An analog of Chang inversion formula
for weighted Radon transforms in multidimensions[1]

F.O.Goncharov, R.G.Novikov

Abstract In this work we study weighted Radon transforms in multidimensions. We introduce an analog of Chang approximate inversion formula for such transforms and describe all weights for which this formula is exact. In addition, we indicate possible tomographical applications of inversion methods for weighted Radon transforms in 3D.

Key words: weighted Radon transforms, inversion formulas

AMS Mathematics Subject Classification: 44A12, 65R32

1   Introduction

We consider the weighted Radon transforms R_W defined by the formula

$$R_W f(s,\theta) \overset{def}{=} \int\limits_{x\theta=s} W(x,\theta)f(x)dx, \tag{1}$$

$$(s,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1},\ x \in \mathbb{R}^n,\ n \geq 2,$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function; see e.g. [1]. Such transforms arise in many domains of pure and applied mathematics; see e.g. [4], [5], [6], [8], [10], [12]. In the present work we assume that

$$W \text{ is complex – valued,}$$
$$W \in C(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1}), \tag{2}$$
$$w_0(x) \overset{def}{=} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} W(x,\theta)d\theta \neq 0,\ x \in \mathbb{R}^n,$$

where $d\theta$ is the element of standard measure on $\mathbb{S}^{n-1}$, $|\mathbb{S}^{n-1}|$ is the standard measure of $\mathbb{S}^{n-1}$.

If $W \equiv 1$, then $R = R_W$ is the classical Radon transform in $\mathbb{R}^n$; see for example [6], [9], [11], [15]. Explicit inversion formulas for R were given for the first time in [15].

In dimension $n = 2$, the transforms R_W are also known as weighted ray transforms on the plane; see e.g. [10], [12]. For several important cases of W satisfying (2) for $n = 2$, explicit (and exact) inversion formulas for R_W were obtained in [2], [7], [13], [14], [16].

On the other hand, it seems that no explicit inversion formulas for R_W were given yet in the literature under assumptions (2) for $n \geq 3$, if $W \neq w_0$.

In the present work we introduce an analog of Chang approximate (but explicit) inversion formula for R_W under assumptions (2), for $n \geq 3$, and describe all W for which this formula is exact. These results are presented in Section 2.

In addition, we indicate possible tomographical applications of inversion methods for R_W in dimension $n = 3$. These considerations are presented in Section 3.

2 Chang-type formulas in multidimensions

We consider the following approximate inversion formulas for R_W under assumptions (2) in dimension $n \geq 2$:

$$f_{appr}(x) \overset{def}{=} \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1}w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \mathbb{H}\left[R_W f\right]^{(n-1)}(x\theta, \theta)d\theta, \tag{3}$$

$x \in \mathbb{R}^n$, n is even,

$$f_{appr}(x) \overset{def}{=} \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \left[R_W f\right]^{(n-1)}(x\theta, \theta)d\theta, \tag{4}$$

$x \in \mathbb{R}^n$, n is odd,

$$\left[R_W f\right]^{(n-1)}(s, \theta) = \frac{d^{n-1}}{ds^{n-1}} R_W f(s, \theta), s \in \mathbb{R}, \ \theta \in \mathbb{S}^{n-1}, \tag{5}$$

$$\mathbb{H}\phi(s) \overset{def}{=} \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{\phi(t)}{s-t} dt, \ s \in \mathbb{R}. \tag{6}$$

For $W \equiv 1$ formulas (3), (4) are exact, i.e. $f_{appr} = f$, and are known as the classical Radon inversion formulas, going back to [15].

As a corollary of the classical Radon inversion formulas and definition (1), formulas (3), (4) for $W \equiv w_0$ are also exact.

Formula (3) for $n = 2$ is known as Chang approximate inversion formula for weighted Radon transforms on the plane. This explicit but approximate inversion formula was suggested for the first time in [3] for the case when

$$W(x, \theta) = \exp\left(-Da(x, \theta^\perp)\right), \tag{7}$$

$$Da(x, \theta^\perp) = \int\limits_{0}^{+\infty} a(x + t\theta^\perp)dt, \tag{8}$$

where a is a non-negative sufficiently regular function on $\mathbb{R}^2$ with compact support, and $\theta = (\theta_1, \theta_2) \in \mathbb{S}^{n-1}$, $\theta^\perp = (\theta_2, -\theta_1)$. We recall that R_W for W given by (7), (8) is known as attenuated Radon transform on the plane and arises, in particular, in the single photon emission tomography (SPECT). In this case an explicit and simultaneously exact inversion formula for R_W was obtained for the first time in [13].

We emphasize that formulas (3), (4) are approximate, in general. In addition, the following result holds:

Theorem 2.1. *Let W satisfy (2). Let f_{appr} be defined by (3), (4) in terms of $R_W f$ and w_0, $n \geq 2$. Then $f_{appr} = f$ (in the sense of distributions) on $\mathbb{R}^n$ for all $f \in C_0(\mathbb{R}^n)$ if and only if*

$$W(x, \theta) - w_0(x) \equiv w_0(x) - W(x, -\theta), \ x \in \mathbb{R}^n, \ \theta \in \mathbb{S}^{n-1}. \tag{9}$$

Here $C_0(\mathbb{R}^n)$ denotes the space of all continous compactly supported functions on $\mathbb{R}^n$.

The result of Theorem 1 for $n = 2$ was obtained for the first time in [14]. Theorem 1 in the general case is proved in Section 4.

If W satisfy (2), $f \in C_0(\mathbb{R}^n)$, but the the symmetry condition (9) does not hold, i.e.

$$w_0(x) \neq \frac{1}{2} \left(W(x, \theta) + W(x, -\theta) \right), \ \text{for some } x \in \mathbb{R}^n, \ \theta \in \mathbb{S}^{n-1},$$

then (3), (4) can be considered as approximate formulas for finding f from $R_W f$.

3 Weighted Radon transforms in 3D in tomographies

In several tomographies the measured data are modeled by weighted ray transforms $P_w f$ defined by the formula

$$P_w f(x, \alpha) = \int_{\mathbb{R}} w(x + \alpha t, \alpha) f(x + \alpha t) \, dt, \ (x, \alpha) \in T\mathbb{S}^2, \tag{10}$$

$$T\mathbb{S}^2 = \{(x, \alpha) \in \mathbb{R}^3 \times \mathbb{S}^2 : x\alpha = 0\},$$

where f is an object function defined on $\mathbb{R}^3$, w is the weight function defined on $\mathbb{R}^3 \times \mathbb{S}^2$, and $T\mathbb{S}^2$ can be considered as the set of all rays (oriented straight lines) in $\mathbb{R}^3$. In particular, in the case of the single-photon emission computed tomography (SPECT) the weight w is given by formulas (7), (8), where $\theta^\perp = \alpha \in \mathbb{S}^2$, $x \in \mathbb{R}^3$.

In practical tomographical considerations $P_w f(x, \alpha)$ usually arises for rays (x, α) parallel to some fixed plane

$$\Sigma_\eta = \{x \in \mathbb{R}^3 : x\eta = 0\}, \ \eta \in \mathbb{S}^2, \tag{11}$$

i.e., for $\alpha\eta = 0$.

The point is that the following formulas hold:

$$R_W f(s, \theta) = \int_{\mathbb{R}} P_w f(s\theta + \tau[\theta, \alpha], \alpha) d\tau, \ s \in \mathbb{R}, \ \theta \in \mathbb{S}^2, \tag{12}$$

$$W(x, \theta) = w(x, \alpha), \ \alpha = \alpha(\eta, \theta) = \frac{[\eta, \theta]}{\|[\eta, \theta]\|}, \ [\eta, \theta] \neq 0, \ x \in \mathbb{R}^3,$$

where $[\cdot, \cdot]$ stands for the standart vector product in $\mathbb{R}^3$.

Due to formula (12) the measured tomographical data modeled by $P_w f$ can be reduced to averaged data modeled by $R_W f$. In particular, this reduction drastically reduces the level of random noise in the initial data.

Therefore, formula (4) for $n = 3$ and other possible methods for finding f from $R_W f$ in 3D may be important for tomographies, where measured data are modeled by $P_w f$ of (10).

Remark 1. *The weight W arising in (12) is not continuous, in general. However, the result of Theorem 2.1 remains valid for this W, at least, under the assumptions that w is bounded and continuous on $\mathbb{R}^3 \times \mathbb{S}^2$, and $w_0(x) \neq 0$, $x \in \mathbb{R}^3$, where w_0 is defined in (2).*

4 Proof of Theorem 1

For W satisfying (2) we also consider its symmetrization defined by

$$W_s(x, \theta) \overset{def}{=} \frac{1}{2}\left(W(x, \theta) + W(x, -\theta)\right), \ x \in \mathbb{R}^n, \ \theta \in \mathbb{S}^{n-1}. \tag{13}$$

Using definitions (1), (13) we obtain

$$R_{W_s} f(s, \theta) = \frac{1}{2}\left(R_W f(s, \theta) + R_W f(-s, -\theta)\right). \tag{14}$$

In addition, if W satisfies (9), then

$$W_s(x, \theta) = w_0(x), \ x \in \mathbb{R}^n, \ \theta \in \mathbb{S}^{n-1}. \tag{15}$$

4.1 Proof of sufficiency

The sufficiency of symmetry (9) follows from formulas (3), (4) for the exact case with $W \equiv w_0$, the identities

$$f_{appr}(x) = \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \mathbb{H}\left[R_{W_s} f\right]^{(n-1)}(x\theta, \theta)d\theta, \tag{16}$$

for even n,

$$f_{appr}(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \left[R_{W_s} f\right]^{(n-1)}(x\theta, \theta)d\theta, \tag{17}$$

for odd n,

and from the identities (14), (15).

In turn, (16) follows from the identities

$$\int\limits_{\mathbb{S}^{n-1}} \mathbb{H}\left[R_W f\right]^{(n-1)}(x\theta, \theta)d\theta$$

$$= \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}} \left(\mathbb{H}\left[R_W f\right]^{(n-1)}(x\theta, \theta) + \mathbb{H}\left[R_W f\right]^{(n-1)}(-x\theta, -\theta)\right) d\theta \tag{18}$$

$$= \int\limits_{\mathbb{S}^{n-1}} \mathbb{H}\left[R_{W_s} f\right]^{(n-1)}(x\theta, \theta)d\theta.$$

In addition, the second of the identities of (18) follows from the identities:

$$\mathbb{H}\left[R_{W_s}f\right]^{(n-1)}(s,\theta) = \frac{1}{2\pi}p.v.\int\limits_{\mathbb{R}}\frac{1}{s-t}\times$$

$$\times\frac{d^{n-1}}{dt^{n-1}}\left[R_Wf(t,\theta)+R_Wf(-t,-\theta)\right]dt$$

$$= \frac{1}{2}\mathbb{H}\left[R_Wf\right]^{(n-1)}(s,\theta) + \frac{(-1)^{n-1}}{2\pi}p.v.\int\limits_{\mathbb{R}}\frac{[R_Wf]^{(n-1)}(-t,-\theta)}{s-t}dt; \qquad (19)$$

$$\frac{(-1)^{n-1}}{\pi}p.v.\int\limits_{\mathbb{R}}\frac{[R_Wf]^{(n-1)}(-t,-\theta)}{s-t}dt = -\frac{(-1)^{n-1}}{\pi}p.v.\int\limits_{\mathbb{R}}\frac{[R_Wf]^{(n-1)}(t,-\theta)}{-s-t}dt$$

$$= (-1)^n\mathbb{H}\left[R_Wf\right]^{(n-1)}(-s,-\theta) = \mathbb{H}\left[R_Wf\right]^{(n-1)}(-s,-\theta). \qquad (20)$$

This concludes the proof of sufficiency for n even.

Finally, (17) follows from the identities

$$\int\limits_{\mathbb{S}^{n-1}}[R_Wf]^{(n-1)}(x\theta,\theta)d\theta$$

$$= \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}}\left([R_Wf]^{(n-1)}(x\theta,\theta)+[R_Wf]^{(n-1)}(-x\theta,-\theta)\right)d\theta, \qquad (21)$$

$$[R_{W_s}f]^{(n-1)}(t,\theta) = \frac{1}{2}\frac{d^{n-1}}{dt^{n-1}}\left[[R_Wf](t,\theta)+[R_Wf](-t,-\theta)\right]$$

$$= \frac{1}{2}\left[[R_Wf]^{(n-1)}(t,\theta)+(-1)^{n-1}[R_Wf]^{(n-1)}(-t,-\theta)\right]$$

$$= \frac{1}{2}\left[[R_Wf]^{(n-1)}(t,\theta)+[R_Wf]^{(n-1)}(-t,-\theta)\right]. \qquad (22)$$

This concludes the proof of sufficiency for odd n.

4.2 Proof of necessity

Using that $f_{appr} = f$ for all $f \in C_0(\mathbb{R}^n)$ and using formulas (3), (4) for the exact case $W \equiv w_0$, we obtain

$$\int\limits_{\mathbb{S}^{n-1}}\left(\mathbb{H}\left[R_Wf\right]^{(n-1)}(x\theta,\theta)-\mathbb{H}\left[R_{w_0}f\right]^{(n-1)}(x\theta,\theta)\right)d\theta = 0 \qquad (23)$$

on $\mathbb{R}^n$ for even n,

$$\int\limits_{\mathbb{S}^{n-1}}[R_Wf - R_{w_0}f]^{(n-1)}(x\theta,\theta)d\theta = 0 \qquad (24)$$

on $\mathbb{R}^n$ for odd n,

for all $f \in C_0(\mathbb{R}^n)$.

Identities (18), (21), (22), (23), (24) imply the identities

$$\int_{\mathbb{S}^{n-1}} \left(\mathbb{H}\left[R_{W_s}f\right]^{(n-1)}(x\theta,\theta) - \mathbb{H}\left[R_{w_0}f\right]^{(n-1)}(x\theta,\theta) \right) d\theta = 0 \tag{25}$$

on $\mathbb{R}^n$ for even n,

$$\int_{\mathbb{S}^{n-1}} \left[R_{W_s}f - R_{w_0}f\right]^{(n-1)}(x\theta,\theta)d\theta = 0 \tag{26}$$

on $\mathbb{R}^n$ for odd n,

for all $f \in C_0(\mathbb{R}^n)$.

The necessity of symmetry (9) follows from the identities (25), (26) and the following lemmas:

Lemma 4.1. *Let (25), (26) be valid for fixed $f \in C_0(\mathbb{R}^n)$ and W satisfying (2), $n \geq 2$. Then*

$$R_{W_s}f = R_{w_0}f. \tag{27}$$

Lemma 4.2. *Let (27) be valid for all $f \in C_0(\mathbb{R}^n)$ and fixed W satisfying (2), $n \geq 2$. Then*

$$W_s = w_0. \tag{28}$$

Lemmas 1 and 2 are proved in Sections 5 and 6.

5 Proof of Lemma 1

We will use the following formulas

$$\int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} g(x\theta,\theta)\, d\theta\, dx$$

$$= \frac{\sqrt{2\pi}}{|\xi|^{n-1}} \left(\hat{g}\left(|\xi|, \frac{\xi}{|\xi|}\right) + \hat{g}\left(-|\xi|, -\frac{\xi}{|\xi|}\right) \right), \tag{29}$$

$$\int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} g^{(n-1)}(x\theta,\theta)\, d\theta\, dx = \int_{\mathbb{R}^n} e^{i\xi x} \int_{\mathbb{S}^{n-1}} (\theta\nabla_x)^{n-1} g(x\theta,\theta)\, d\theta\, dx$$

$$= (-i)^{n-1}\sqrt{2\pi} \left(\hat{g}\left(|\xi|, \frac{\xi}{|\xi|}\right) + (-1)^{n-1}\hat{g}\left(-|\xi|, -\frac{\xi}{|\xi|}\right) \right), \tag{30}$$

$$\hat{g}(\tau,\theta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau s} g(s,\theta)ds, \ \tau \in \mathbb{R}, \ \theta \in \mathbb{S}^{n-1}, \tag{31}$$

where $g \in C(\mathbb{S}^{n-1}, L^2(\mathbb{R}))$, $\xi \in \mathbb{R}^n$. The validity of formulas (29), (30) (in the sense of distributions) follows from Theorem 1.4 of [12].

5.1 The case of odd n

Using identity (14) we get

$$g(s, \theta) = g(-s, -\theta), \text{ for all } s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \tag{32}$$

where

$$g(s, \theta) = [R_{W_s} f(s, \theta) - R_{w_0} f(s, \theta)]. \tag{33}$$

From (32), we obtain the same symmetry for the Fourier transform $\hat{g}(\cdot, \theta)$ of $g(\cdot, \theta)$:

$$
\begin{aligned}
\hat{g}(t, \theta) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s, \theta) e^{its} ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(-s, -\theta) e^{i(-s)(-t)} ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s, -\theta) e^{-its} ds = \hat{g}(-t, -\theta), t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}.
\end{aligned}
\tag{34}
$$

For odd n, from identities (26), (30) it follows that

$$\hat{g}\left(|p|, \frac{p}{|p|}\right) + \hat{g}\left(-|p|, -\frac{p}{|p|}\right) = 0 \text{ in } L_{loc}^2(\mathbb{R}^n). \tag{35}$$

Using (34), (35) we obtain

$$
\begin{cases}
\hat{g}\left(|p|, \dfrac{p}{|p|}\right) = 0, \\
\hat{g}\left(-|p|, -\dfrac{p}{|p|}\right) = 0
\end{cases}
\Leftrightarrow \hat{g} = 0 \Leftrightarrow g = 0.
\tag{36}
$$

Formula (27) for odd n follows from (33), (36).

5.2 The case of even n

We consider

$$g(s, \theta) = \mathbb{H}\left[R_{W_s} f - R_{w_0} f\right](s, \theta), s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}, \tag{37}$$

arising in (25). Using the identity

$$
\begin{aligned}
\mathbb{H}\left[R_{W_s} f - R_{w_0} f\right](-s, -\theta) &= \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s} f(t, -\theta) - R_{w_0} f(t, -\theta)}{-s - t} dt \\
&= \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s} f(-t, -\theta) - R_{w_0} f(-t, -\theta)}{-s + t} dt \\
&= -\frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s} f(t, \theta) - R_{w_0} f(t, \theta)}{s - t} dt = -\mathbb{H}\left[R_{W_s} f - R_{w_0}\right](s, \theta),
\end{aligned}
\tag{38}
$$

we obtain
$$g(s, \theta) = -g(-s, -\theta), \text{ for all } s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}. \tag{39}$$

From (39), similarly with (34), we obtain the same symmetry for the Fourier transform $\hat{g}(\cdot, \theta)$ of $g(\cdot, \theta)$:

$$\hat{g}(t, \theta) = -\hat{g}(-t, -\theta), t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}. \tag{40}$$

For n even, from the property of the Hilbert transform

$$\mathbb{H}\left[\phi^{(k)}\right] = (\mathbb{H}\left[\phi\right])^{(k)}, \phi \in C_0^k(\mathbb{R}),$$

where this identity holds in the sense of distributions if $\phi \in C_0(\mathbb{R})$, and identities (25), (30) it follows that

$$\hat{g}\left(|p|, \frac{p}{|p|}\right) - \hat{g}\left(-|p|, -\frac{p}{|p|}\right) = 0 \text{ in } L_{loc}^2(\mathbb{R}^n). \tag{41}$$

Using (40), (41) we again obtain (36) but already for even n. Due to (36), (37) we have
$$\mathbb{H}\left[R_{W_s}f - R_{w_0}f\right] = 0. \tag{42}$$

Formula (27) for even n follows from (42), invertibility of the Hilbert transform on L^p, $p > 1$ and the fact that $R_W f \in C_0(\mathbb{R} \times \mathbb{S}^{n-1})$.

Lemma 1 is proved.

6   Proof of Lemma 2

Suppose that
$$W_s(y, \theta) - w_0(y) = z \neq 0 \tag{43}$$

for some $y \in \mathbb{R}^n$, $\theta \in \mathbb{S}^{n-1}$, $z \in \mathbb{C}$. Since W satisfies (2), then for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\forall y' : |y' - y| < \delta \rightarrow |W_s(y', \theta) - w_0(y') - z| < \varepsilon, \tag{44}$$

for fixed y, θ.

Let $f \in C_0(\mathbb{R}^n)$, $f \geq 0$ and satisfies the conditions

$$f(y') \equiv 1, y' \in B_{\delta/2}(y), \tag{45}$$
$$supp f \subset B_\delta(y), \tag{46}$$

where $B_\delta(y)$ is the open ball with radius δ, centered at y, $\delta = \delta(\varepsilon)$, $0 < \varepsilon < |z|$. It suffices to show that
$$|R_{W_s}f(y\theta, \theta) - R_{w_0}f(y\theta, \theta)| > 0, \tag{47}$$

which contradicts the condition of the lemma.

The identity (47) follows from the formulas

$$
\begin{aligned}
|R_{W_s}(y\theta,\theta) - R_{w_0}(y\theta,\theta)| &= \left| \int\limits_{x\theta=y\theta} f(x)(W_s(x,\theta) - w_0(x))dx \right| \\
&= \left| \int\limits_{x\theta=y\theta} f(x)(W_s(x,\theta) - w_0(x) - z)dx + z \int\limits_{x\theta=y\theta} f(x)dx \right| \\
&\geq |z| \int\limits_{x\theta=y\theta} f(x)dx - \int\limits_{x\theta=y\theta} f(x) \, |W_s(x,\theta) - w_0(x) - z| \, dx \\
&\geq (|z| - \varepsilon) \int\limits_{x\theta=y\theta} f(x)dx > 0, \ \text{for } 0 < \varepsilon < |z|.
\end{aligned}
\tag{48}
$$

Lemma 2 is proved.

References

[1] J. Boman and E. T. Quinto, *Support theorems for real-analytic Radon transforms*, Duke Mathematical J., 55(4), (1987), 943-948.

[2] J. Boman and J. Strömberg, Novikov's inversion formula for the attenuated Radon transform – a new approach, *The Journal of Geometric Analysis*, 14(2), (2004), 185–198.

[3] L. Chang, A method for attenuation correction in radionuclide computed tomography, *IEEE Transactions on Nuclear Science*, 25(1), (1978), 638–643.

[4] S. R. Deans, The Radon Transform and some of Its Applications, *Courier Corporation*, 2007.

[5] T. Durrani and D. Bisset, The Radon transform and its properties, *Geophysics*, 49(8), (1984), 1180–1187.

[6] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, Integral Geometry and Representation Theory, *Academic press,* Vol. 5, 2014.

[7] S. Gindikin, A remark on the weighted Radon transform on the plane, *Inverse Problems and Imaging*, 4, (2010), 649–653.

[8] P. Grangeat, Mathematical framework of cone beam 3D reconstruction via the first derivative of the Radon transform, *Mathematical methods in tomography, Springer*, (1991), 66-97.

[9] S. Helgason, The Radon Transform on R^n, *Springer*, 2011.

[10] L. A. Kunyansky, Generalized and attenuated Radon transforms: restorative approach to the numerical inversion, *Inverse Problems*, 8(5), (1992), 809-819.

[11] D. Ludwig, The Radon transform on Euclidean space, *Communications on Pure and Applied Mathematics*, 19(1), (1996), 49–81.

[12] F. Natterer, The Mathematics of Computerized Tomography, Vol. 32, *SIAM*, 1986.

[13] R. G. Novikov, An inversion formula for the attenuated X-ray transformation, *Arkiv för matematik*, 40(1), (2002), 145–167.

[14] R. G. Novikov, Weighted Radon transforms for which Chang's approximate inversion formula is exact, *Russian Mathematical Surveys*, 66(2), (2011), 442–443.

[15] J. Radon, Uber die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten [On the determination of functions by their integral values along certain manifolds], 75 years of Radon transform (Vienna, 1992), conf. *Proc. Lecture Notes Math. Phys., IV, Int. Press, Cambridge, MA*, (1994), 324–339.

[16] O. Tretiak and C. Metz, The exponential Radon transform, *SIAM Journal on Applied Mathematics*, 39(2), (1980), 341–354.

F. O. Goncharov,
Moscow Institute of Physics and Technology,
Dolgoprudny, Moscow Region, 141700, Russian Federation;
Institute for Information Transmission Problems (RAS),
Bolshoy Karetny per. 19, buld.1, Moscow 127051, Russian Federation;
Email: `fedor.goncharov.ol@gmail.com`

R. G. Novikov,
CNRS (UMR 7641), Centre de Mathématiques Appliqueés, Ecole Polytechnique,
91128 Palaiseau, France;
Institute of Earthquake Prediction Theory and Mathematical Geophysics (RAS),
117997 Moscow, Russia;
Email: `novikov@cmap.polytechnique.fr`