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AN ALGORITHM FOR WAVE PROPAGATION ANALYSIS IN STRATIFIED POROELASTIC MEDIA

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Abstract The classic poroelastic theory of Biot, developed in 1950's, describes the propagation of elastic waves through a porous media containing a fluid. This theory has been extensively used in various fields dealing with porous media: seismic exploration, oil/gas reservoir characterization, environmental geophysics, earthquake seismology, etc. In this work we use the Ursin formalism to derive explicit formulas for the analysis of propagation of elastic waves through a stratified 3D porous media, where the parameters of the media are characterized by piece-wise constant functions of only one spatial variable, depth.

Key words: poroelasticity, Biot system, low-frequency range, layered media, Ursin algorithm

AMS Mathematics Subject Classification: 86-06, 86A15

1 Introduction

Poroelastic models are used in geophysics and petroleum engineering, where porous media filled with fluid and/or gas is of great interest. The best-known poroelastic theory was developed by Maurice Biot, see [5, 6].

There are many works devoted to the development and application of analytical/semi-analytical methods for wave propagation analysis in stratified elastic media, see, for instance, [1, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18].

The development of similar methods in the case of stratified porous media is very important too, see [2, 4, 8, 16].

To construct explicit formulas for the analytical/numerical analysis of the elastic waves propagation in stratified 3D porous media, we use the Ursin formalism, which was initially used to give a unified treatment of electromagnetic waves, acoustic waves, and the isotropic elastic waves in plane layered media. Recently, this formalism was applied to the Pride equations for simulation of the electrokinetic phenomena in layered media, see [19].

In this work we apply Ursin's method for solving the Biot system in the case of the 3D poroelastic plane layered media. In the exposition of results, we follow basically to the works [17, 19]. Although the results obtained in the work [19] allow, under certain conditions, to split Pride's equations and select only the poroelastic part, we examine the case of a more complete poroelastic system, characterised by presence in the Darcy law of an inertial force connected with the effective density of pore fluid, see [3] for details.

2 Statement of the problem

We shall consider wave propagation in a porous medium $\mathcal{R} = \cup_{k=0}^{N}\mathcal{R}_k$, composed by stratified layers identified with $\mathcal{R}_k = \{x = (x_1, x_2, x_3 \equiv z) \in \mathbb{R}^3 : \ z_k < z < z_{k+1}\}$, with $0 = z_0 < z_1 \cdots < z_{N+1} = \infty$. Let $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$ be the solid and relative fluid displacements, respectively. The Biot equations in the time frequency (ω) domain, at each point $x \in \mathcal{R}$, are (time dependence of $e^{-i\omega t}$ is assumed)

$$
\begin{aligned}
&-i\omega(\rho v + \rho_f q) = \nabla \cdot \tau + f\,,\\
&q = \frac{\kappa}{\eta}(-\nabla p + i\omega\rho_f v + i\omega\rho_E q + g)\,,\\
&-i\omega\tau = (\lambda\nabla\cdot v + C\nabla\cdot q)I + G(\nabla v + \nabla v^T)\,,\\
&i\omega p = C\nabla\cdot v + M\nabla\cdot q\,.
\end{aligned}
\tag{1}
$$

Here $v = -i\omega u, q = -i\omega w$ are the solid and relative fluid velocities, $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ are the forces imposed on the solid and on the pore fluid, respectively, τ is the stress tensor, p is the pressure in the pore fluid, λ, G are the Lamé coefficients, C, M are the Biot moduli, ρ is the bulk density, ρ_f is the density of the pore fluid, ρ_E is the effective density of the pore fluid, κ is the permeability, η is the pore fluid viscosity, I is the 3×3 identity matrix. We assume that all material parameters are represented by piece-wise constant functions depended only the depth coordinate z, with the discontinuities at the points $z = z_k$, $k = 1, 2, \ldots, N$.

At the internal layer boundaries $z = z_k$, we suppose that the following functions are continuous:

$$
v\,, q\,, p\,, \tau_{13}\,, \tau_{23}\,, \tau_{33}\,. \tag{2}
$$

The boundary conditions at the free surface $z = 0$ are

$$
p = \tau_{13} = \tau_{23} = \tau_{33} = 0\,. \tag{3}
$$

And finally, at the infinity the solution satisfy the following radiation conditions:

$$
\lim_{|x|\to\infty}(u, w) = 0\,. \tag{4}
$$

3 Method

3.1 Ursin format

Consider the Fourier transform in the two coordinates x_1, x_2

$$
\hat{X}(k_1, k_2, z) = F_{x_1 x_2}(X) \equiv \int_{\mathbb{R}^2} e^{-i(k_1 x_1 + k_2 x_2)} X(x_1, x_2, z)\,dx_1 dx_2\,.
$$

Let $(k_1, k_2)^T$ be the horizontal wave number and $k = \sqrt{k_1^2 + k_2^2}, \gamma = k\omega^{-1}$. Applying the Fourier transform to (1) we obtain the system of ordinary differential equations (ODE's) represented in the terms of $\hat{f}, \hat{g}, \hat{v}, \hat{q}, \hat{\tau}, \hat{p}$. For each horizontal wave number, plane wave sources of the form $e^{i(k_1x_1+k_2x_2)}\hat{f}$, $e^{i(k_1x_1+k_2x_2)}\hat{g}$ will produce plane wave responses with spatial dependence of the form $e^{i(k_1x_1+k_2x_2)}$. The equations can be simplified if we rotate to a coordinate system $(\hat{x}_1, \hat{x}_2, \hat{x}_3)^T$ with the first coordinate oriented in the direction of the horizontal wave number, so that all of these plane waves have a spatial dependence of the form $e^{ik\hat{x}_1}$. Therefore, let

$$\Omega = \frac{1}{k}\begin{pmatrix} k_1 & k_2 & 0 \\ -k_2 & k_1 & 0 \\ 0 & 0 & k \end{pmatrix}. \tag{5}$$

The ODE's obtained after application of the Fourier transform can be simplified if we define

$$\tilde{x} = \Omega x\,, \tilde{v} = \Omega\hat{v},\ \tilde{q} = \Omega\hat{q}\,, \tilde{\tau} = \Omega\hat{\tau}\Omega^T\,, \tilde{p} = \hat{p}\,, \tilde{f} = \Omega\hat{f}\,, \tilde{g} = \Omega\hat{g}\,. \tag{6}$$

A straightforward calculation uncouples this system

$$\frac{d\Phi^{(m)}}{dz} = -i\omega M^{(m)}\Phi^{(m)} + S^{(m)}\,, m = 1, 2\,, \tag{7}$$

where $\Phi^{(m)}$, $m = 1, 2$, are the $2n_m$-vectors $(n_1 = 3, n_2 = 1)$ defined as

$$\Phi^{(1)} = (\tilde{v}_3, \tilde{\tau}_{13}, -\tilde{q}_3, \tilde{\tau}_{33}, \tilde{v}_1, \tilde{p})^T\,, \Phi^{(2)} = (\tilde{v}_2, \tilde{\tau}_{23})^T\,,$$

$S^{(m)}$ are the source $2n_m$-vectors, and $M^{(m)}$ are the $2n_m \times 2n_m$-matrices

$$M^{(m)} = \begin{pmatrix} 0 & M_1^{(m)} \\ M_2^{(m)} & 0 \end{pmatrix} \tag{8}$$

with symmetric $n_m \times n_m$-matrices $M_1^{(m)}, M_2^{(m)}$. For Systems 1 and 2 the sub-matrices and the corresponding source vectors are

$$M_1^{(1)} = \begin{pmatrix} -\beta M & \beta\gamma(C^2 - \lambda M) & -\beta C \\ \beta\gamma(C^2 - \lambda M) & \rho + \frac{i\omega\rho_f^2\kappa}{\eta - i\omega\rho_E\kappa} - 4\beta\gamma^2 G(C^2 - M(\lambda + G)) & 2\beta\gamma GC - \frac{i\omega\rho_f\gamma\kappa}{\eta - i\omega\rho_E\kappa} \\ -\beta C & 2\beta\gamma GC - \frac{i\omega\rho_f\gamma\kappa}{\eta - i\omega\rho_E\kappa} & -\beta(\lambda + 2G) + \frac{i\omega\gamma^2\kappa}{\eta - i\omega\rho_E\kappa} \end{pmatrix}$$

$$M_2^{(1)} = \begin{pmatrix} \rho & \gamma & -\rho_f \\ \gamma & G^{-1} & 0 \\ -\rho_f & 0 & -\frac{\eta - i\omega\rho_E\kappa}{i\omega\kappa} \end{pmatrix}, S^{(1)} = (0, -\tilde{f}_1 - \frac{i\omega\rho_f\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_1, \frac{ik\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_1, -\tilde{f}_3, 0, \tilde{g}_3)^T \tag{9}$$

and

$$M_1^{(2)} = G^{-1}\,, M_2^{(2)} = \rho - G\gamma^2 + \frac{i\omega\rho_f^2\kappa}{\eta - i\omega\rho_E\kappa}\,,$$
$$S^{(2)} = (0, -\tilde{f}_2 - \frac{i\omega\rho_f\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_2)^T\,. \tag{10}$$

Here $\beta = (C^2 - M(\lambda + 2G))^{-1}$. Once $\Phi^{(1)}$ and $\Phi^{(2)}$ have been determined, we may compute

$$\begin{aligned}
\tilde{q}_1 &= \frac{\kappa}{\eta - i\omega\rho_E\kappa}(-ik\tilde{p} + i\omega\rho_f\tilde{v}_1 + \tilde{g}_1)\,,\\
\tilde{\tau}_{11} &= \beta\big(-4\gamma G(C^2 - M(\lambda+G))\tilde{v}_1 + (C^2 - \lambda M)\tilde{\tau}_{33} + 2GC\tilde{p}\big)\,,\\
\tilde{\tau}_{22} &= \beta\big(-2\gamma G(C^2 - \lambda M)\tilde{v}_1 + (C^2 - \lambda M)\tilde{\tau}_{33} + 2GC\tilde{p}\big)\,,\\
\tilde{q}_2 &= \frac{\kappa}{\eta - i\omega\rho_E\kappa}(i\omega\rho_f\tilde{v}_2 + \tilde{g}_2)\,, \dot{\tilde{\tau}}_{12} = -C\gamma\tilde{v}_2\,.
\end{aligned} \tag{11}$$

The boundary conditions for Systems 1 and 2 at the free surface $z = 0$ are

$$\tilde{p} = \tilde{\tau}_{13} = \tilde{\tau}_{23} = \tilde{\tau}_{33} = 0\,. \tag{12}$$

Note that (12) gives $n_1 = 3$ conditions for System 1 having $2n_1 = 6$ variables, and gives $n_2 = 1$ condition for System 2, which has $2n_2 = 2$ variables. It means that for each system we need n_m, $m = 1, 2$ additional conditions to completely specify the solution. These relations we obtain using the radiation condition (4), which means that there are no up-going waves from $z = \infty$.

3.2 Ursin diagonalization

Let's give briefly a derivation of the Ursin diagonalization procedure. We consider matrices of the form (8), where for simplicity we drop the superscript $^{(m)}$.

Assume that M_1M_2 has n distinct nonzero eigenvalues λ_j^2, $j = 1, 2, \ldots, n$, with associated eigenvectors a_j, such that $a_j^T M_2 a_j = \lambda_j$. Here $\lambda_j = \sqrt{\lambda_j^2}$ with the branch chosen so that $\mathrm{Im}(\lambda_j) \geq 0$ and $\lambda_j > 0$ if λ_j is real. Define $b_j = \lambda_j^{-1} M_2 a_j$. This vector is an eigenvector of M_2M_1 with eigenvalue λ_j^2. Using symmetricity of M_1, M_2 we obtain $a_j^T b_i = \delta_j^i$, where δ_j^i is the Kronecker delta.

Let L_1 be the $n \times n$-matrix whose j-th column is a_j, and let L_2 be the $n \times n$-matrix whose i-th column is b_i, then $L_1^{-1} = L_2^T$, $L_2^{-1} = L_1^T$. Introduce $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $L_2\Lambda = M_2L_1$ and $M_1L_2 = L_1\Lambda$, which implies

$$M_1 = L_1\Lambda L_1^T\,, M_2 = L_2\Lambda L_2^T\,. \tag{13}$$

Introducing the diagonal matrix $\tilde{\Lambda} = \mathrm{diag}(\Lambda, -\Lambda)$ and using (13), we finally obtain

$$M = L\tilde{\Lambda}L^{-1}\,, \tag{14}$$

where

$$L = \frac{1}{\sqrt{2}}\begin{pmatrix} L_1 & L_1 \\ L_2 & -L_2 \end{pmatrix}\,,\ L^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} L_2^T & L_1^T \\ L_2^T & -L_1^T \end{pmatrix}\,.$$

In our case the explicit formulas for λ_j, a_j, b_j, are:

System 1. There are three modes: fast compressional wave ($\lambda_1^{(1)}$), Biot slow wave

$(\lambda_2^{(1)})$ and vertical shear wave $(\lambda_3^{(1)})$.

$$(\lambda_j^{(1)})^2 = -\gamma^2 + \beta\big(C\rho_f - \frac{M\rho}{2} - i(\lambda+2G)\frac{\eta - i\omega\rho_E\kappa}{2\omega\kappa}\big) \pm$$
$$\pm \frac{\beta}{2}\sqrt{\big(M\rho - i(\lambda+2G)\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}\big)^2 - 4(M\rho_f - iC\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa})(C\rho - (\lambda+2G)\rho_f)}$$

$j = 1, 2,$ with $(+)$ for $j = 1$ and (-) for $j = 2,$ and $(\lambda_3^{(1)})^2 = -\gamma^2 + G^{-1}\big(\rho + \dfrac{i\omega\kappa\rho_f^2}{\eta - i\omega\rho_E\kappa}\big)$

$$a_j^{(1)} = \overline{a}_j(-1, 2G\gamma, \xi_j)^T\,, j = 1, 2\,,\ a_3^{(1)} = \frac{\overline{a}_3}{\lambda_3^{(1)}}\big(\gamma, G(\lambda_3^{(1)})^2 - G\gamma^2, -\frac{i\omega\kappa\gamma\rho_f}{\eta - i\omega\rho_E\kappa}\big)^T$$

$$b_j^{(1)} = \frac{\overline{a}_j}{\lambda_j^{(1)}}\big(2G\gamma^2 - \rho - \rho_f\xi_j, \gamma, \rho_f + i\xi_j\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}\big)^T\,, j = 1, 2\,,\ b_3^{(1)} = \overline{a}_3(2G\gamma, 1, 0)^T$$

where

$$\xi_j = \frac{C\rho - (\lambda+2G)\rho_f}{\frac{(\lambda_j^{(1)})^2+\gamma^2}{\beta} - C\rho_f + i(\lambda+2G)\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}}\,, j = 1, 2\,,$$

$$\overline{a}_j = \sqrt{\frac{\lambda_j^{(1)}}{\rho + 2\rho_f\xi_j + i\xi_j^2\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}}}\,, j = 1, 2\,,\ \overline{a}_3 = \sqrt{\frac{\lambda_3^{(1)}}{G(\lambda_3^{(1)})^2 + G\gamma^2}}\,. \tag{15}$$

System 2. There is the horizontal shear wave $(\lambda^{(2)})$ only.

$$(\lambda^{(2)})^2 = -\gamma^2 + G^{-1}\big(\rho + \frac{i\omega\kappa\rho_f^2}{\eta - i\omega\rho_E\kappa}\big)\,,\ a^{(2)} = \sqrt{\frac{1}{G\lambda^{(2)}}}\,,\ b^{(2)} = \sqrt{G\lambda^{(2)}}\,. \tag{16}$$

3.3 Reflection and transmission matrices

Firstly, we consider a homogeneous source-free region of space. Dropping $^{(m)}$ we have a $2n$-dimensional system of the form (7) with M constant and $S = 0$. Let

$$\Phi = L\Psi \text{ and } \Psi = \big(U, D\big)^T\,, \tag{17}$$

where U, D are n-vectors. Inserting (17) into (7) and using (14) we arrive at

$$\frac{d}{dz}\Psi = -i\omega\tilde{\Lambda}\Psi\,.$$

Then

$$\Psi(z) = \big(e^{-i\omega\Lambda(z-z_0)}U(z_0), e^{i\omega\Lambda(z-z_0)}D(z_0)\big)^T\,, \tag{18}$$

where z_0 is a fixed point in the same source-free region. The vectors U, D characterise up-going (U) and down-going (D) waves. Next, consider an interface at $z = \overline{z}$, where the material parameters vary discontinuously across $\overline{z}$. We denote by $^\pm$ quantities evaluated at $\overline{z}^\pm = \overline{z} \pm 0$. Since Φ is continuous across $\overline{z}$, we obtain

$$\Psi^+ = J\Psi^-\,,\ \Psi^- = J^{-1}\Psi^+\,, \tag{19}$$

where the jump matrix is

$$J = (L^+)^{-1}L^- = \begin{pmatrix} J_A & J_B \\ J_B & J_A \end{pmatrix} \;, \; J^{-1} = \begin{pmatrix} J_A^T & -J_B^T \\ -J_B^T & J_A^T \end{pmatrix}$$

and J_A, J_B are the $n \times n$-matrices

$$J_A = \frac{1}{2}\big[\big(L_2^+\big)^T L_1^- + \big(L_1^+\big)^T L_2^-\big] \;, \; J_B = \frac{1}{2}\big[\big(L_2^+\big)^T L_1^- - \big(L_1^+\big)^T L_2^-\big] \,.$$

Next, we consider a stack of layers $0 < z_1 < \cdots < z_N < \infty$. We denote by subscript j a quantity at interface $z = z_j$, with superscripts $^\pm$ as before. Then

$$\big(U_N^-, D_N^-\big)^T = J_N^{-1}\big(0, D_N^+\big)^T \,,$$

where we have used that there is no up-going wave below the last interface at $z = z_N$. So, we obtain

$$U_N^- = \Gamma_N D_N^- \;, \; D_N^+ = T_N D_N^- \,, \tag{20}$$

where

$$\Gamma_N = -J_{B,N}^T\big(J_{A,N}^T\big)^{-1} \;, \; T_N = \big(J_{A,N}^T\big)^{-1} \,. \tag{21}$$

Here Γ_N is the reflection matrix and T_N is the transmission matrix from the last interface $z = z_N$.

Let $j < N$ and $\triangle z_j = z_{j+1} - z_j$, $j = 0, 1, \ldots, N-1$, is the layer thickness. Then by jumping across the layer boundary and using (18), (19) we obtain

$$\begin{aligned} U_j^- &= J_{A,j}^T e^{i\omega\Lambda_j \triangle z_j} U_{j+1}^- - J_{B,j}^T e^{-i\omega\Lambda_j \triangle z_j} D_{j+1}^- \,, \\ D_j^- &= -J_{B,j}^T e^{i\omega\Lambda_j \triangle z_j} U_{j+1}^- + J_{A,j}^T e^{-i\omega\Lambda_j \triangle z_j} D_{j+1}^- \,. \end{aligned} \tag{22}$$

Define reflection and transmission matrices Γ_j, T_j by the relations that for any incident wave D_j^- at the top of stack of layers underlying $z = z_j$

$$U_j^- = \Gamma_j D_j^- \;, \; D_j^+ = T_j D_j^- \,. \tag{23}$$

Therefore Γ_j computes the reflected wave from the stack and T_j computes the transmitted wave below the stack, when the incident wave is known. From (22), (23) we obtain by induction

$$\begin{aligned} \Gamma_j &= \big(J_{A,j}^T \tilde{\Gamma}_{j+1} - J_{B,j}^T\big)\big(- J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T\big)^{-1} \,, \\ T_j &= T_{j+1} e^{i\omega\Lambda_j \triangle z_j}\big(- J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T\big)^{-1} \,, \end{aligned} \tag{24}$$

where $\tilde{\Gamma}_{j+1} = e^{i\omega\Lambda_j \triangle z_j}\Gamma_{j+1} e^{i\omega\Lambda_j \triangle z_j}$. Again, by induction it can be shown that Γ_j is symmetric.

Thus all the reflection and transmission matrices can be calculated by (24), starting with (21).

3.4 Sources and boundary conditions

Consider a $2n$-dimensional system of the form (7) with $^{(m)}$ omitted. Let the source be of the form

$$S = S_0\delta(z - z_s) + S_1\delta'(z - z_s) \tag{25}$$

with S_0, S_1 independent of z. Here δ is the Dirac function. Define n-vectors S_A, S_B by the following formula

$$\big(S_A, S_B\big)^T = i\omega M S_1 - S_0\,. \tag{26}$$

Applying the standard procedure we obtain the following jump condition across the source

$$\Phi(z_s^-) = \Phi(z_s^+) + \big(S_A, S_B\big)^T. \tag{27}$$

Inserting a fictitious layer boundary at $z = z_s^+$, we compute the reflection matrix $\Gamma_s \equiv \Gamma(z_s^+)$ from the top of this layer. Note that at z_s^+, $J_A = I, J_B = 0$, since the material properties do not change at z_s. Then the up-going wave $U_s \equiv U_s(z_s^+)$ is related to the down-going wave $D_s \equiv D_s(z_s^+)$ there by (23). Then we have

$$\Psi(z_s^+) = \big(\Gamma_s D_s, D_s\big)^T. \tag{28}$$

Using (17), (27) and (28) we obtain

$$\Psi(z_s^-) = \big(\Gamma_s D_s, D_s\big)^T + \frac{1}{\sqrt{2}}\big(L_2^T S_A + L_1^T S_B, L_2^T S_A - L_1^T S_B\big)^T. \tag{29}$$

This expression may now propagated upwards through layers, using (18) and jumped upwards across layers boundaries using (19) until we reach the free surface at $z = 0^+$. Then the n boundary conditions at $z = 0$ can be used to find the n unknowns D_s.

Consider now one particular case when $z_s \in (0, z_1)$. In this case

$$\begin{aligned}\Psi(0^+) = {} & \big(e^{i\omega\Lambda_s z_s}\Gamma_s D_s, e^{-i\omega\Lambda_s z_s} D_s\big)^T + \\ & + \frac{1}{\sqrt{2}}\big(e^{i\omega\Lambda_s z_s}(L_2^T S_A + L_1^T S_B), e^{-i\omega\Lambda_s z_s}(L_2^T S_A - L_1^T S_B)\big)^T.\end{aligned} \tag{30}$$

We next write

$$\Phi(0^+) = \big(G_A\Phi_0, G_B\Phi_0\big)^T, \tag{31}$$

where Φ_0 is an n-vector of unknowns at $z = 0$ and G_A, G_B are $n \times n$ matrices. For System 1, let

$$\Phi_0^{(1)} = \big(\tilde{v}_3, -\tilde{q}_3, \tilde{v}_1\big)^T_{z=0^+}\,,\ G_A^{(1)} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\,,\ G_B^{(1)} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}. \tag{32}$$

We can check that (31) holds for System 1 with the boundary conditions given by (12). For System 2, let

$$\Phi_0^{(2)} = \tilde{v}_2(0^+)\,,\ G_A^{(2)} = 1\,,\ G_B^{(2)} = 0\,. \tag{33}$$

It may be checked that (31) holds for System 2 with the boundary conditions given by (12).

Using (17), (30) and (31) we obtain

$$
\begin{aligned}
\Phi_0 &= \Big(e^{i\omega\Lambda z_s}\Gamma_s e^{i\omega\Lambda z_s}\big(L_2^T G_A - L_1^T G_B\big) - \big(L_2^T G_A + L_1^T G_B\big)\Big)^{-1} \times \\
&\times e^{i\omega\Lambda z_s}\Big(\Gamma_s\big(L_2^T S_A - L_1^T S_B\big) - \big(L_2^T S_A + L_1^T S_B\big)\Big)\,, \\
D_s &= \frac{1}{\sqrt{2}} e^{i\omega\Lambda z_s}\big(L_2^T G_A - L_1^T G_B\big)\Phi_0 - \frac{1}{\sqrt{2}}\big(L_2^T S_A - L_1^T S_B\big)\,.
\end{aligned}
\tag{34}
$$

In particular, when the source is situated just below the surface we get

$$
\begin{aligned}
\Phi_0 &= \Big(\big(\Gamma_s - I\big)L_2^T G_A - \big(\Gamma_s + I\big)L_1^T G_B\Big)^{-1} \times \\
&\times \Big(\big(\Gamma_s - I\big)L_2^T S_A - \big(\Gamma_s + I\big)L_1^T S_B\Big) \text{ as } z_s \to 0^+\,.
\end{aligned}
\tag{35}
$$

Φ_0 defines all of Φ at the free surface, and $D_s, U_s = \Gamma_s D_s$ give all of Φ just below the source. Now we are able to compute Φ in any $z \in \mathbb{R}_+$ by propagating through the layers using (18) and (19).

Remark. *Propagation of an upward-going wave in the downward direction will be unstable numerically using* (18)*, because the complex exponentials grow rather than decay with distance. Therefore, numerically one has to obtain U from D using the reflection or transmission matrices.*

Inverting (6), we can calculate the hat (ˆ) variables, i.e.,

$$
\hat{v} = \Omega^T \tilde{v},\ \hat{q} = \Omega^T \tilde{q}\,, \hat{\tau} = \Omega^T \tilde{\tau}\Omega\,, \hat{p} = \tilde{p}\,. \tag{36}
$$

The matrices for Systems 1 and 2 depend only on the magnitude k. However, factors k_1 and k_2 are introduced by (5) and possibly by the directionality of the source. For any function $\hat{h}(k)$ let

$$
\mathcal{T}_{j_1,j_2}\big(\hat{h}\big) \equiv F^{-1}_{x_1x_2}\big(k_1^{j_1}k_2^{j_2}\hat{h}(k)\big) = (-i)^{j_1+j_2}\partial^{j_1}_{x_1}\partial^{j_2}_{x_2}F^{-1}_{x_1x_2}\big(\hat{h}(k)\big)\,.
$$

We can compute these quantities as Hankel transforms in the cylindrical coordinates r, θ, z. Define

$$
B_{j_1,j_2}\big(\hat{h}\big) = \frac{1}{2\pi}\int_0^\infty k^{j_1} J_{j_2}(kr)\hat{h}(k)dk\,,
$$

where J_{j_2} is the Bessel function and j_1, j_2 are nonnegative integers. Then

$$
\begin{aligned}
&\mathcal{T}_{0,0} = B_{1,0}\,,\ \mathcal{T}_{1,0} = i\cos\theta B_{2,1}\,,\ \mathcal{T}_{0,1} = i\sin\theta B_{2,1}\,, \\
&\mathcal{T}_{1,1} = \sin\theta\cos\theta\big(B_{3,0} - \frac{2}{r}B_{2,1}\big)\,,\ \mathcal{T}_{2,0} = \cos^2\theta B_{3,0} - \frac{\cos 2\theta}{r}B_{2,1}\,, \\
&\mathcal{T}_{0,2} = \sin^2\theta B_{3,0} + \frac{\cos 2\theta}{r}B_{2,1}\,.
\end{aligned}
\tag{37}
$$

4 Example of dynamite source

A dynamite source imposed on the solid and the fluid can be defined in the following form

$$f(x) = g(x) = -s(\omega)\nabla\delta(x - x_s)\,,$$

where $x_s = (0, 0, z_s)^T$ is the source position and $s(\omega)$ is the spectrum of the seismic moment. Applying the Fourier transform $F_{x_1x_2}$ we obtain

$$\hat{f} = \hat{g} = -s(\omega)\big(ik_1\delta(z - z_s), ik_2\delta(z - z_s), \delta'(z - z_s)\big)^T\,.$$

The rotation by Ω yields

$$\tilde{f} = \tilde{g} = -s(\omega)\big(ik\delta(z - z_s), 0, \delta'(z - z_s)\big)^T\,. \tag{38}$$

Substitution of (38) into (9) gives the source for System 1 in the form (25) with

$$\begin{aligned} S_0^{(1)} &= s(\omega)\Big(0, ik - \frac{\omega\rho_f k\kappa}{\eta - i\omega\rho_E\kappa}, \frac{k^2\kappa}{\eta - i\omega\rho_E\kappa}, 0, 0, 0\Big)^T\,, \\ S_1^{(1)} &= s(\omega)(0, 0, 0, 1, 0, -1)^T\,. \end{aligned} \tag{39}$$

Substitution of (38) into (10) shows that $S^{(2)}$ is zero, then $\tilde{v}_2, \tilde{\tau}_{23}$ associated with System 2 are zero too. This is to be expected result because System 2 is related to SH-waves, which are not excited by the dynamic source. Substitution of (39) into (26) gives

$$\begin{aligned} S_A^{(1)} &= i\beta s(w)\big(\omega(C - M), 2kG(M - C), \omega(\lambda + 2G - C)\big)^T\,, \\ S_B^{(1)} &= (0, 0, 0)^T\,. \end{aligned} \tag{40}$$

Formulas (40) may be used in (34) or (35) for a shallow source, to obtain all the tilde ($\tilde{}$) functions.

To invert the rotation Ω, using (36), note that from (11) and the vanishing of System 2, $\tilde{v}_2, \tilde{q}_2, \tilde{\tau}_{12}, \tilde{\tau}_{23}$ are identically zero. All the remaining tilde functions depend of k only and can be calculated by the following formulas

$$\begin{aligned} &\hat{v}_1 = \frac{k_1}{k}\tilde{v}_1\,, \quad \hat{v}_2 = \frac{k_2}{k}\tilde{v}_1\,, \quad \hat{v}_3 = \tilde{v}_3\,, \\ &\hat{q}_1 = \frac{k_1}{k}\tilde{q}_1\,, \quad \hat{q}_2 = \frac{k_2}{k}\tilde{q}_1\,, \quad \hat{q}_3 = \tilde{q}_3\,, \\ &\hat{\tau}_{11} = \frac{k_1^2\tilde{\tau}_{11} + k_2^2\tilde{\tau}_{22}}{k^2}\,, \quad \hat{\tau}_{12} = \frac{k_1k_2(\tilde{\tau}_{11} - \tilde{\tau}_{22})}{k^2}\,, \\ &\hat{\tau}_{22} = \frac{k_2^2\tilde{\tau}_{11} + k_1^2\tilde{\tau}_{22}}{k^2}\,, \quad \hat{\tau}_{13} = \frac{k_1\tilde{\tau}_{13}}{k}\,, \quad \hat{\tau}_{23} = \frac{k_2\tilde{\tau}_{13}}{k}\,, \quad \hat{\tau}_{33} = \tilde{\tau}_{33}\,. \end{aligned} \tag{41}$$

Then the Fourier transform $F_{x_1x_2}$ can be inverted in cylindrical coordinates (r, θ, z) using (37) to obtain the solid and fluid velocities

$$v = \big(iB_{1,1}(\tilde{v}_1)\big)e_r + \big(B_{1,0}(\tilde{v}_3)\big)e_z\,, \quad q = \big(iB_{1,1}(\tilde{q}_1)\big)e_r + \big(B_{1,0}(\tilde{q}_3)\big)e_z \tag{42}$$

and the stress tensor components

$$\begin{aligned}
&\tau_{11} = \mathcal{T}_{2,0}\big(k^{-2}\tilde{\tau}_{11}\big) + \mathcal{T}_{0,2}\big(k^{-2}\tilde{\tau}_{22}\big)\,, \quad \tau_{12} = \mathcal{T}_{1,1}\big(k^{-2}(\tilde{\tau}_{11} - \tilde{\tau}_{22})\big)\,,\\
&\tau_{22} = \mathcal{T}_{0,2}\big(k^{-2}\tilde{\tau}_{11}\big) + \mathcal{T}_{2,0}\big(k^{-2}\tilde{\tau}_{22}\big)\,, \quad \tau_{13} = \mathcal{T}_{1,0}\big(k^{-1}\tilde{\tau}_{13}\big)\,,\\
&\tau_{23} = \mathcal{T}_{0,1}\big(k^{-1}\tilde{\tau}_{13}\big)\,, \quad \tau_{33} = \mathcal{T}_{0,0}(\tilde{\tau}_{33})\,.
\end{aligned} \tag{43}$$

These stresses may now be computed in cylindrical coordinates from (37) using the Hankel transforms of the appropriate tilde functions.

4 Example of vertical source

We next consider a vertical point force acting on the free surface $z = 0$, i.e.,

$$f(x) = g(x) = (0,0,1)^T s(\omega)\delta(x_1)\delta(x_2)\delta(z - z_s)\,,$$

where $z_s \to 0^+$ puts the force on the free surface. This models hammer, weight drop, and vibroseis sources. Applying the Fourier transform $F_{x_1 x_2}$ and rotation Ω we arrive at

$$\tilde{f} = \tilde{g} = \hat{f} = \hat{g} = (0,0,1)^T s(\omega)\delta(z - z_s)\,. \tag{44}$$

Substitution of (44) into (9), (10) yields the source for Systems 1 and 2 in the form

$$S^{(1)} = (0,0,0,-1,0,1)^T s(\omega)\delta(z - z_s)\,,\ S^{(2)} = (0,0)^T\,. \tag{45}$$

Thus, all variables in System 2 are zero, as it was in the case of the dynamite source. From (25), (26) and (45) we obtain

$$S_A^{(1)} = (0,0,0)^T\,,\ S_B^{(1)} = (1,0,-1)^T s(\omega)\,. \tag{46}$$

Now all the tilde variables at the free surface may be computed from equations (35) as $z_s \to 0^+$ and propagated anywhere else in space. Note that $S_A^{(1)}, S_B^{(1)}$ are independent of k_1, k_2, so that the tilde variables depend only on k and not on wavenumber direction. Therefore, similar to dynamite we can transform to the hat variables using (41) and transform back to the spatial coordinates using (42) and (43).

5 Conclusions

We have shown how the complete Biot equations (low-frequency range) can be put into the Ursin form in a plane-layered medium. We have derived explicit formulas of the solution to a boundary-value problem formulated for Biot's system, which can be used as the basis of a numerical algorithm and studying the propagation of elastic waves in porous media.

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