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A PROBLEM OF RECOVERING A SPECIAL TWO-DIMENSIONAL POTENTIAL IN A HYPERBOLIC EQUATION

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Abstract We consider an inverse problem for partial differential equations of the second order related to recovering a coefficient (potential) in the lower term of this equations. It is supposed that the unknown potential is a trigonometric polynomial with respect to one of space variables with continuous coefficients of the other variable. The direct problem for the hyperbolic equation is the initial-boundary value problem for half-space $x > 0$ with zero initial Cauchy data and a special Neumann data at $x = 0$. We prove a local existence theorem for the inverse problem. The used method gives stability estimates for the solution to the direct and inverse problems and proposes a method of solving them.

Key words: inverse problem, hyperbolic equation, uniqueness, existence.

AMS Mathematics Subject Classification: 35R30

1 Introduction

Inverse problems for hyperbolic equations intensively studied beginning with the second part of the last century (see [1]-[4], [7]-[13], [17], [21]-[24]). Some of this investigations were represented later in the books [5], [6], [14]- [16], [18]-[20], [25]-[30]. Uniqueness and stability of solutions are usually main questions in a study of inverse problems. For some one-dimension inverse problems existence theorems can be also stated. But for multidimensional inverse problems such theorems are almost absent. An exception here is a class of analytical functions. If unknown coefficients and data of an inverse problem are analytical functions by some of variables, then sometimes the local existence theorems can be proved (see, for example, [24], [27]). Numerical methods for solving inverse problems based on the minimization of residual functionals and regularization procedures were developed (see the books [15], [16], [29] and references therein). Very often these methods use a finite-dimensional approximation for unknown coefficients and a finite-dimensional approximation for solutions of direct problems.

Below we consider an inverse problem of recovering a coefficient in the lower term of a hyperbolic equation. We suppose that the unknown coefficient is a polynomial of a fixed order with respect to the independent variable y with continuous coefficients dependent on x. In the next section we formulate the inverse problem and study properties of the solution of a direct problem. The latter problem contains infinitely many components of Fourier series for the solution. We demonstrate that for the corresponding infinitely system of equations the method of successive approximations is converged and allows estimate the solution. In section 3 we prove a local existence theorem for the posed inverse problem. The presented results give a convenient approach for numerical solving the inverse problem.

2 Posing the problem and some lemmas

For the function $u(x,y,t)$ consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - q(x,y)u = 0,\ (x,y,t) \in \mathbb{R}^3_+; \quad u|_{t<0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \delta(t), \tag{1}$$

where $\mathbb{R}^3_+ = \{(x,y,t) \in \mathbb{R}^3 |\, x > 0\}$. Assume that the potential $q(x,y)$ can be represented in the form of a finite Fourier series

$$q(x,y) = \sum_{s=-N}^{N} q_s(x) e^{isy} \tag{2}$$

with a fixed integer $N \geq 0$. Denote by $\mathcal{Q}(N,L,Q)$ the set of functions $q(x,y)$ for which the coefficients $q_s(x)$, $|s| \leq N$, are continuous functions on the interval $[0,L]$ and satisfy the conditions

$$|q_s(x)| \leq Q, \quad x \in [0,L], \quad -N \leq s \leq N. \tag{3}$$

For $q(x,y) \in \mathcal{Q}(N,L,Q)$ the solution of the problem (1) is a 2π periodic function of y and can be represented as a Fourier series

$$u(x,y,t) = \sum_{m=-\infty}^{\infty} u_m(x,t) e^{imy}, \tag{4}$$

where the coefficients $u_m(x,t)$ satisfy the following relations

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right) u_m(x,t) - \sum_{s=-N}^{N} q_s(x) u_{m-s}(x,t) = 0,\ (x,t) \in \mathbb{R}^2_+;$$
$$u_m|_{t<0} = 0, \quad \left.\frac{\partial u_m}{\partial x}\right|_{x=0} = \delta(t)\delta_{0m} \quad m = 0, \pm 1, \pm 2, \ldots. \tag{5}$$

In the latter equations $\mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 |\, x > 0\}$ and δ_{0m} is the Kronecker delta.

We shall consider

The inverse problem. Find the coefficients $q_s(x)$, $s = 0, \pm 1, \pm, \ldots, \pm N$, from the given

$$u_m(0,t) = f_m(t),\ t \in [0,T], \quad m = 0, \pm 1, \pm 2, \ldots, \pm N, \tag{6}$$

where T is a fixed positive number.

We begin studying this problem with consideration of some properties of the solution to the direct problem (5).

Lemma 2.1. *Let T be an arbitrary positive number, $q(x,y) \in \mathcal{Q}(N,T/2,Q)$ and $D(T) = \{(x,t) |\, 0 \leq x \leq T - t\}$. Then the solution to the problem (5) exists and can be represented in $D(T)$ in the form*

$$u_m(x,t) = \overline{u}_m(x,t) H(t-x), \quad m = 0, \pm 1, \pm 2, \ldots. \tag{7}$$

where $H(t)$ is the Heaviside step-function and $\overline{u}_m(x,t)$ are continuous functions together with its derivatives up to the second order in the domain $D'(T)=\{(x,t)|\,0\le x\le t\le T-x\}$. Moreover, this solution is unique and there exist positive constants $C_1=C_1(N,T,Q)$ and $C_2=C_2(N,T,Q)$ such that the following estimates hold

$$\begin{aligned}
|u_0(x,t)-1| &\le C_1\frac{QTt}{2\cdot 1!},\\
|u_m(x,t)| &\le C_1\frac{Q^{n+1}T^{n+1}(2N+1)^n t^{n+1}}{2^{n+1}\cdot(n+1)!},\\
&\qquad (x,t)\in D'(T),\quad Nn<|m|\le(n+1)N,\quad n=0,1,2,\dots,
\end{aligned}\tag{8}$$

$$\begin{aligned}
\left|\frac{\partial u_m(x,t)}{\partial t}\right| &\le \frac{1}{4}C_2(4+m^2T^2)Q(2N+1)t,\quad -N\le m\le N,\\
\left|\frac{\partial u_m(x,t)}{\partial t}\right| &\le C_2(4+m^2T^2)\frac{Q^{n+1}T^n(2N+1)^n t^{n+1}}{2^{n+2}\cdot(n+1)!},\\
&\qquad nN<|m|\le(n+1)N,\quad n=1,2,\dots,,\quad (x,t)\in\overline{D'}(T).
\end{aligned}\tag{9}$$

Proof. The representation (7) follows from the well known fact that the solution to the problem (1) vanishes for all (x,y,t) satisfying the condition $x>t>0$ because the initial data are zero and the boundary source is located on the axis $x=0,t=0$. Hence, all $u_m(x,t)\equiv 0$ for $x>t>0$. Therefore, $u_m(x,t)=\overline{u}_m(x,t)$ in $D'(T)$. For the sake of convenience, we continue all functions $u_m(x,t)$, $q_s(x)$ for $x<0$ as even functions: $u_m(-x,t)=u_m(x,t)$, $q_s(-x)=q_s(x)$ and define

$$\overline{D}(T)=\{(x,t)|\,t\le T-|x|\},\quad \overline{D'}(T)=\{(x,t)|\,0\le|x|\le t\le T-|x|\}.$$

Then the problem (5) is equivalent to the following integral equations

$$u_m(x,t)=\delta_{0m}+\frac{1}{2}\int\limits_{\Diamond(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\sum_{s=-N}^{N}q_s(\xi)u_{m-s}(\xi,\tau)d\xi d\tau,\tag{10}$$
$$(x,t)\in\overline{D'}(T),\quad m=0,\pm1,\pm2,\dots.$$

Here $J_0(\zeta)$ is the Bessel function and

$$\Diamond(x,t)=\{(\xi,\tau)|\,|\xi|\le\tau\le t-|\xi-x|\}.$$

Recall that the Bessel function $J_\nu(\zeta)$ for a fixed integer $\nu\ge0$ is defined by the formula

$$J_\nu(\zeta)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{\zeta}{2}\right)^{2k+\nu}.$$

From this formula follows the estimate

$$\left|\frac{J_\nu(\zeta)}{\zeta^\nu}\right|\le\frac{1}{2^\nu\nu!},\quad |\zeta|\le2.\tag{11}$$

Moreover, for the Bessel function the following representation holds:

$$J_\nu(\zeta) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{-i\zeta\cos\varphi + i\nu\varphi} d\varphi,$$

From here follows that

$$|J_\nu(\zeta)| \leq 1 \text{ for all } \zeta \in \mathbb{R}. \tag{12}$$

Consider for equations (10) the method of successive approximations. Define

$$u_m(x,t) = \sum_{k=0}^{\infty} u_m^k(x,t), \tag{13}$$

where

$$\begin{aligned} &u_m^0(x,t) = \delta_{0m}, \\ &u_m^k(x,t) = \frac{1}{2} \int\limits_{\Diamond(x,t)} J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) \sum_{s=-N}^{N} q_s(\xi) u_{m-s}^{k-1}(\xi,\tau) d\xi d\tau, \\ &(x,t) \in \overline{D'}(T), \quad k = 1,2,\ldots, \quad m = 0, \pm1, \pm2, \ldots. \end{aligned} \tag{14}$$

It is obvious that all functions $u_m^k(x,t)$ are continuous in $\overline{D'}(T)$. Moreover, the following estimates hold

$$\begin{aligned} |u_m^1(x,t)| &\leq \frac{QT}{2} \int\limits_0^t \sum_{s=-N}^{N} \max_{|\xi|\leq T/2} |u_{m-s}^0(\xi,\tau)| d\tau \\ &\leq \frac{QTt}{2\cdot 1!} \begin{cases} 1, & |m| \leq N, \\ 0, & |m| > N, \end{cases} \\ |u_m^2(x,t)| &\leq \frac{(QT)^2}{2^2} \int\limits_0^t \sum_{s=-N}^{N} \max_{|\xi|\leq T/2} |u_{m-s}^1(\xi,\tau)| d\tau \\ &\leq \frac{(QT)^2(2N+1)t^2}{2^2\cdot 2!} \begin{cases} 1, & |m| \leq 2N, \\ 0, & |m| > 2N, \end{cases} \\ &\quad (x,t) \in \overline{D'}(T). \end{aligned} \tag{15}$$

Continuing these estimates, we easily prove that

$$\begin{aligned} |u_m^k(x,t)| &\leq \frac{(QT)_m^k(2N+1)^{k-1}t^k}{2^k\cdot k!} \begin{cases} 1, & |m| \leq kN, \\ 0, & |m| > kN, \end{cases} \\ &\quad (x,t) \in \overline{D'}(T), \quad k = 1,2,\ldots. \end{aligned} \tag{16}$$

Since $t \leq T$ in $\overline{D'}(T)$ the series (13) is uniformly converged in $\overline{D'}(T)$ for all m. Hence, its sum is a continuous function in $\overline{D'}(T)$. Moreover, the following estimates hold

$$
\begin{aligned}
|u_0(x,t)-1| &\le \sum_{k=1}^{\infty} |u_m^k(x,t)| \le \sum_{k=1}^{\infty} \frac{(Q\,T)^k(2N+1)^{k-1}t^k}{2^k\cdot k!} \\
&\le \frac{QTt}{2\cdot 1!}C_1, \\
|u_m(x,t)| &\le \sum_{k=n+1}^{\infty} |u_m^k(x,t)| \le \sum_{k=n+1}^{\infty} \frac{(Q\,T)^k(2N+1)^{k-1}t^k}{2^k\cdot k!} \\
&\le \frac{(Q\,T)^{n+1}(2N+1)^n t^{n+1}}{2^{n+1}\cdot(n+1)!}C_1, \\
&\quad (x,t)\in\overline{D'}(T), \quad Nn<|m|\le(n+1)N, \quad n=0,1,2,\dots,
\end{aligned}
\tag{17}
$$

where $C_1 = exp\,(QT^2(2N+1)/2)$. Now differentiating equations (10) with respect to x and t, we easily check that functions $u_m(x,t)$ are twice differentiable in $\overline{D'}(T)$. We check it for the derivatives with respect to t only. The expressions for these derivatives will be useful in the analysis of the inverse problem. Using (10), we find

$$
\begin{aligned}
\frac{\partial u_m(x,t)}{\partial t} &= \frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2} \sum_{s=-N}^{N} q_s(\xi)u_{m-s}(\xi,t-|x-\xi|)d\xi \\
&\quad +\frac{1}{2}\int\limits_{\Diamond(x,t)} K_m(t-\tau,x-\xi)\sum_{s=-N}^{N} q_s(\xi)u_{m-s}(\xi,\tau)d\xi d\tau, \\
&\quad (x,t)\in\overline{D'}(T), \quad m=0,\pm1,\pm2,\dots.
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
K_m(t-\tau,x-\xi) &= \frac{\partial}{\partial t}J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right) \\
&= -m^2(t-\tau)\,\frac{J_1(\zeta)}{\zeta}\bigg|_{\zeta=m\sqrt{(t-\tau)^2-(x-\xi)^2}}.
\end{aligned}
$$

Here $J_1(\zeta)$ is the Bessel function. Note that $J_1(\zeta)/\zeta$ is a continuous function for all $\zeta\in[0,\infty)$ and $J_1(\zeta)/\zeta\to 1/2$ as $\zeta\to 0$. From the relation (18) we see that the derivatives $\partial u_m(x,t)/\partial t$ are, indeed, continuous in $\overline{D'}(T)$ for all m.

To obtain estimates (9), we denote $\partial u_m(x,t)/\partial t = v_m(x,t)$. Since from (11) and (12) follows that $|J_1(\zeta)/\zeta|\le 1/2$ for all $\zeta\in\mathbb{R}$, we have $|K_m(t-\tau,x-\xi)|\le m^2T/2$. Therefore we can estimate $v_m(x,t)$ as follows

$$
\begin{aligned}
|v_m(x,t)| &\le (4+m^2T^2)\frac{Q}{4}\int\limits_0^t \sum_{s=-N}^{N} \max_{\xi\in\Sigma(x,t,\tau)} |u_{m-s}(\xi,\tau)|d\tau, \\
&\quad (x,t)\in\overline{D'}(T), \quad m=0,\pm1,\pm2,\dots,
\end{aligned}
\tag{19}
$$

where $\Sigma(x,t,\tau)=\{\xi|\,(\xi,\tau)\in\Diamond(x,t)\}$. Using (8) and the obvious inequality $C_1\geq 1$, we get that the following estimates hold

$$
\begin{aligned}
|v_0(x,t)| &\leq tQ(2N+1)\left(1+C_1\frac{QTt}{2\cdot 2!}\right)\leq tQ(2N+1)C_1\left(1+\frac{QTt}{4}\right),\\
|v_m(x,t)| &\leq (4+m^2T^2)\frac{Q(2N+1)}{4}C_1\\
&\times\max\left(\frac{t}{1!}+\frac{QTt^2}{2\cdot 2!},\frac{Q^2T^2(2N+1)t^3}{2^2\cdot 3!}\right),\quad 0<|m|\leq N, \qquad (20)\\
|v_m(x,t)| &\leq C_1(4+m^2T^2)\frac{Q(2N+1)}{4}\max\left(\frac{Q^nT^n(2N+1)^{n-1}t^{n+1}}{2^n\cdot(n+1)!},\right.\\
&\left.\frac{Q^{n+1}T^{(n+1)}(2N+1)^nt^{(n+2)}}{2^{n+1}\cdot(n+2)!},\frac{Q^{n+2}T^{(n+2)}(2N+1)^{n+1}t^{(n+3)}}{2^{n+2}\cdot(n+3)!}\right),\\
&nN<|m|\leq(n+1)N,\quad n\geq 1,\quad (x,t)\in\overline{D'}(T).
\end{aligned}
$$

Denoting

$$
C_2=C_1\max\left(1+\frac{QT^2(2N+1)}{3},\frac{Q^2T^4(2N+1)^2}{2^2\cdot 3!}\right),
$$

we come to the estimates

$$
\begin{aligned}
|v_m(x,t)| &\leq \frac{1}{4}C_2(4+m^2T^2)Q(2N+1)t,\quad -N\leq m\leq N,\\
|v_m(x,t)| &\leq C_2(4+m^2T^2)\frac{Q^{n+1}T^n(2N+1)^nt^{n+1}}{2^{n+2}\cdot(n+1)!},\\
&nN<|m|\leq(n+1)N,\quad n=1,2,\ldots,\quad (x,t)\in\overline{D'}(T). \qquad (21)
\end{aligned}
$$

which coincide with (9).

It follows from equation (10), that $u_m(x,|x|+0)=\delta_{0m}$. Using this fact and differentiating the equalities (18) with respect to t, we obtain

$$
\begin{aligned}
\frac{\partial^2u_m(x,t)}{\partial t^2} &= \frac{1}{4}\left[q_m\left(\frac{x+t}{2}\right)+q_m\left(\frac{x-t}{2}\right)\right]\\
&+\frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}\sum_{s=-N}^{N}q_s(\xi)v_{m-s}(\xi,t-|x-\xi|)d\xi\\
&-\frac{m^2}{4}\int\limits_{(x-t)/2}^{(x+t)/2}|x-\xi|\sum_{s=-N}^{N}q_s(\xi)u_{m-s}(\xi,t-|x-\xi|)d\xi\\
&+\frac{1}{2}\int\limits_{\Diamond(x,t)}K'_m(t-\tau,x-\xi)\sum_{s=-N}^{N}q_s(\xi)u_{m-s}(\xi,\tau)d\xi d\tau, \qquad (22)\\
&(x,t)\in\overline{D'}(T),\quad m=0,\pm1,\pm2,\ldots,
\end{aligned}
$$

where $v_m(x.t) = \partial u_m(x,t)/\partial t$ and

$$\begin{aligned} K'_m(t-\tau, x-\xi) &= \frac{\partial}{\partial t} K_m(t-\tau, x-\xi) \\ &= -m^2 \left[\frac{J_1(\zeta)}{\zeta} - m^2(t-\tau)^2 \frac{J_2(\zeta)}{\zeta^2} \right] \Bigg|_{\zeta = m\sqrt{(t-\tau)^2-(x-\xi)^2}}. \end{aligned}$$

We have used here that
$$\frac{\partial}{\partial \zeta}\left(\frac{J_1(\zeta)}{\zeta}\right) = -\frac{J_2(\zeta)}{\zeta},$$
where $J_2(\zeta)$ is the Bessel function. Since the function $J_1(\zeta)/\zeta$ and $J_2(\zeta)/\zeta^2$ are continuous for all $\zeta \in [0,\infty)$, the function $K'_m(t,x)$ is continuous for $(x,t) \in \overline{D'}(T)$. Hence, the second derivatives $\partial^2 u_m(x,t)/\partial t^2$ are also continuous functions in $\overline{D'}(T)$.

Similarly one can check that the derivatives $\partial u_m(x,t)/\partial x$ and $\partial^2 u_m(x,t)/\partial x \partial t$ are continuous functions in $\overline{D'}(T)$.

Concluding the Lemma, we note that the uniqueness of the constructed solution follows from the uniqueness theorem to the problem (1). $\square$

Corollary. If $q(x,y) \in \mathcal{Q}(N, T/2, Q)$ the data (6) of the inverse problem must satisfy the following requirements

$$\begin{gathered} f_m(t) \in C^2[0,T], \quad f'_m(0) = 0, \quad m = 0, \pm 1, \pm 2, \ldots, \pm N, \\ f_0(0) = 1, \quad f_m(0) = 0, \quad m = \pm 1, \pm 2, \ldots, \pm N. \end{gathered} \tag{23}$$

For the problem (5) the following lemma holds.

Lemma 2.2. *Let $q(x,y)$ and $\hat{q}(x,y)$ be two arbitrary functions of the set $\mathcal{Q}(N, T/2, Q)$ and $u_m(x,t)$ and $\hat{u}_m(x,t)$, $m = 0, \pm 1, \pm 2, \ldots$, be the solutions to the problem (5) which correspond to $q(x,y)$ and $\hat{q}(x,y)$, respectively. Then there exist constants $C_3 = C_3(N,T,Q)$ and $C_4 = C_4(N,T,Q)$ such that*

$$\begin{gathered} |u_0(x,t) - \hat{u}_0(x,t)| \le C_3 \tilde{Q} t, \\ |u_m(x,t) - \hat{u}_m(x,t)| \le C_3 \tilde{Q} \frac{Q^n T^n (2N+1)^n t^{n+1}}{2^n \cdot (n+1)!}, \\ nN < |m| \le (n+1)N, \quad n = 0,1,2,\ldots, \quad (x,t) \in \overline{D'}(T). \end{gathered} \tag{24}$$

$$\begin{aligned} \left| \frac{\partial u_0(x,t)}{\partial t} - \frac{\partial \hat{u}_0(x,t))}{\partial t} \right| &\le C_4 \tilde{Q} (2N+1) \frac{t^2}{2!} \\ \left| \frac{\partial u_m(x,t)}{\partial t} - \frac{\partial \hat{u}_m(x,t))}{\partial t} \right| &\le C_4 \tilde{Q} \frac{(4+m^2T^2) Q^n T^n (2N+1)^n}{2^{n+2}} \frac{t^{n+2}}{(n+2)!} \\ nN < |m| \le (n+1)N, \qquad & n = 0,1,2,\ldots,\ (x,t) \in \overline{D'}(T). \end{aligned} \tag{25}$$

where

$$\tilde{Q} = \max_{-N \le s \le N} \max_{0 \le x \le T/2} |q_s(x) - \hat{q}_s(x)|, \tag{26}$$

Proof. Denote

$$\tilde{u}_m(x,t) = u_m(x,t) - \hat{u}_m(x,t), \quad \tilde{q}_s(x) = q_s(x) - \hat{q}_s(x). \tag{27}$$

Using the equations (10) with (u_m, q_s) and $(\hat{u}_m, \hat{q}_s)$ and extracting one from other, we find

$$\begin{aligned} \tilde{u}_m(x,t) = \frac{1}{2} \sum_{s=-N}^{N} \int\limits_{\Diamond(x,t)} & J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) \\ & \times [q_s(\xi)\tilde{u}_{m-s}(\xi,\tau) + \tilde{q}_s(\xi)\hat{u}_{m-s}(\xi,\tau)] d\xi d\tau, \\ & (x,t) \in \overline{D'}(T), \quad m = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{28}$$

Represent $\tilde{u}_m(x,t)$ in the form

$$\tilde{u}_m(x,t) = \sum_{k=0}^{\infty} \tilde{u}_m^k(x,t), \tag{29}$$

where

$$\tilde{u}_m^0(x,t) = \frac{1}{2} \sum_{s=-N}^{N} \int\limits_{\Diamond(x,t)} J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) \tilde{q}_s(\xi)\hat{u}_{m-s}(\xi,\tau) d\xi d\tau, \tag{30}$$

$$(x,t) \in \overline{D'}(T), \quad m = 0, \pm 1, \pm 2, \ldots,$$

$$\tilde{u}_m^k(x,t) = \frac{1}{2} \sum_{s=-N}^{N} \int\limits_{\Diamond(x,t)} J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) q_s(\xi)\tilde{u}_{m-s}^{k-1}(\xi,\tau) d\xi d\tau, \tag{31}$$

$$(x,t) \in \overline{D'}(T), \quad k = 1, 2, \ldots, \quad m = 0, \pm 1, \pm 2, \ldots.$$

From here we obtain

$$\begin{aligned} |\tilde{u}_m^0(x,t)| &\le \frac{T\tilde{Q}}{2} \sum_{s=-N}^{N} \int\limits_0^t \max_{|\xi| \le T/2} |\hat{u}_{m-s}(\xi,\tau)| d\tau, \\ |\tilde{u}_m^k(x,t)| &\le \frac{TQ}{2} \sum_{s=-N}^{N} \int\limits_0^t \max_{|\xi| \le T/2} |\tilde{u}_{m-s}^{k-1}(\xi,\tau)| d\tau, \\ & (x,t) \in \overline{D'}(T), \quad m = 0, \pm 1, \pm 2, \ldots, k = 1, 2, \ldots. \end{aligned} \tag{32}$$

Use the estimates (8) for functions $\hat{u}_m(x,t)$ here. Then we get

$$|\tilde{u}_0^0(x,t)| \le \frac{T\tilde{Q}}{2} C_1 \left(\frac{t}{1!} + \frac{QTt^2}{2 \cdot 2!}\right)(2N+1),$$

$$|\tilde{u}_m^0(x,t)| \le \frac{T\tilde{Q}}{2} C_1 \max\left(\frac{t}{1!} + \frac{QTt^2}{2 \cdot 2!}, \frac{Q^2T^2(2N+1)t^3}{2^2 \cdot 3!}\right)(2N+1)t,$$

$$0 < |m| \le N,$$

$$|\tilde{u}_m^0(x,t)| \le \frac{T\tilde{Q}}{2} C_1 \frac{Q^nT^n(2N+1)^n t^{n+1}}{2^n \cdot (n+1)!} \max\left(1, \frac{Q^2T^2(2N+1)^2t^2}{2^2 \cdot (n+2)(n+3)}\right),$$

$$nN < |m| \le (n+1)N, \quad n = 1, 2, \ldots, \quad (x,t) \in \overline{D'}(T).$$

These estimates we can write it in the more compact form as follows

$$
\begin{aligned}
&|\tilde{u}_m^0(x,t)| \le \tilde{Q}\frac{T(2N+1)t}{2}C_2, \quad -N < m \le N,\\
&|\tilde{u}_m^0(x,t)| \le \tilde{Q}\frac{Q^n T^{n+1}(2N+1)^n t^{n+1}}{2^{n+1}\cdot(n+1)!}C_2,\\
&\qquad nN < |m| \le (n+1)N, \quad n = 1,2,\dots, \quad (x,t) \in \overline{D'}(T).
\end{aligned} \tag{33}
$$

For $k \ge 1$ we find that

$$
\begin{aligned}
&|\tilde{u}_0^k(x,t)| \le C_2\tilde{Q}\frac{Q^k(T(2N+1)t)^{k+1}}{2^{k+1}\cdot(k+1)!},\\
&|\tilde{u}_m^k(x,t)| \le C_2\tilde{Q}\frac{Q^{n+k}(T(2N+1)t)^{n+k+1}}{2^{n+k+1}\cdot(n+k+1)!}\max\left(1, \frac{QTt}{2(n+k+2)}\right)\\
&\qquad (x,t) \in \overline{D'}(T), \quad nN < |m| \le (n+1)N, \quad n = 0,1,2,\dots.
\end{aligned} \tag{34}
$$

Hence,

$$
\begin{aligned}
&|\tilde{u}_0(x,t)| \le C_2\tilde{Q}\frac{T(2N+1)t}{2\cdot 1!}\exp\left(QT(2N+1)t\right),\\
&|\tilde{u}_m(x,t)| \le C_2\tilde{Q}\frac{Q^n(T(2N+1)t)^{n+1}}{2^{n+k+1}\cdot(n+1)!}\max\left(1, \frac{QT^2}{4}\right)\exp\left(QT(2N+1)t\right),\\
&\qquad nN < |m| \le (n+1)N, \quad n = 0,1,2,\dots, \quad (x,t) \in \overline{D'}(T).
\end{aligned} \tag{35}
$$

Thus, the estimate (24) holds with

$$
R_0 = C_2\frac{T(2N+1)}{2}\max\left(1, \frac{QT^2}{4}\right)\exp\left(QT^2(2N+1)\right).
$$

For proving (25), we use the relation (18) for (u_m, q_s) and $(\hat{u}_m, \hat{q}_s)$. Subtracting one from other, we find

$$
\begin{aligned}
\frac{\partial \tilde{u}_m(x,t)}{\partial t} &= \frac{1}{2}\int\limits_{(x-t)/2}^{(x+t)/2}\sum_{s=-N}^{N}\left[q_s(\xi)\tilde{u}_{m-s}(\xi, t-|x-\xi|) + \tilde{q}(\xi)\hat{u}_{m-s}(\xi, t-|x-\xi|)\right]d\xi\\
&+\frac{1}{2}\int\limits_{\Diamond(x,t)} K_m(t-\tau, x-\xi)\sum_{s=-N}^{N}\left[q_s(\xi)\tilde{u}_{m-s}(\xi,\tau) + \tilde{q}(\xi)\hat{u}_{m-s}(\xi,\tau)\right]d\xi d\tau,\\
&\qquad m = 0, \pm1, \pm2, \dots, \quad (x,t) \in \overline{D'}(T).
\end{aligned} \tag{36}
$$

Then

$$
\begin{aligned}
&\left|\frac{\partial \tilde{u}_m(x,t)}{\partial t}\right| \le \frac{4+m^2T^2}{4}\sum_{s=-N}^{N}\int\limits_0^t \max_{\xi\in\Sigma(x,t,\tau)}\left(Q|\tilde{u}_{m-s}(\xi,\tau)| + \tilde{Q}|\hat{u}_{m-s}(\xi,\tau)|\right)d\tau,\\
&\qquad m = 0, \pm1, \pm2, \dots, \quad (x,t) \in \overline{D'}(T).
\end{aligned} \tag{37}
$$

Using estimates (8) and (24), we find

$$\begin{aligned}
\left|\frac{\partial \tilde{u}_0(x,t)}{\partial t}\right| &\leq \tilde{Q}(2N+1)\left(C_3 + C_1\left(1+\frac{QT}{4}\right)\right)\frac{t^2}{2!} \\
\left|\frac{\partial \tilde{u}_m(x,t)}{\partial t}\right| &\leq \tilde{Q}\frac{(4+m^2T^2)Q^nT^n(2N+1)^n t^{n+1}}{2^{n+2}}\frac{t^{n+2}}{(n+2)!} \\
&\times\left(C_3 + C_1\frac{QT}{2}\max\left(1+\frac{QT}{4}, \frac{Q^2T^4(2N+!)^2}{4}\right)\right), \qquad (38) \\
&nN < |m| \leq (n+1)N,\ n = 0,1,2,\ldots,\ (x,t) \in \overline{D'}(T).
\end{aligned}$$

Hence. estimates (25) hold with

$$C_4 = C_3 + C_1\frac{QT}{2}\max\left(1+\frac{QT}{4}, \frac{Q^2T^4(2N+!)^2}{4}\right).$$

The lemma is proven. □

3 The existence and uniqueness theorem

Set $x = 0$ in the equation (22) and use the condition (6). Then we obtain

$$\begin{aligned}
f''_m(t) &= \frac{1}{2}q_m\left(\frac{t}{2}\right) + \int\limits_0^{t/2}\sum_{s=-N}^{N} q_s(\xi)v_{m-s}(\xi, t-|\xi|)d\xi \\
&-\frac{m^2}{2}\int\limits_0^{t/2}|\xi|\sum_{s=-N}^{N} q_s(\xi)u_{m-s}(\xi, t-|\xi|)d\xi \\
&+\int\limits_{\triangleright(t)} K'_m(t-\tau,\xi)\sum_{s=-N}^{N} q_s(\xi)u_{m-s}(\xi,\tau)d\xi d\tau, \qquad (39) \\
&\qquad t\in[0,T], \quad m = 0, \pm 1, \pm 2, \pm N,
\end{aligned}$$

where

$$f''_m(t) = \frac{d^2 f_m(t)}{dt^2}, \quad \triangleright(t) = \{(\xi,\tau)|0\leq\xi\leq t/2, \xi\leq\tau\leq t-\xi\}.$$

The equations (10), (19), (39) form the system of integral relations for finding unknown functions $u_m(x,t)$, $v_m(x,t)$ and $q_s(x)$ in the domain $\overline{D}'(T)$. The equations (10) determine u_m as the operator functions of q_s, $s = -N,\ldots,N$, i.e., $u_m = u_m(x,t;q_{-N},\ldots,q_N)$. Similarly equations (19) determine $v_m = v_m(x,t;q_{-N},\ldots,q_N)$. Then the equations (39) we can consider as the operator equations

$$q_m(x) = A_m(x;q_{-N},\ldots,q_N), \quad m = 0,\pm 1,\ldots,\pm N, \quad x\in[0,T/2], \qquad (40)$$

where operators $A_m(x; q_{-N}, \dots, q_N)$ are defined by the formulae

$$A_m(x; q_{-N}, \dots, q_N) = q_m^0(x) - 2\int_0^x \sum_{s=-N}^{N} q_s(\xi) v_{m-s}(\xi, 2x - |\xi|; q_{-N}, \dots, q_N)d\xi$$
$$+m^2 \int_0^x |\xi| \sum_{s=-N}^{N} q_s(\xi) u_{m-s}(\xi, 2x - |\xi|; q_{-N}, \dots, q_N)d\xi$$
$$-2\int_{\triangleright(2x)} K'_m(2x-\tau, \xi) \sum_{s=-N}^{N} q_s(\xi) u_{m-s}(\xi, \tau; q_{-N}, \dots, q_N)d\xi d\tau, \tag{41}$$
$$x \in [0, T/2], \quad m = 0, \pm1, \pm2, \pm N,$$

and

$$q_m^0(x) = 2f''_m(2x) \subset C[0, T/2], \quad m = 0, \pm1, \pm2, \pm N. \tag{42}$$

Denote by $\mathcal{Q}_0(N, L, Q_0)$ the set of functions $q_s(s)$, $-N \le s \le N$, satisfying the conditions

$$\|q_m - q_m^0\|_{C[0,L]} \le Q_0, \quad Q_0 = \max_{-N\le s\le N} \|q_s^0\|_{C[0,L]}, \quad m = 0, \pm1, \pm2, \pm N. \tag{43}$$

If functions $q_s \in \mathcal{Q}_0(N, L, Q_0)$ for $-N \le s \le N$ then, obviously, $q_s \in \mathcal{Q}(N, L, 2Q_0)$. For the operator equations the following theorem holds.

Theorem 3.1. *Let the data (6) satisfy the conditions (23) and*

$$F = \max_{-N\le m\le N} \|f''_m\|_{C[0,T]}. \tag{44}$$

Then there exists a number $T_0 \in (0, T]$ such that the operator equations (40) have one and only one solution on the set $\mathcal{Q}_0(N, T_0/2, 2F)$.

Proof. It is obviously that $Q_0 = 2F$. We prove that operator $A = (A_{-N}, \dots, A_N)$ maps the set $\mathcal{Q}_0(N, T/2, 2F)$ into itself and it is a contracted operator if T satisfies a smallness condition. Let $q_s \in \mathcal{Q}_0(N, T/2, 2F)$, $-N \le s \le N$. Then $q_s \in \mathcal{Q}(N, T/2, 4F)$, $-N \le s \le N$. Then from relations (40) we obtain

$$|q_m(x) - q_m^0(x)| \le 8F(2N+1)\int_0^x \max_{-N\le s\le N} |v_{m-s}(\xi, 2x - |\xi|; q_{-N}, \dots, q_N)|d\xi$$
$$+4F(2N+1)N^2 \int_0^x |\xi| \max_{-N\le s\le N} |u_{m-s}(\xi, 2x - |\xi|; q_{-N}, \dots, q_N)|d\xi$$
$$+8F(2N+1)\int_{\triangleright(2x)} |K'_m(2x-\tau, \xi)| \max_{-N\le s\le N} |u_{m-s}(\xi, \tau; q_{-N}, \dots, q_N)|d\xi d\tau, \tag{45}$$
$$x \in [0, T/2], \quad m = 0, \pm1, \pm2, \pm N.$$

Estimate first the function $K'_m(2x-\tau,\xi)$. Since from (11) and (12) follows that

$$\left|\frac{J_1(\zeta)}{\zeta}\right| \le \frac{1}{2}, \quad \left|\frac{J_2(\zeta)}{\zeta^2}\right| \le \frac{1}{4}$$

for all $\zeta \in \mathbb{R}$, then the following estimate holds

$$|K'_m(2x-\tau,\xi)| \le \frac{m^2(2+m^2T^2)}{4}, \quad (\xi,\tau) \in \triangleright(2x),\ x \in [0,T/2]. \tag{46}$$

Then from (45) we get

$$\begin{aligned} |q_m(x)-q_m^0(x)| &\le 4F(2N+1)T \max_{-2N\le j\le 2N} \max_{(\xi,\tau)\in D'(T)} |v_j(\xi,\tau|;q_{-N},\dots,q_N)| \\ &+\frac{1}{2}F(2N+1)T^2N^2[4+N^2T^2] \max_{-2N\le j\le 2N} \max_{(\xi,\tau)\in D'(T)} |u_j(\xi,\tau|;q_{-N},\dots,q_N)|, \\ &x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N. \end{aligned} \tag{47}$$

For functions $u_m(x,t;q_{-N},\dots,q_N)$ and $v_m(x,t;q_{-N},\dots,q_N)$ the estimates (8) and (9) valid with $Q=4F$. Using them, we find

$$\begin{aligned} |q_m(x)-q_m^0(x)| &\le 8F^2(2N+1)^2T^2C_2\max(1,2FT^3(2N+1)) \\ &+\frac{1}{2}F(2N+1)T^2N^2[4+N^2T^2]C_1\max(1+2FT^2,2F^2T^4(2N+1)) \quad (48) \\ &\equiv 2FT^2C_5(N,T,F), \quad (49) \\ &x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N. \end{aligned}$$

Choosing $T_1=T_1(N,F)$ as a positive root of the equation

$$T^2C_5(N,T,F)=1,$$

we obtain that

$$|q_m(x)-q_m^0(x)| \le 2F, \quad x\in[0,T_1/2], \quad m=0,\pm1,\pm2,\pm N,$$

i.e. the operator $A=(A_{-N},\dots,A_N)$ maps the set $\mathcal{Q}_0(N,T_1/2,2F)$ into itself.

Let us demonstrate now that this mapping is contracted if $T\le T_1$ and it is enough small. Let $(q_{-N},\dots,q_N)$ and $(\hat q_{-N},\dots,\hat q_N)$ be two solutions of the inverse problem belonging the set $\mathcal{Q}_0(N,T/2,2F)$. Denote corresponding them solutions of the problem (10) by u_m and $\hat u_m$, respectively, and its derivatives with respect to t by v_m and $\hat v_m$. Then we can write relations (40) for $(q_{-N},\dots,q_N)$ and $(\hat q_{-N},\dots,\hat q_N)$. Denoting

$$\tilde u_m=u_m-\hat u_m, \quad \tilde v_m=v_m-\hat v_m, \quad \tilde q_m=q_m-\hat q_m, \quad \tilde Q=\max_{-N\le m\le N}\max_{x\in[0,T/2]}|\tilde q_m(x)|,$$

we find

$$\tilde q_m(x) = -2\int\limits_0^x \sum_{s=-N}^{N} [\tilde q_s(\xi) v_{m-s}(\xi, 2x-|\xi|) + \hat q_s(\xi)\tilde v_{m-s}(\xi, 2x-|\xi|)]d\xi$$
$$+m^2\int\limits_0^x |\xi| \sum_{s=-N}^{N} [\tilde q_s(\xi) u_{m-s}(\xi, 2x-|\xi|) + \hat q_s(\xi)\tilde u_{m-s}(\xi, 2x-|\xi|)]d\xi$$
$$-2\int\limits_{\triangleright(2x)} K'_m(2x-\tau,\xi) \sum_{s=-N}^{N} [\tilde q_s(\xi) u_{m-s}(\xi,\tau) + \hat q_s(\xi)\tilde u_{m-s}(\xi,\tau)]d\xi d\tau, \qquad (50)$$
$$x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N.$$

From here

$$|\tilde q_m(x)| \le T(2N+1) \max_{-2N\le j\le 2N}\ \max_{(\xi,\tau)\in D'(T)} \left[\tilde Q|v_j(\xi,\tau)| + 4F|\tilde v_j(\xi,\tau)|\right]$$
$$+\frac{T^2N^2(4+N^2T^2)}{8}(2N+1) \max_{-2N\le j\le 2N}\ \max_{(\xi,\tau)\in D'(T)} \left[\tilde Q|u_j(\xi,\tau)| + 4F|\tilde u_j(\xi,\tau)|\right], \qquad (51)$$
$$x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N.$$

Using estimates (8), (9) and (24), (25), we find

$$|\tilde q_m(x)| \le 2T^2(2N+1)^2F\tilde Q(1+N^2T^2)\,[2C_2+C_4T]\max(1,FT^2)$$
$$+\frac{T^2N^2(4+N^2T^2)}{8}(2N+1)\tilde Q$$
$$\times\left[C_1\max(1+2FT^2, 2F^2T^4(2N+1) + 4FC_3T\max(1,FT(2N+1))\right]$$
$$\equiv \tilde Q T^2 C_6(N,T,F), \qquad (52)$$
$$x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N.$$

Set a fixed $\rho\in(0,1)$ and define $T_2 = T_2(N,F,\rho)$ as the posirive root of the equation

$$T^2C_6(N,T,F) = \rho. \qquad (53)$$

Then for $T\le T_2$ the estimates hold

$$|\tilde q_m(x)| \le \rho\,\tilde Q, \quad x\in[0,T/2], \quad m=0,\pm1,\pm2,\pm N. \qquad (54)$$

It means that operator $A=(A_{-N},\ldots,A_N)$ is contractive on the set $\mathcal{Q}_0(N,T_2/2,2F)$. Taking $T_0=\min(T_1,T_2)$ we get that this operator maps the set $\mathcal{Q}_0(N,T_0/2,2F)$ into itself and it is a contracted operator on this set. By the Banach's principle the operator equation (40) has one and only one solution on the set $\mathcal{Q}_0(N,T_0/2,2F)$. $\square$

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