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SOME GEOMETRIC ASPECTS IN INVERSE PROBLEMS

V.G.Romanov

Abstract We consider inverse problems for partial differential equations of the second order related to recovering a coefficient in these equations. It is supposed that some measurements of solutions to direct problems are made on convenient sets. A study of some inverse problems leads to geometric problems: recovering a function from its integrals along geodesic lines of the Riemannian metric which determines by the leading part of the differential equations or recovering the metric. Our main goal here is to demonstrate how inverse problems for equations of different types are reduced to the well known geometrical problems.

Key words: inverse problems, hyperbolic, parabolic, elliptic equations, tomography, inverse kinematic problem.

AMS Mathematics Subject Classification: 35R30

1 Introduction

It was noted about 50 years ago that some inverse problems for hyperbolic equations are closely related to the problems of the integral geometry that consist in recovering a function from its integrals along a family of given curves or given surfaces [14, 15, 16, 27, 28]. The geometric objects connected to the latter problems are the rays or fronts of the hyperbolic equations. They are sufficiently complicated if coefficients in the leading terms of the differential operators are not constants. But in the simplest case, when the leading part is the wave operator and an incident source is located at a fixed point, the rays are segments of strait lines and the fronts are spheres. Then the problem of recovering a variable spatial coefficient in the lower term of the equation is often reduced to the tomography problem. The problem of recovering a variable speed of sound in the wave equation is also reduced to the similar problem, if one considers this inverse problem in a linear setting and the linearization is given for a constant speed. In the next section we consider the relations of some inverse problems for hyperbolic equations with the tomography problem, the integral geometry problem and the inverse kinematic problem.

Some later it was opened that inverse problems for linear parabolic equations can be reduced to analogical problems for associating hyperbolic equations [13]. It turns out that a solution of a parabolic equation can be expressed via the solution of a hyperbolic equation and vice versa. This result introduces the typical for hyperbolic equations geometric objects and for parabolic equations also. Particularly, some inverse problems for parabolic equations generate the problem of the integral geometry. But to make it effectively, one needs to express a solution of the hyperbolic equation via a solution of the parabolic one. It is possible produce on the base of an analytical continuation of the solution to the parabolic equation with respect to the time variable t into the complex plane. The latter problem is strongly unstable. Therefore this way is practically impossible. Recently (see [33]) it was suggested an other way of using the relation between solutions to the both equations hyperbolic and parabolic. The new approach uses a special expansion of the fundamental solution for the parabolic equation with respect to t as $t \to 0$. In section 3 we explain how some inverse problems for the parabolic equation arrive at the inverse kinematic and the integral geometry problems.

In section 4 a three-dimensional inverse scattering problem for the Schrödinger equation with a compactly supported unknown potential in the frequency domain is considered. This problem was subject of studying in many papers (see, e.g., [3, 5, 6, 8, 9, 20], [21]-[25] and references therein). We consider here a phaseless inverse problem when only the modulus of a scattering field is given for large frequencies. We demonstrate that the problem of the potential recovering is reduced to the tomography problem. For the case when leading part of the equation is a linear elliptic operator with unknown refraction coefficient, the phaseless inverse problem is reduced to the inverse kinematic problem.

2 Hyperbolic equations

Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - Lu = \delta(x-y,t),\ x \in \mathbb{R}^3; \quad u|_{t<0} = 0, \tag{1}$$

where $y \in \mathbb{R}^3$ is a fixed point (parameter of the problem), L is the linear elliptic operator

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + q(x)u,$$

in which $(a_{ij}(x)) = A(x)$ is an uniformly positive matrix. Assume that all coefficients of the operator L are uniformly bounded and, for simplicity, they belong $C^\infty(\mathbb{R}^3)$.

Below we consider inverse problems related to the operator L that arrive at some geometric problems.

2.1 The tomography problem

We assume here that $L = \Delta + q(x)$. Consider the structure of the solution to the problem (1).

Lemma 2.1. *Let $T > 0$ is an arbitrary fixed number, $D(T,y) = \{(x,t) |\, |x-y| \le t \le T - |x-y|\}$, $L = \Delta + q(x)$ and $q(x) \in C(\mathbb{R}^3)$. Then the solution to the problem (1) has the following form*

$$u(x,t;y) = \frac{1}{4\pi}\delta(t-|x-y|) + \hat{u}(x,t;y)H(t-|x-y|), \tag{2}$$

where $\hat{u}(x,t;y)$ is a continuous function in $D(T,y)$, $H(t)$ is the Heaviside step-function: $H(t)=0$ for $t<0$ and $H(t)=1$ for $t\geq 0$. Moreover, the following formula holds

$$\lim_{t\to|x-y|+0}\hat{u}(x,t;y)=\frac{1}{8\pi|x-y|}\int\limits_{L(x,y)} q(\xi)ds, \tag{3}$$

in which $L(x,y)$ is the segment of the strait line passing through points x, y, and ξ is a variable point on this line and s is the arc length.

Proof. Using the Kirchhoff formula we find that the function $\hat{u}$ is solution to the integral equation

$$\hat{u}(x,t;y)=\frac{1}{4\pi}\int\limits_{|\xi-x|\leq t}^{t}\frac{q(\xi)}{|\xi-x|}\left[\hat{u}(\xi,t-|x-\xi|;y)+\frac{\delta(t-|\xi-y|-|x-\xi|)}{4\pi|\xi-y|}\right]d\xi. \tag{4}$$

Represent the function $\hat{u}$ in the form

$$\hat{u}(x,t;y)=\sum_{n=1}^{\infty}u_n(x,t;y), \tag{5}$$

where the functions $u_n(x,t;y)$ are defined by the formulae

$$u_1(x,t;y)=\frac{1}{16\pi^2}\int\limits_{|\xi-x|\leq t}^{t}\frac{q(\xi)\delta(t-|\xi-y|-|x-\xi|)}{|\xi-x||\xi-y|}d\xi, \tag{6}$$

$$u_n(x,t;y) \;=\; \frac{1}{4\pi}\int\limits_{|\xi-x|\leq t}^{t}\frac{q(\xi)u_{n-1}(\xi,t-|x-\xi|;y)}{|\xi-x|}d\xi, \quad n=1,2,\ldots. \tag{7}$$

We shall prove that all functions $u_n(x,t;y)$, $n\geq 1$, are continuous in $D(T,y)$ up to boundary $t=|x-y|$ and the series (5) is uniformly converged in $D(T,y)$ for any $T>0$. Moreover, the following relations hold

$$u_n(x,t;y)\to\frac{\delta_{n1}}{8\pi|x-y|}\int\limits_{L(x.y)} q(\xi)ds, \quad \text{as } t\to|x-y|. \tag{8}$$

Here δ_{n1} is the Kronecker delta: $\delta_{n1}=1$, if $n=1$, and $\delta_{n1}=0$, if $n\neq 1$. Formula (3) follows from the latter relations.

Indeed, consider the function $u_1(x,t;y)$ defined by (6). In order to study properties of this function, it is convenient introduce some curvilinear coordinates of the variable point ξ. For $\tau>|x-y|$ consider the ellipsoid with focuses at points x and y:

$$E(x,y,\tau)=\{\xi\in\mathbb{R}^3|\,|\xi-y|+|\xi-x|=\tau\}.$$

Note that the integral (6) is taken along the ellipsoid $E(x, y, t)$. An arbitrary point $\xi \in \mathbb{R}^3$ for fixed x and y can be represent via curvilinear coordinates τ, z, ψ in the form

$$\xi(\tau, z, \psi) = y + \frac{\rho + z\tau}{2} e_1 + \frac{\sqrt{(\tau^2 - \rho^2)(1 - z^2)}}{2}(e_2 \cos\psi + e_3 \sin\psi),$$
$$z \in [-1, 1], \psi \in [0, 2\pi], \tau \in [\rho, \infty). \tag{9}$$

where $\rho = |x - y|$, $e_1 = (x - y)/|x - y|$ and the unit vectors e_2, e_3 are orthogonal to e_1 and one to other. Then

$$|\xi - y| = \frac{\tau + \rho z}{2}, \quad |\xi - x| = \frac{\tau - \rho z}{2},$$
$$J = \Big|\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\tau, z, \psi)}\Big| = \frac{\tau^2 - \rho^2 z^2}{8}, \qquad \frac{J}{|\xi - x^0||\xi - x|} = \frac{1}{2}.$$

Therefore

$$u_1(x, t; y) = \frac{1}{2(4\pi)^2} \int_0^{2\pi} \int_{-1}^{1} q(\xi(t, z.\psi)) dz d\psi. \tag{10}$$

In particular, from this formula follows that the ellipsoid $E(x, y, t)$ degenerates into the segment $L(x, y) = \{\xi = y + e_1\rho(1 + z)/2, z \in [-1, 1]\}$ as $t \to \rho = |x - y|$ and

$$\begin{aligned} u_1(x, |x - y| + 0; y) &= \frac{1}{16\pi} \int_{-1}^{1} q(y + e_1\rho(1 + z)/2) dz \\ &= \frac{1}{8\pi\rho} \int_0^{\rho} q(y + s e_1) ds. \end{aligned}$$

The latter formula coincides with one given by (8). It follows from (10) that $u_1(x, t; y)$ is continuous in $D(T, y)$ and

$$\|u_1\|_{C(D(T,y))} \le \frac{q_0}{8\pi}, \tag{11}$$

where

$$q_0 = q_0(y) = \max_{|x-y| \le T/2} |q(x)|.$$

Formulae for u_n, $n \ge 2$, have the form

$$u_n(x, t; y) = \frac{1}{16\pi} \int_\rho^t \int_0^{2\pi} \int_{-1}^{1} q(\xi(\tau, z, \psi))$$
$$\times u_{n-1}(\xi(\tau, z, \psi), t - (\tau - \rho z)/2; y)(\tau + \rho z) dz d\psi d\tau. \tag{12}$$

It follows from these formulae that all functions u_n are continuous in $D(T,y)$ and for them the following relations hold

$$|u_n(x,t;y)| \le \frac{3q_0T}{32\pi}\int\limits_{\rho}^{t}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1}|w_{n-1}(\xi(\tau,z,\psi),t-(\tau-\rho z)/2;y)|dzd\psi d\tau, \qquad (13)$$
$$(x,t)\in D(T,y).$$

We used here that $0\le\tau+\rho z\le\tau+\rho\le 3T/2$ for $(x,t)\in D(T,y)$. Using inequalities (13) one can prove that the following estimates hold

$$|u_n(x,t;y)| \le \|u_1\|_{C(D(T,y))}\Big(\frac{3q_0T}{8}\Big)^{n-1}\frac{(t-|x-y|)^{n-1}}{(n-1)!}, \quad (x,t)\in D(T,y),$$
$$n=2,3,\dots. \qquad (14)$$

Use for this goal the mathematical induction method. Assume that the formulae (14) hold for all $n\le k$, $k\ge 1$. Then

$$|u_{k+1}(x,t;y)| \le \frac{3q_0T}{32\pi}\int\limits_{\rho}^{t}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1}|u_k(\xi(\tau,z,\psi),t-(\tau-\rho z)/2;y)|dzd\psi d\tau$$
$$\le \frac{3q_0T}{32\pi}\int\limits_{\rho}^{t}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1}\|u_1\|_{C(D(T,y))}\Big(\frac{q_0T}{4}\Big)^{k-1}\frac{(t-\tau)^{k-1}}{(k-1)!}dzd\psi d\tau$$
$$\le \|u_1\|_{C(D(T,y))}\Big(\frac{3q_0T}{8}\Big)^{k}\frac{(t-|x-y|)^{k}}{k!}, \quad (x,t)\in D(T,y). \qquad (15)$$

Here the following equalities $t-|\xi-y|-(\tau-\rho z)/2=t-|\xi-y|-|\xi-x|=t-\tau$ were used. Hence, the estimates (14), indeed, take place. From these estimates follow the relations (8) for $n\ge 2$. Because $0\le t-|x-y|\le T$ for all $(x,t)\in D(T,y)$, the series (5) is uniformly converged in $D(T,y)$. Hence its sum $\hat u(x,t;y)$ belongs to $C(D(T,y))$ for any T and y. □

Consider now the following inverse problem. Let Ω be the ball of radius R centered at the origin, $\Omega=\{x\in\mathbb{R}^3|\,|x|<R\}$, and S is its boundary, $S=\{x\in\mathbb{R}^3|\,|x|=R\}$. Assume that the solution of the problem (1) with $L=\Delta+q(x)$ is given for all $x\in S$, $y\in S$ and $t\in[0,T]$, where $T>2R$, i.e.,

$$u(x,t;y)=f(x,t;y), \quad (x,y)\in S\times S,\ t\in[0,T]. \qquad (16)$$

The inverse problem is: find $q(x)$ in Ω from given $f(x,t;y)$.

Using Lemma 2.1 we obtain that function $f(x,t;y)$ should be represent in the form

$$f(x,t;y)=\frac{1}{4\pi}\delta(t-|x-y|)+\hat f(x,t;y)H(t-|x-y|), \qquad (17)$$

where function $\hat f(x,t;y)$ is a continuous function in $D^*(T,y)=\{(x,t)|\,x\in S,|x-y|\le t\le T-|x-y|\}$ for any $y\in S$. Moreover,

$$\lim_{t\to|x-y|+0} f(x,t;y)\equiv g(x,y)=\frac{1}{8\pi|x-y|}\int\limits_{L(x,y)}q(\xi)ds. \qquad (18)$$

Hence, we immediately arrive at the well known tomography problem: given the integrals

$$\int\limits_{L(x,y)} q(\xi)ds = \hat{g}(x,y), \quad (x,y) \in S \times S, \tag{19}$$

where

$$\hat{g}(x,y) = 8\pi|x-y|g(x,y), \tag{20}$$

find $q(x)$ in $\overline{\Omega} = \Omega \cup S$.

There are exist various stable algorithms for solving this problem (see for instance [19] and references therein). Following the original paper by R.G. Mukhometov [17] we give below only a stability estimate for this problem. Note that the tomography problem can be solved for a cross-section of Ω by an arbitrary plane Σ. Denote the cross-section of Ω with Σ by $\Omega(\Sigma)$ and the cross-section of S with Σ by $S(\Sigma)$. Represent the equation of $S(\Sigma)$ as $\xi = h(\varphi)$, where $h(\varphi)$ is the smooth 2π periodic function, and let $x = h(\varphi_1)$ and $y = h(\varphi_2)$.Then the following estimate holds

$$\|q\|^2_{L^2(\Omega(\Sigma))} \leq \frac{1}{4\pi} \int\limits_{S(\Sigma)\times S(\Sigma)} \left|\nabla_{\varphi_1,\varphi_2}\hat{g}(h(\varphi_1),h(\varphi_2))\right|^2 d\varphi_1 d\varphi_2. \tag{21}$$

2.2 The integral geometry problem

Consider now the more general case when operator L is given by (2). Let $a^{ij}(x)$ be elements of the matrix $A^{-1}(x)$ inverse to $A(x) = (a_{ij}(x)$ and the length element $d\tau$ of the Riemannian metric be determine by the formula

$$d\tau = \left(\sum_{i,j=1}^{3} a^{ij}(x)dx_i dx_j\right)^{1/2}.$$

It is well known that the Riemannian distance $\tau(x,y)$ between points x and y is the solution to the Cauchy problem

$$\sum_{i,j=1}^{3} a_{ij}(x)\tau_{x_i}\tau_{x_j} = 1, \quad \tau(x,y) = O(|x-y|) \quad \text{as} \quad x \to y. \tag{22}$$

Below we shall use the following assumption.

Assumption. We assume that geodesic lines of the Riemannian metric satisfy the regularity condition, i.e. for each two points $x, y \in \mathbb{R}^3$ there exists a single geodesic line $\Gamma(x,y)$ connecting these points.

Introduce the following functions:

$$\begin{aligned} \theta_0(t) &:= \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \\ \theta_k(t) &:= \frac{t^k}{k!}\,\theta_0(t), \qquad k = 1, 2, \ldots. \end{aligned} \tag{23}$$

Then the following lemma holds (see Lemma 2.2.1 in the book [31])

Lemma 2.2. *Let a_{ij} and q be $\mathbf{C}^\infty(\mathbb{R}^3)$ functions and the Assumption holds. Then the solution to problem (1) can be represented in the form of the asymptotic series*

$$u(x,t;y)=\theta_0(t)\left[\alpha_{-1}(x,y)\delta(t^2-\tau^2(x,y))+\sum_{k=0}^{\infty}\alpha_k(x,y)\theta_k(t^2-\tau^2(x,y))\right], \tag{24}$$

where $\tau^2(x,y)$, $\alpha_k(x,y)$, $k=-1,0,1,\ldots$, are infinitely smooth functions of x,y and, moreover, $\alpha_{-1}(x,y)>0$.

Let $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ be the Riemannian coordinates of a point x with respect to a fixed point y. They can be calculated through function $\tau^2(x,y)$ by the formula (see formula (2.2.28) in [31]):

$$\zeta=-\frac{1}{2}\left(\nabla_y\tau^2(x,y)\right)A(y). \tag{25}$$

Denote by $J(x,y)$ the Jacobian of the transformation of the Riemannian coordinates into Cartesian ones, i.e.,

$$J=\det\left(\frac{\partial\zeta}{\partial x}\right).$$

Then coefficients of the expansion (24) are defined by the formulae (see (2.2.44), (2.2.45) in [31])

$$a_{-1}(x,y)=\frac{\sqrt{J(x,y)}}{2\pi\sqrt{\det A(y)}}, \tag{26}$$

$$a_k(x,y)=\frac{a_{-1}(x,y)}{4\tau^{k+1}(x,y)}\int\limits_{\Gamma(x,y)}\tau^k(\xi,y)\frac{L_\xi a_{k-1}(\xi,y)}{a_{-1}(\xi,y)}d\tau,\quad k=0,1,2,\ldots, \tag{27}$$

where $\Gamma(x,y)$ is the geodesic line connecting x and y and $d\tau$ is the element of the Riemannian length and $\xi\in\Gamma(x,y)$ is a variable point.

Suppose that the coefficients $a_{ij}(x)$ are given for all $x\in\mathbb{R}^3$ and the ball Ω is convex with respect to geodesics $\Gamma(x,y)$, $(x,y)\in(S\times S)$. Consider again the inverse problem of recovering $q(x)$ inside the ball Ω assuming that the following information is known

$$u(x,t;y)=f(x,t;y),\quad (x,y)\in(S\times S),\ t\in[0,T], \tag{28}$$

where T is a positive number such that

$$T>\max_{(x,y)\in(S\times S)}\tau(x,y). \tag{29}$$

Since the coefficients $a_{ij}(x)$ are given the function $\tau(x,y)$, $\zeta(x,y)$, $J(x,y)$ and geodesic lines $\Gamma(x,y)$ are known for all $x\in\overline{\Omega}$ and $y\in\overline{\Omega}$. Therefore the coefficient $a_{-1}(x,y)$ in the expansion (24) is also known for all $(x,y)\in(\Omega\times\Omega)$. Then putting $k=0$ in formulae (27), we find

$$\int\limits_{\Gamma(x,y)}q(\xi)d\tau=g(x,y),\quad (x,y)\in(S\times S), \tag{30}$$

where

$$g(x,y)=\frac{4\tau(x,y)a_0(x,y)}{a_{-1}(x,y)}-\int\limits_{\Gamma(x,y)}\frac{L'_\xi a_{-1}(\xi,y)}{a_{-1}(\xi,y)}d\tau \tag{31}$$

and

$$L'=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial}{\partial x_i}\Big). \tag{32}$$

Because $a_0(x,y)$ is defined by the given information,

$$a_0(x,y)=\lim_{t\to\tau(x,y)+0}f(x,t;y),\quad (x,y)\in(S\times S), \tag{33}$$

the function $g(x,y)$ is known. Hence, we come to integral geometry problem: find $q(x)$ inside Ω from given its integrals along the geodesic lines joining points x,y belonging to S.

This problem arise in vary inverse problems (see [14, 15, 16, 27, 28], [30] - [32]). It was intensively studied in 70-th of the last century. In the papers [1, 2, 17, 29] stability estimates for this problem were found. The integral geometry problem for tensor fields was studied in [4, 26, 34]. Below we give the estimate of solution to problem (30) following the book [31]).

Let θ,φ be the spherical coordinates on S, $\nu(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$, and $x=R\nu(\theta_1,\varphi_1)$, $y=R\nu(\theta_2,\varphi_2)$. Put $\tau(R\nu(\theta_1,\varphi_1),R\nu(\theta_2,\varphi_2))=\hat\tau(\theta_1,\varphi_1,\theta_2,\varphi_2)$, $g(R\nu(\theta_1,\varphi_1),R\nu(\theta_2,\varphi_2))=\hat g(\theta_1,\varphi_1,\theta_2,\varphi_2)$ and

$$I(\hat g,\hat\tau)=\det\begin{pmatrix}0 & \hat g_{\theta_1} & \hat g_{\varphi_1}\\ \hat g_{\theta_2} & \hat\tau_{\theta_1\theta_2} & \hat\tau_{\varphi_1\theta_2}\\ \hat g_{\varphi_2} & \hat\tau_{\theta_1\varphi_2} & \hat\tau_{\varphi_1\varphi_2}\end{pmatrix}.$$

Then (see formula (2.3.19) in [31]) the following estimate holds

$$\int\limits_{\Omega}q(x)(\det A(x))^{-1/2}dx\le\frac{1}{8\pi}\int\limits_{S\times S}|I(\hat g,\hat\tau)|d\theta_1 d\varphi_1 d\theta_2 d\varphi_2. \tag{34}$$

2.3 The inverse kinematic problem

Assume here that $a_{ij}=n^{-2}(x)\delta_{ij}$. Consider the problem: find $n(x)$ in Ω given the function $f(x,t,;y)$ in (28). Fix $x\in S$ and $y\in S$. Using the representation (24) we easily find

$$\tau(x,y)=\sup_{\tau\ge0}\{\tau|\ f(x,t;y)\equiv 0 \text{ for } t<\tau\},\quad \forall(x,y)\in(S\times S). \tag{35}$$

Hence. the function $\tau(x,y)$ is uniquely determined for all $(x,y)\in(S\times S)$ by the given information. Then we come to the following problem: find $n(x)$ in Ω given $\tau(x,y)$ for all $(x,y)\in(S\times S)$. This problem is called the inverse kinematic problem. It is widely used in the seismology, the electromagnetic prospecting. The function $\tau(x,y)$ solves the Cauchy problem for the eikonal equation

$$|\nabla_x \tau(x,y)|^2 = n^2(x),\ x \in \Omega, \quad \tau(x,y) = O(|x-y|) \text{ as } x \to y. \tag{36}$$

Moreover, the following formula holds

$$\tau(x,y) = \int\limits_{\Gamma(x,y)} n(\xi) ds, \tag{37}$$

where s is arc length. In means that $\tau(x,y)$ is the Riemannian length of the geodesic $\Gamma(x,y)$. The inverse kinematic problem is nonlinear one. If $n(x) = n_0(x) + \beta(x)$, where $n_0(x)$ is a positive known function and $\|\beta(x)\|_{C^1(\Omega)} << \|n_0(x)\|_{C^1(\Omega)}$ one can linearize the problem. Assume that $\tau(x,y) = \tau_0(x,y) + \tau_1(x,y)$. where $\tau_0(x,y)$ corresponds to the function $n_0(x)$, i.e., $\tau_0(x,y)$ is the solution to problem (36) with $n = n_0(x)$. Let $\Gamma_0(x,y)$ be the geodesic line corresponding $n_0(x)$. Then

$$\tau_1(x,y) = \int\limits_{\Gamma_0(x,y)} \beta(\xi) ds. \tag{38}$$

The formula (38) was derived in [14] for the first time. The derivation is quite simple. If one substitutes $\tau(x,y) = \tau_0(x,y) + \tau_1(x,y)$ and $n(x) = n_0(x) + \beta(x)$ in (36) and neglects by terms $\beta^2(x)$ and $|\nabla_x \tau_1(x,y)|^2$, one obtains the relations

$$\nabla_x \tau_1(x,y) \cdot \nabla_x \tau_0(x,y) = n_0(x)\beta(x), \quad \tau_1(x,y) \to 0 \text{ as } x \to y. \tag{39}$$

Because the vector $\nabla_x \tau_0(x,y)$ is directed along the tangent lines to $\Gamma_0(x,y)$ at the point x and $|\nabla_x \tau_0(x,y)|^2 = n_0(x)$, the left hand side in (39) coincides with the product $n_0(x)$ and the derivative of $\tau_1(x,y)$ with respect to s. Then. dividing both side of (39) on $n_0(x)$ and integrating the result along $\Gamma_0(x,y)$, one obtains (38).

The formula (38) defines the Frechet derivative on the element $n_0(x)$ of nonlinear operator $\tau(n)$ and it lies in a base of obtaining the stability estimate for the inverse kinematic problem. For two-dimensional case the stability estimate was found in the paper [17] and has the form

$$\|n_1 - n_2\|_{L^2(\Omega)} \le \frac{1}{4\pi} \|\tau_1 - \tau_2\|_{H^1(S\times S)}, \tag{40}$$

where n_1 and n_2 two different positive functions $n(x)$ and τ_1 and τ_2 are corresponding them solutions to the problem (36) with $n(x) = n_k(x)$, $k = 1, 2$. For three dimensional case the stability estimate was found in the papers [1, 2, 18] and has the form

$$\|n_1 - n_2\|_{L^2(\Omega)} \le C \|\tau_1 - \tau_2\|_{H^2(S\times S)}, \tag{41}$$

where the positive constant C depends on the lower bond of n_1 and n_2 in Ω.

3 The parabolic equations

Consider the Cauchy problem for the parabolic equation

$$\frac{\partial v}{\partial t} - Lv = \delta(x-y,t),\ x \in \mathbb{R}^3; \quad v|_{t<0} = 0, \tag{42}$$

where L is the uniformly elliptic operator defined by the formula (2) and $x \in \mathbb{R}^3$. Suppose that the solutions of problems (1) and (42) do not increase as $t \to \infty$. Then the Laplace transforms of the functions $u(x,t;y)$ and $v(x,t;y)$ with respect to t exist and the Laplace images of these functions are related by the equality $\tilde{v}(x,p;y) = \tilde{u}(x,\sqrt{p};y)$ for all complex p with positive real part. Therefore, we have (see, e.g., [35]).

$$v(x,t,y) = \frac{1}{2\sqrt{\pi t^3}} \int\limits_0^\infty e^{-\frac{z^2}{4t}} u(x,z,y) z dz, \quad t > 0. \tag{43}$$

Let us apply (43) to obtain an asymptotic expansion of $v(x,t;y)$ as $t \to +0$. Substituting representation (24) into (43), we obtain

$$v(x,t;y) = \frac{e^{-\frac{\tau^2(x,y)}{4t}}}{4\sqrt{\pi t^3}} \int\limits_0^\infty e^{-\frac{s}{4t}} \left[\alpha_{-1}(x,y)\delta(s) + \sum_{n=0}^{\infty} \alpha_n(x,y)\theta_n(s) \right] ds, \quad t > 0. \tag{44}$$

Elementary calculations yield the relation

$$v(x,t;y) = \frac{e^{-\frac{\tau^2(x,y)}{4t}}}{4\sqrt{\pi t^3}} \sum_{n=-1}^{\infty} \alpha_n(x,y)(4t)^{n+1}, \quad t > 0. \tag{45}$$

The obtained above relations make it possible to bridge the gap between a number of settings of inverse problems for parabolic equations and similar settings of inverse problems for hyperbolic equations, which have been studied earlier. To demonstrate this, we first obtain relations between the solutions of problem (42) and the coefficients in the expansion (45). Let Ω be the same domain as in section 2 with boundary S. Suppose that, for some $T > 0$, the solution $v(x,t;y)$ of problem (42) is known for all $(x,t,y) \in G(\Omega,T)$, where $G(\Omega,T) = \{(x,t,y) | (x,y) \in (S \times S), t \in [0,T]\}$. Let us find expressions for $\tau(x,y)$ and $\alpha_{-1}(x,y)$, $\alpha_0(x,y)$ for $(x,y) \in (S \times S)$ in terms of the given function. It follows from (45) that we have

$$\tau(x,y) = \Big(\lim_{t\to+0} (-4t \ln v(x,t,y)) \Big)^{1/2}, \quad (x,y) \in (S \times S). \tag{46}$$

Given the function $\tau(x,y)$, the coefficients $\alpha_{-1}(x,y)$ and $\alpha_0(x,y)$ are determined by

$$\alpha_{-1}(x,y) = \lim_{t\to+0} \Big(4v(x,t,y) e^{\frac{\tau^2(x,y)}{4t}} \sqrt{\pi t^3} \Big), \quad (x,y) \in (S \times S), \tag{47}$$

$$\alpha_0(x,y) = \lim_{t\to+0} \Big[\Big(4v(x,t,y) e^{\frac{\tau^2(x,y)}{4t}} \sqrt{\pi t^3} - \alpha_{-1}(x,y) \Big) / (4t) \Big]. \tag{48}$$

Relations (46)-(48) can be used in problems of determining the coefficients of the operator L inside Ω from the solution of problem (42) given for $(x,t,y)\in G(\Omega,T)$. Suppose that $a_{ij}=n^{-2}(x)\delta_{ij}$, where $n(x)>0$ and δ_{ij} is the Kronecker delta, and it is required to determine $n(x)$ in Ω. Calculating the function $\tau(x,y)$ by formula (46), we arrive at the inverse kinematic problem of finding $n(x)$ in Ω from the given function $\tau(x,y)$ for $(x,y)\in(S\times S)$. This problem was considered in subsection 2.3.

Suppose now that the coefficients $a_{ij}(x)$ are given and it is required to find $q(x)$ from the given function $\alpha_0(x,y)$ for $(x,y)\in(S\times S)$. Since the coefficients $a_{ij}(x)$ are given the function $\tau(x,y)$, $\alpha_{-1}(x,y)$ and geodesic lines $\Gamma(x,y)$ are known for all $(x,y)\in(\Omega\times\Omega)$. Then we obtain the relations (30), (31). Hence, we again arrive at the same integral geometry problem as in subsection 2.2.

Thus, expansions (45) obtained above for the solution of problem (42) directly imply a whole series of new results about the uniqueness and stability of solutions for of inverse problems for parabolic equations. In this way, numerical methods for solving such inverse problems can also be developed.

4 The elliptic equations

Let $w(x,y,k)$ be solution of the Schrödinger equation

$$-\Delta w-k^2w+q(x)w=\delta(x-y),\ x\in\mathbb{R}^3, \tag{49}$$

satisfying the Sömmerfeld conditions

$$w(x,y,k)=O(r^{-1}),\quad \frac{\partial w}{\partial r}-ikw=o(r^{-1})\quad \text{as}\, r\to\infty, \tag{50}$$

where $r=|x|$. Here the frequency $k>0$ and conditions (50) are valid for every fixed source position y. We assume here that potential $q(x)$ is $C^4(\mathbb{R}^3)$ smooth function satisfying the conditions

$$q(x)\geq 0,\quad q(x)\equiv 0\ \ \forall x\in(\mathbb{R}^3\setminus\Omega) \tag{51}$$

and Ω is the same ball as earlier with boundary S. The solution of the problem (49), (50) can be represented in the form

$$w(x,y,k)=w_0(x,y,k)+w_{sc}(x,y,k), \tag{52}$$

where $w_0(x,y,k)$ given by the formula

$$w_0(x,y,k)=\frac{e^{ik|x-y|}}{4\pi|x-y|} \tag{53}$$

is the fundamental solution for the Helmholtz operator $-\Delta-k^2$ with the conditions (50) and $w_{sc}(x,y,k)$ is the scattering field on the potential $q(x)$.

Let k_0 be a positive number. Consider the following phaseless inverse scattering problem: the function $|w_{sc}(x,y,k)|$ is given for $(x,y)\in(S\times S)$ and $k\geq k_0$, i.e.,

$$|w_{sc}(x,y,k)|=f(x,y,k),\quad (x,y)\in(S\times S),\ k\geq k_0, \tag{54}$$

it is required to find the potential $q(x)$ in Ω.

This problem was studied in the paper [10]. It turned out that this problem is closely related to the asymptotic expansion of the solution to (49), (50) with respect to k as $k \to \infty$. The such expansion can be found if we compare the solutions of the problem (49), (50) with the solution $u(x,t;y)$ of the Cauchy problem (1) with $L = \Delta + q(x)$. It was stated in the book [36], that the function $u(x,t;y)$ exponentially decay with respect to $t \to \infty$ together with the second partial derivatives if x belongs any bounded domain. Moreover, in [10] was proved the following lemma.

Lemma 4.1. *Let $T > 0$ be an arbitrary fixed number, $D(T,y) = \{(x,t) |\, |x-y| \le t \le T - |x-y|\}$, $L = \Delta + q(x)$, and $q(x) \in C^4(\mathbb{R}^3)$ and satisfies conditions (51). Then the solution to the problem (1) has the form (2) and the function $\hat{u}(x,t;y)$ is continuous in $D(T,y)$ together with $\partial^k \hat{u}(x,t;y)/\partial t^k$, $k = 1,2$, for any T and y.*

Using this Lemma and L. Vainberg's results [36], we state that

$$w_{sc}(x,y,k) = \int_{-\infty}^{\infty} e^{ikt} \hat{u}(x,t;y) dt. \tag{55}$$

Integrating by parts we get

$$\begin{aligned} w_{sc}(x,y,k) &= \int_{|x-y|}^{\infty} e^{ikt} \hat{u}(x,t;y) dt \\ &= -\frac{e^{ik|x-y|} \hat{u}(x, |x-y|+0; y)}{ik} + \frac{e^{ik|x-y|} \hat{u}_t(x, |x-y|+0; y)}{(ik)^2} \\ &\quad + \frac{1}{(ik)^2} \int_{|x-y|}^{\infty} e^{ikt} \hat{u}_{tt}(x,t;y) dt. \end{aligned}$$

From here, using formula (3), we obtain

$$w_{sc}(x,y,k) = -e^{ik|x-y|} \left[\frac{1}{8ik\pi|x-y|} \int_{L(x,y)} q(\xi) ds + O\left(\frac{1}{k^2}\right) \right], \quad \text{as } k \to \infty.$$

Thus, the given function $f(x,y,k)$ has the asymptotic

$$f(x,y,k) = \frac{1}{8k\pi|x-y|} \int_{L(x,y)} q(\xi) ds + O\left(\frac{1}{k^2}\right), \quad \text{as } k \to \infty. \tag{56}$$

From the latter formula we find

$$\int_{L(x,y)} q(\xi) ds = g(x,y), \quad (x,y) \in (S \times S), \tag{57}$$

where

$$g(x,y) = 8\pi|x-y|\lim_{k\to\infty}[kf(x,y,k)]. \tag{58}$$

is a known function.

Hence, to obtain $q(x)$ we should solve the tomography problem. Note that the similar problem occurs, if instead of the point sources, one uses incident plane waves going from infinity [11].

Consider now the more general equation

$$-Lw - k^2 w = \delta(x-y),\ x\in\mathbb{R}^3, \tag{59}$$

where $L = \operatorname{div}(n^{-2}\nabla) + q(x)$ and the function $n(x)$ is $C^\infty(\mathbb{R}^3)$ smooth function and can be represented in the form

$$n(x) = 1+\beta(x), \quad \beta(x)\geq 0, \quad \beta(x)\equiv 0 \text{ for } x\in(\mathbb{R}^3\setminus\Omega). \tag{60}$$

We assume that the potential $q(x)$ satisfies the previous conditions (51). Let function $w(x,y,k)$ solves the problem (59), (50). Assume that y is an arbitrary point of S, represent the function $w(x,y,k)$ in the form (52) and consider the inverse problem of recovering $\beta(x)$ inside Ω (see also [12]) from given function $f(x,y,k)$ defined by (54).

Consider again the auxiliary problem (1) and use the Lemma 2.2. Then the function $u(x,t;y)$ can be represented in the form

$$u(x,t;y) = \frac{\alpha_{-1}(x,y)}{2\tau(x,y)}\delta(t-\tau(x,y)) + \hat{u}(x,t;y)H(t-\tau(x,y)), \tag{61}$$

where $\hat{u}(x,t;y)$ is $C^\infty(\mathbb{R}^3)$ smooth function. and

$$\hat{u}(x,\tau(x,y)+0;y) = a_0(x,y). \tag{62}$$

Again, using the Vainberg's results [36], we get

$$w(x,y,k) = \int\limits_{-\infty}^{\infty} e^{ikt}u(x,t;y)dt.$$

Taking into account the representation (61), we find

$$w(x,y,k) = e^{ik\tau(x,y)}\left[\frac{\alpha_{-1}(x,y)}{2\tau(x,y)} - \frac{\hat{u}(x,\tau(x,y)+0;y)}{ik} + \frac{\hat{u}_t(x,\tau(x,y)+0;y)}{(ik)^2}\right] + \frac{1}{(ik)^2}\int\limits_{\tau(x,y)}^{\infty} e^{ikt}\hat{u}_{tt}(x,t;y)dt.$$

From here, using formula (62), we obtain

$$w(x,y,k) = e^{ik\tau(x,y)}\left[\frac{\alpha_{-1}(x,y)}{2\tau(x,y)} - \frac{\alpha_0(x,y)}{ik} + O\left(\frac{1}{k^2}\right)\right], \quad \text{as } k\to\infty. \tag{63}$$

Hence,

$$w_{sc}(x,y,k) = e^{ik\tau(x,y)}\frac{\alpha_{-1}(x,y)}{2\tau(x,y)} - \frac{e^{ik|x-y|}}{4\pi|x-y|} + O\left(\frac{1}{k}\right), \quad \text{as } k \to \infty. \tag{64}$$

Fix here $x \in S$ and $y \in S$. Then $|w_{sc}(x,y,k)| = f(x,y,k)$ for $k \geq k_0$ is the given function of k. Consider the limit of $f(x,y,k)$ as $k \to \infty$. It is exists if and only if $\tau(x,y) = |x-y|$. If $\tau(x,y) \neq |x-y|$, then $\tau(x,y) > |x-y|$ because $n(x) \geq 1$ in Ω. The function $f^2(x,y)$ is represented in the form

$$\begin{aligned} f^2(x,y,k) = &\left(\frac{\alpha_{-1}(x,y)}{2\tau(x,y)}\right)^2 + \left(\frac{1}{4\pi|x-y|}\right)^2 \\ &- \frac{\alpha_{-1}(x,y)}{2\pi|x-y|\tau(x,y)}\cos\left[k(\tau(x,y) - |x-y|)\right] + O\left(\frac{1}{k}\right), \quad \text{as } k \to \infty. \end{aligned} \tag{65}$$

This formula implies that

$$f^*(x,y) = \limsup_{k\to\infty} f(x,y,k) = \frac{\alpha_{-1}(x,y)}{2\tau(x,y)} + \frac{1}{4\pi|x-y|}, \tag{66}$$

$$f^{**}(x,y) = \liminf_{k\to\infty} f(x,y,k) = \left|\frac{\alpha_{-1}(x,y)}{2\tau(x,y)} - \frac{1}{4\pi|x-y|}\right|. \tag{67}$$

Hence, we find

$$\frac{\alpha_{-1}(x,y)}{2\tau(x,y)} = f^*(x,y) - \frac{1}{4\pi|x-y|}, \quad (x,y) \in (S \times S).$$

Now we can determine $\tau(x,y)$ for $(x,y) \in (S \times S)$. Fix again x and y on S and take an arbitrary positive ε such that $\varepsilon < f^*(x,y) - f^{**}(x,y)$. Then there exist the numbers $k_1(\varepsilon) > \max(k_0, \varepsilon^{-1/2})$ and $k_2(\varepsilon) > k_1(\varepsilon)$ such that

$$f(x,y,k_1(\varepsilon)) = f^*(x,y) - \varepsilon, \tag{68}$$

and

$$k_2(\varepsilon) = \min_{k > k_1(\varepsilon)}\{k |\ f(x,y,k) = f^*(x,y) - \varepsilon\}. \tag{69}$$

From relation (65) we derive that

$$[k_2(\varepsilon) - k_1(\varepsilon)](\tau(x,y) - |x-y|) = 2\pi + O(\varepsilon), \quad \text{as } \varepsilon \to +0.$$

Hence,

$$\tau(x,y) = |x-y| + \lim_{\varepsilon\to+0}\frac{2\pi}{k_2(\varepsilon) - k_1(\varepsilon)}, \quad (x,y) \in (S \times S). \tag{70}$$

Thus, the function $\tau(x,y)$ becomes known for all $(x,y) \in (S \times S)$. So, we arrive to the inverse kinematic problem. Solving it, we recover $n(x)$ inside Ω.

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Vladimir G. Romanov,
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,
Email: romanov@math.nsc.ru

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