h\, M_\eta^\alpha - B_\delta\|^2 - \big(\delta + h\|M_\eta^\alpha\|\big)^2 - \Delta^2 \sum_{s=1}^{N_{optimal}} \big(\sigma^{(s)}\big)^2 = 0.$$

As a result, we will be able to suppress the additional volatility and get a solution, which will be able to trust.

4 Some results of model calculations

As an example we considered the following problem [10]. A ship passes over a system of triaxial sensors (Figure 1) which measure the value of the induced magnetic field. According to these values of induced magnetic field it is necessary to restore the magnetization parameters over the hull of the ship [1, 2, 3, 4, 10, 11]. This formulation of the problem is equal to a situation where the ship stands over the system of sensor arrays (Figure 2).

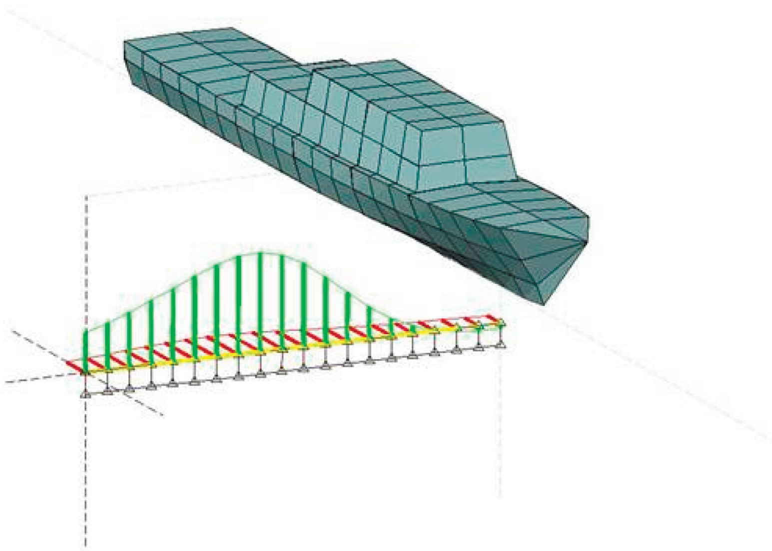

Figure 1: The ship passes over the system of triaxial sensors.

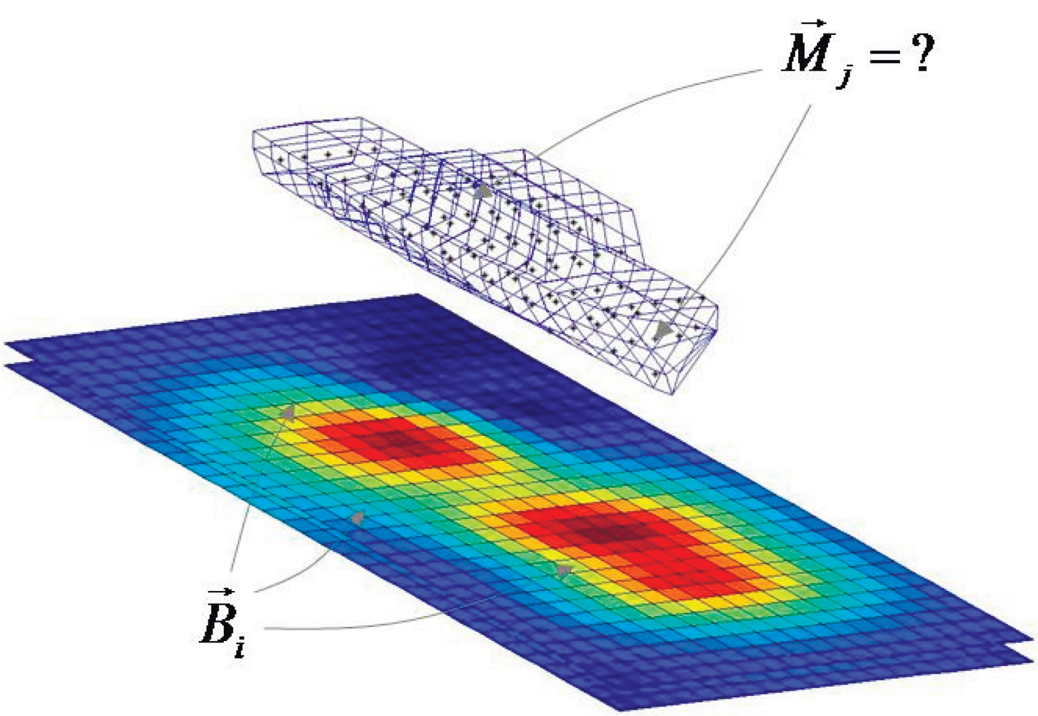

Figure 2: The ship stands over the system of sensor arrays.

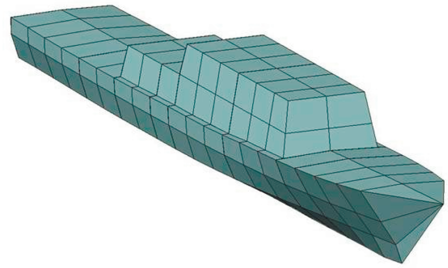

Figure 3: Model of the ship.

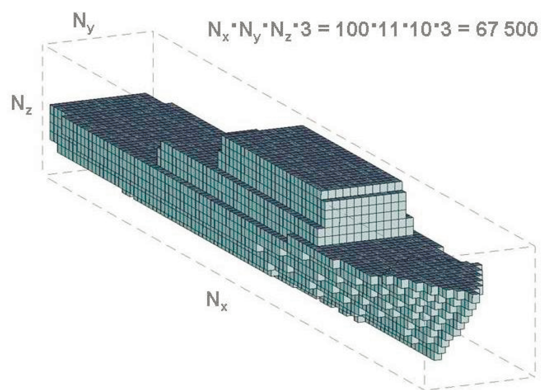

Figure 4: Parallelepiped segmentation of the volume of the ship.

Typical dimensions on each spatial variables that correspond to real applications are $N_x = 100$, $N_y = 15$, $N_z = 15$ (Figure 3 and 4). Input data simulated a real experiment (in the case of total magnetic intensity data) and correspond to grids $N_x = 200$, $N_y = 15$, $N_z = 15$, $N_s = 4000$, $N_t = 3$, $N_r = 2$. This corresponds to 67500 unknowns and 72000 equations.

As a result of implementation of described method a distribution of the magnetization parameters over the volume of the ship was obtained. Some results of calculations are represented on Figure 5 [10]. Input data were specified with error equal to 0,5%.

Conclusion

In this paper we discussed the modern model of identifying of the magnetization parameters of some object and the features of numerical solving of this problem. Using magnetic gradient tensor data to invert the interested parameters is rather new in literature, this is partly due to the fact that the field data is hard to obtain [5]. But this work could initiate corresponding discussion about some method that are able to help in solving of the mentioned problem.

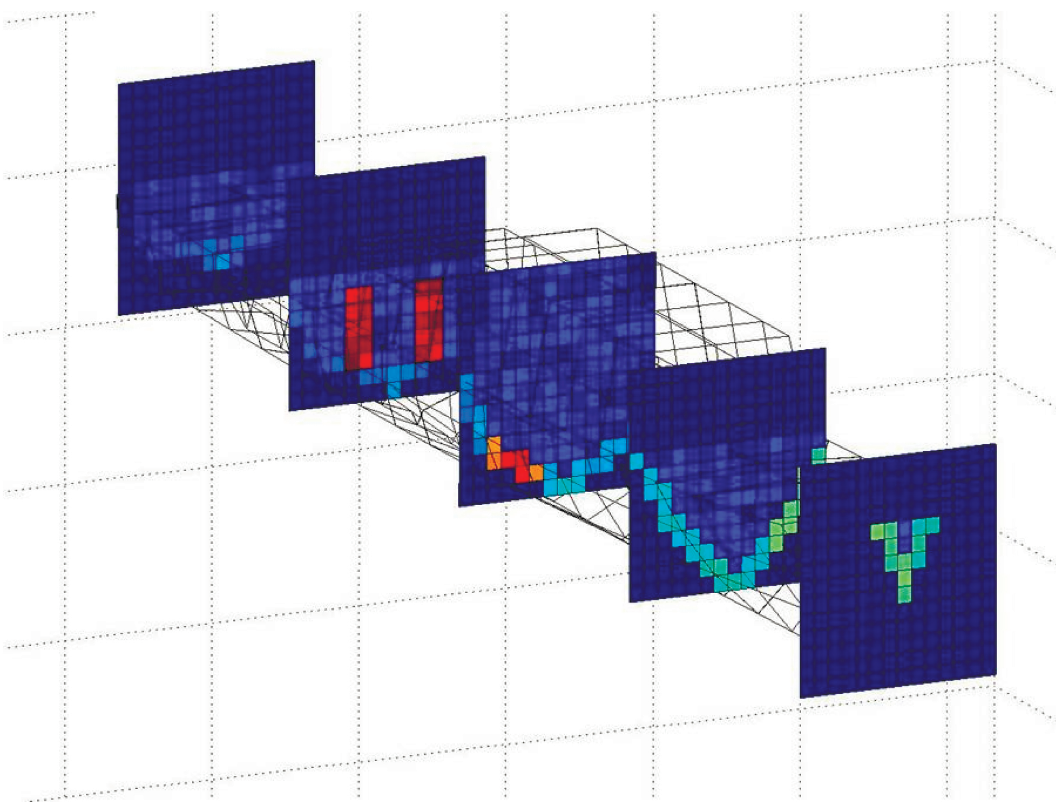

Figure 5: The results of the inversion of the magnetization parameters over the volume of the ship (it represented 5 slices of the module of inverted vector function $\boldsymbol{M}$).

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