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TWO-DIMENSIONAL ANALOGS OF THE EQUATIONS OF GELFAND, LEVITAN, KREIN, AND MARCHENKO

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Abstract. An algorithm for regularization of two-dimensional inverse coefficient problems is considered on the basis of a projection method and an approach proposed by I.M. Gelfand, B.M. Levitan, M.G. Krein, and V.A. Marchenko. An algorithm to reconstruct the potential, density, and velocity in two-dimensional inverse coefficient problems is described. Two-dimensional analogs for the equations of Gelfand, Levitan, and Krein are obtained. A two-dimensional analog of the Marchenko equation is considered for the Kadomtsev–Petviashvili equation. This approach can be generalized for corresponding multi-dimensional inverse problems. The results of numerical calculations are presented.

Key words: Gelfand-Levitan equation, Krein equation, Marchenko equation, inverse coefficient problem, inverse scattering problem

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1 Introduction

The Gelfand–Levitan method is one of the most widely used tools in inverse problem theory. It consists in reducing a nonlinear inverse problem to a one-parameter family of linear integral equations of the first or second kind.

In 1951, I.M. Gelfand and B.M. Levitan published a paper presenting a method to reconstruct a Sturm–Liouville operator by the spectral function and providing conditions sufficient for a given monotonic function to be the spectral function of the operator. One should also mention a 1954 paper by M.G. Krein with a physical statement of the vibrating string problem and theorems for solving an inverse boundary value problem.

The ideas of the Gelfand–Levitan method have been actively used in inverse dynamic problems of seismology (first by A.S. Alekseev, G. Kunetz, and B.S. Pariiskii). A dynamic (in time) version of the Gelfand–Levitan method for the inverse problem of acoustics was developed by A.S. Blagoveshchenskii.

A. S. Alekseev and V.I. Dobrinskii used a discrete analog of the Gelfand–Levitan method to investigate numerical algorithms for solving the one-dimensional inverse dynamic problem of seismology. A detailed review of numerical methods for solving equations of the Gelfand–Levitan type is presented in the paper of B.S. Pariiskii [99]. In his paper [46], S.I. Kabanikhin proposed a new algorithm for solving the Gelfand–Levitan equation based on a condition sufficient for solvability of the inverse problem. In the monograph of V.G. Romanov and S.I. Kabanikhin [105], a dynamic version of the Gelfand–Levitan method is applied to the one-dimensional inverse problem of geoelectrics for a quasi-steady-state approximation of the system of Maxwell equations. A multi-dimensional analog of the Gelfand–Levitan equation was obtained by M.I. Belishev [11], S.I. Kabanikhin [47] and M.I. Belishev, A.S. Blagoveshchenskii [12].

The inverse scattering method was introduced by C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura in 1967 to study some nonlinear equations of mathematical physics. Some elements of this method, for instance, the Backlund transformation, were known as early as the 19th century. In this method, the nonlinear equation under study is represented as a compatibility condition for a system of linear equations. An initial version of the method based on scattering theory for differential operators was applied to the Korteweg–de Vries equation.

2 Brief historical review

2.1 Inverse spectral problems and inverse scattering problems

In 1929, it was shown by V.M. Ambartsumyan [6] that "a homogeneous string is uniquely determined by the set of eigenfrequencies". Specifically, if eigenvalues of the boundary value problem λ_n

$$
\begin{gathered}
-y'' + q(x)y = \lambda y, \quad x \in [0, \pi]; \\
y'(0) = y'(\pi) = 0
\end{gathered}
$$

are $\lambda_n = n^2$ and $q(x)$ is real continuous function, then we have $q(x) \equiv 0$.

The results of this paper had remained unnoticed till 1943, when the next paper on this subject appeared. V.M. Ambartsumyan's ironic remark on this is as follows: "When an astronomer publishes a mathematical paper in a physical journal, one cannot expect that it will attract many readers".

In 1943, W. Heisenberg considered an inverse scattering problem [44, 45] and proved that for solution inverse scattering problem it is sufficient to know the asymptotic behavior of the wave function and not its behavior for all endpoints.

In 1946, an inverse problem for a Sturm–Liouville operator was considered by G. Borg [26].

Theorem. Let $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ be the eigenvalues of the operator

$$
\begin{gathered}
-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \\
y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \quad h, H \in \mathbb{R},
\end{gathered}
$$

Let $\mu_0 < \mu_1 < \mu_2 < \ldots$ be the eigenvalues of the operator

$$
\begin{gathered}
-y'' + q(x)y = \mu y, \quad 0 \le x \le \pi, \\
y'(0) - hy(0) = 0, \quad y'(\pi) + H_1 y(\pi) = 0, \quad H_1 \ne H \in \mathbb{R},
\end{gathered}
$$

Then the sequences $\{\lambda_m\}_{m=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ uniquely determine the function $q(x)$ and the numbers h, H, and H_1.

In 1949, N. Levinson [80] provided simpler proofs for some of the results obtained by G. Borg [5].

In the same year, N. Levinson tackled an inverse problem of the quantum theory of scattering [81]. He proved that in the absence of negative eigenvalues a scattering phase given for all positive energies and any fixed angular momentum uniquely determines the potential.

In 1949, it was shown by V. Bargmann that in the general case the spherically symmetric potential (at any fixed angular momentum) is not uniquely determined by the scattering phase [7, 8].

In the same year, A.N. Tikhonov proved the theorem of uniqueness of solving the inverse Sturm–Liouville problem on a semi-axis by a given Weil function [113].

In 1951, I.M. Gelfand and B.M. Levitan [39] developed a method to reconstruct a Sturm–Liouville operator by the spectral function and formulated conditions sufficient for a given monotonic function to be the spectral function of the operator (in a half-line or finite interval).

It follows from these conditions that for two sequences of real numbers $\{\lambda_n\}_0^\infty$, $\{\alpha_n\}_0^\infty$, $\alpha_n > 0$, to be the spectrum and normalization numbers of the classical Sturm–Liouville operator, it is sufficient that the following classical asymptotics be valid:

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + \ldots, \qquad \alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + \frac{b_1}{n^4} + \ldots$$

In 1950 and 1952, V.A. Marchenko used transformation operators to investigate inverse problems. He proved that the spectral function of a Sturm–Liouville operator (given in a half-line or finite interval) uniquely determines the operator [85, 86]. These operators resulted from general ideas of the theory of generalized transmutation operators, whose foundations were laid by the French mathematician Jean Frederic Auguste Delsarte.

Marchenko's theorem contains both G. Borg's and N. Levinson's theorems, and fully explains the following phenomenon discovered by Bargmann: the spectral function is determined by the scattering phase only for positive values of the spectral parameter.

P' 1951, 1952, and 1953 M.G. Krein developed an efficient method to construct the Sturm–Liouville operator by the spectral function and two spectra [62, 63, 64, 65, 66].

In 1953, M.Sh. Blokh investigated the inverse problem in the entire axis using a spectral matrix-function [23].

In 1956, B.Ya. Levin introduced transformation operators preserving asymptotics of solutions at infinity [78].

In 1955, V.A. Marchenko used the new transformation operators introduced by B.Ya. Levin to solve the inverse problem of scattering [87].

In 1960, Z.S. Agranovich and V.A. Marchenko studied the inverse problem of the quantum theory of scattering in an axis and semi-axis [2].

In 1962, A.S. Alekseev tackled a one-dimensional inverse dynamic problem of elasticity theory in a spectral domain [3].

In 1964, B.M. Levitan and M.G. Gasymov constructed a Sturm–Liouville operator by two spectra on a finite interval [79].

2.2 Inverse scattering method

In 1954, E. Fermi, J. Pasta, and S. Ulam computationally found an anomalously slow stochastization of a dynamic system in the form of a chain of nonlinear oscillators [34].

In 1965, M.D. Kruskal and N.J. Zabussky detected, by means of numerical simulation, that the collisions of solitons in the Korteweg–de Vries equation are elastic. As a result, infinite series of conservation laws were discovered soon after that [68].

In 1967, C.S. Gardner, J.M. Green, M.D. Kruskal, and R.M. Miura developed the method of inverse scattering [37] and integrated the Korteweg–de Vries equation:

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \in \mathbb{R}, t > 0$$

with the initial condition

$$u(x, 0) = f(x).$$

The Korteweg–de Vries equation is integrated by going from the potential of the one-dimensional Schrodinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + q(x)\psi = k^2\psi$$

to the reflection coefficient $r(k)$ on this potential.

In 1968, P. Lax discovered an algebraic mechanism forming a basis of the method of inverse scattering [74, 75].

In 1971, C.S. Gardner constructed a theory of the Korteweg–de Vries equation as a Hamiltonian system [38].

In 1971, the method of inverse scattering was applied by V.E. Zakharov and A.B. Shabat to the nonlinear Schrodinger equation [117].

In the same year, V.E. Zakharov and L.D. Faddeev, independently of C.S. Gardner, constructed a theory of the Korteweg–de Vries equation as a Hamiltonian system [118].

In 1973, A.B. Shabat constructed a class of quasi-linear equations that can be reduced to linear equations [108].

In 1974, S.P. Novikov [97], P.D. Lax [76], and V.A. Marchenko [89] studied a periodic Cauchy problem for the Korteweg–de Vries equation.

Simultaneously with the first Novikov's paper, there appeared a paper by P. Lax whose main result contained a part of the main Novikov's theorem. The proof proposed by P. Lax was inefficient and differed from Novikov's method. In two papers published by Marchenko, a method of successive approximations based on the spectral theory of the Schrodinger operator L was developed and used to solve the Korteweg–de Vries equation. Some ideas of Marchenko's paper overlap with some technical details of Novikov's paper.

The only new result for the periodic problem obtained from the method introduced by C.S. Gardner, J.M. Green, M.D. Kruskal, and R.M. Miura was the Faddeev–Zakharov theorem: the eigenvalues of the operator L are commutating integrals of the Korteweg–de Vries equation as a Hamiltonian system.

In 1974, V.E. Zakharov and A.B. Shabat proposed a general scheme to apply the method of inverse scattering for the integration of nonlinear differential equations and a search algorithm for equations allowing such integration [121].

In the same year, it was shown by V.E. Zakharov and S.V. Manakov that the nonlinear Shrodinger equation considered as a Hamiltonian system is fully integrable. This can be done using a scattering matrix of the one-dimensional Dirac operator [122].

In 1976, Manakov generalized the Lax pair for two-dimensional time dependent equations [84].

In 1976, it was shown by Zakharov and Manakov that every one-dimensional differential operator whose coefficient functions depend on an arbitrary set of parameters can be associated with a series of multidimensional nonlinear partial differential equations integrable by the method of inverse scattering [119].

In 1976, P. Lax considered almost periodic behavior (in time) of the periodic solutions to the Korteweg–de Vries equation. He presented a new proof based on Lennart's recursion [77].

In 1979, the method developed in 1976 was extended by Zakharov and Shabat to spectral problems as rational functions of the spectral parameter. They obtained a description of new classes of equations integrable by the method of inverse scattering and an algorithm to construct their exact solutions [120].

In the early 1970s, new nonlinear equations integrable by the method of inverse scattering (in particular, the nonlinear Shrodinger equation, the sin-Gordon equation, etc.) were found.

In 1980, V.E. Zakharov, S.V. Manakov, S.P. Novikov, and L.P. Pitaevskii provided a systematic description of the method of inverse scattering with all available mathematical information in their book "Theory of Solitons: The Inverse Scattering Method" [116].

In 1982, L.P. Nizhnik and M.D. Pochinayko investigated the nonlinear two-dimensional (in space) Shrodinger equation, and used the inverse scattering method for its integration [96].

In 1984, A.P. Veselov and S.P. Novikov considered a two-dimensional generalization of the Korteweg–de Vries equation (the Veselov–Novikov equation) with the help of the two-dimensional potential Shrodinger operator [114].

In 1984, R.G. Novikov and G.M. Henkin applied (and adapted) the inverse scattering method to obtain weakly localized solutions to a Korteweg–de Vries equation in which the transmission coefficient of the scattering matrix may be zero for a finite set of pulses [98].

In 1992, J-P Francoise and R.G. Novikov investigated the hierarchy of the Calogero–Moser system for the Kadomtsev–Petviashvili and Veselov–Novikov equations [36].

In 2011, A.V. Kazeykina and R.G. Novikov, studying the asymptotics of solutions to a Cauchy problem for the Veselov–Novikov equation with positive energy, showed that there are no isolated soliton-type waves at large times, in comparison to the well-known asymptotics of the solutions to a Korteweg–de Vries equation with reflectionless initial data [59].

2.3 Inverse problems for hyperbolic equations

In 1954, M.G. Krein considered the so-called string problem and formulated theorems on the solvability of an inverse problem [67]. The nonlinear inverse problem for the string equation was reduced to an integral equation (the Krein equation).

In 1968, B.S. Pariiskii studied a one-dimensional inverse problem for the wave equation [99] with a perturbation at some depth, and derived the Krein equation.

In 1971, A.S. Blagoveshchenskii provided another proof of Krein's results in the theory of inverse problems for the string equation. He showed that the dependence of the sought-for coefficient on the additional information is local [21]. Blagoveshchenskii showed that Pariiskii had not finished the proof and the proof contained errors.

Earlier Fourier (or Laplace) transforms in time had been used for such problems, and the differential equation coefficient had actually been reconstructed from the properties of the eigenfunctions of the corresponding spectral problem.

In 1970 and 1971, B. Gopinath and M. Sondi, independently of each other, proposed an alternative integral equation (also in a time domain) for reconstructing human speech from acoustic measurements [41, 42].

In 1975, A.S. Alekseev and V.I. Dobrinskii used a discrete analog of the Gelfand–Levitan method to study numerical algorithms for solving the one-dimensional inverse dynamic problem of seismology [4].

In 1977, B.S. Pariiskii published a detailed review of the numerical methods for solving Gelfand–Levitan equations [100].

In 1979, W. Symes applied nonlinear integral equations in a time domain [112].

In 1980, R. Burridge attempted to apply the Gelfand–Levitan–Marchenko equations for elasticity theory in a time domain, and found a relation between them and the Gopinath–Sondi equation [27].

In 1982, Santosa [107] developed an exact method for solving an inverse problem of plane wave propagation by the Gelfand–Levitan method, tested a numerical scheme for solving the integral equation, investigated the stability, and analyzed the numerical errors and approximations.

In 1988, S.I. Kabanikhin proposed a new algorithm for solving the Gelfand–Levitan equation using a condition sufficient for solvability of the inverse problem [47].

In 1991, V.G. Romanov and S.I. Kabanikhin applied a dynamic version of the Gelfand–Levitan method to the one-dimensional inverse problem of geoelectrics for a quasi-steady approximation of the system of Maxwell equations [105].

In 1998, the spectral method was used by A.S. Alekseev and V.S. Belonosov to reconstruct the acoustic impedance in a one-dimensional problem of wave propagation theory [5].

First multi-dimensional analogs of the Gelfand–Levitan–Krein equations were developed in 1987 by M.I. Belishev, who created a boundary control method [11].

In 1988 and 1989, S.I. Kabanikhin proposed a multi-dimensional analog of the Gelfand–Levitan–Krein equations [47, 48].

In 1992, M.I. Belishev and A.S. Blagoveshchenskii proposed a multi-dimensional analog of the Gelfand–Levitan equation on the basis of the boundary control method [12].

In 2004, it was shown by S.I. Kabanikhin and M.A. Shishlenin [53] that the discrete analogs of the Krein and boundary control equations are the same for the one-dimensional coefficient inverse problem of acoustics.

In 2005, S.I. Kabanikhin, M.A. Satybaev, and M.A. Shishlenin published a paper on numerical methods for solving two-dimensional analogs of the Gelfand–Levitan–Krein equation for coefficient inverse problems for the wave and acoustics equations [54].

In 2011, S.I. Kabanikhin and M.A. Shishlenin developed a numerical method for solving the Krein equation for the Nth approximation of the two-dimensional inverse problem of acoustics [56]. The Krein equation for the Nth approximation was obtained in a matrix form for which a numerical method was constructed on the basis of the singular value decomposition method.

In 2011, M.A. Shishlenin and N.S. Novikov performed a comparative analysis of two numerical methods for solving the Gelfand–Levitan equation [109], and developed a Monte Carlo method for solving the Gelfand–Levitan equation.

An advantage of the Gelfand–Levitan–Krein approach for solving coefficient inverse problems for hyperbolic equations is that the direct problems need not be solved many times. Here one should note the boundary control method [13, 15, 16] and a globally convergent method of [9, 10].

3 One-dimensional problems

3.1 The Gelfand–Levitan equation

The Gelfand–Levitan equation results from solving an inverse Sturm–Liouville problem and a dynamic coefficient inverse problem for the string equation.

Consider the direct Sturm–Liouville problem, which is in studying the spectrum of operator l_q,

$$l_q y(x) := -y'' + q(x)y,$$

defined on a set of functions $y \in W_2^2(0,\pi)$ and satisfying the relations

$$l_q y(x) = \lambda y, \quad x \in (0,\pi), \tag{1}$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0. \tag{2}$$

Here $h,\ H \in \mathbb{R}$, $q(x) \in L_2(0,\pi)$.

Let λ_n be an eigenvalue and $\varphi(x,\lambda_n)$, an eigenfunction of the operator l_q and $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$. Let

$$\alpha_n = \int\limits_0^\pi \varphi^2(x,\lambda_n)dx,$$

The set $\{\lambda_n, \alpha_n\}_{n\ge 0}$ is called the spectral data of the operator l_q,
with the following asymptotic properties [115]:

$$\sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\beta_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{\beta_{1n}}{n}, \quad \{\beta_n\}, \{\beta_{1n}\} \in l_2,$$

$$\alpha_n > 0, \qquad \lambda_n \neq \lambda_m, \quad (n \neq m).$$

The inverse Sturm–Liouville problem is in reconstructing the potential $q(x)$ and coefficients h, H from the spectral data.

Define a function $F(x,t)$:

$$F(x,t)=\sum_{n=0}^{\infty}\Big(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n}-\frac{\cos nx\cos nt}{\alpha_n^0}\Big).$$

Here

$$\alpha_n^0=\begin{cases}\frac{\pi}{2}, & n>0,\\ \pi, & n=0.\end{cases}$$

The inverse Sturm–Liouville problem is reduced to solving the Gelfand–Levitan equation

$$G(x,t)+F(x,t)+\int_0^x G(x,s)F(s,t)ds=0,\quad 0<t<x.$$

The solution to the Gelfand–Levitan equation makes it possible to determine the solution to the inverse problem by the formula

$$q(x)=2\frac{d}{dx}G(x,x),\quad h=G(+0,+0),\quad H=\omega-h-\frac{1}{2}\int_0^\pi q(t)\mathrm{d}t.$$

Consider the following coefficient dynamic inverse problem: find an even $q(x)$ from the relations

$$u_{tt}=u_{xx}-q(x)u,\quad x\in\mathbb{R},\ t>0; \tag{3}$$

$$u|_{t=0}=0,\quad u_t|_{t=0}=\delta(x); \tag{4}$$

$$u|_{x=0}=f(t). \tag{5}$$

In [104], the inverse problem is reduced to the Gelfand–Levitan equation:

$$\tilde{w}(x,t)+\int_0^x[f'(t-\tau)+f'(t+\tau)]\tilde{w}(x,\tau)d\tau=$$

$$=-\frac{1}{2}[f'(t-x)+f'(t+x)],\quad t\in[0,x).$$

Here $f(t)$ is oddly extended to the negative t and the derivative of $f(t)$ is taken at the points of continuity.

The solution to the inverse problem (3)—(5) is found by the formula

$$q(x)=4\frac{\mathrm{d}}{\mathrm{d}x}\tilde{w}(x,x-0),\quad x>0.$$

3.2 Krein's equation

Consider the inverse problem of finding the acoustic stiffness $\sigma(x)$ from the relations

$$u_{tt}=u_{xx}-\frac{\sigma'(x)}{\sigma(x)}u_x,\qquad x>0,\quad t>0; \tag{6}$$

$$u\Big|_{t<0}\equiv 0; \tag{7}$$

$$u_x\Big|_{x=+0}=\delta(t); \tag{8}$$

$$u(+0,t)=r(t). \tag{9}$$

In paper [21], the inverse problem of acoustics is reduced to the Krein equation

$$-2r(+0)\Phi(x,t) - \int_{-x}^{x} r'(t-s)\Phi(x,s)\mathrm{d}s = \frac{1}{\sigma(+0)}, \qquad |t| < x,$$

in which the function $r(t)$ is oddly extended to the negative t.

The solution to the inverse problem (6)—(9) is found by the formula

$$\sigma(x) = \frac{1}{4\sigma(+0)\Phi^2(x,x-0)}.$$

3.3 Marchenko's equation. The inverse scattering method

Marchenko's equation emerges when solving inverse problems of scattering and integrating nonlinear wave processes.

Let us consider the direct scattering problem: by given $q(x)$ such that $q(x) < 0$ for $x \to -\infty$ and $q(x) \to 0$ for $x \to \infty$ find eigenfunctions and eigenvalues of the problem according to [69]:

$$-y'' + q(x)y = k^2 y. \tag{10}$$

The equation (10) for $q(x) < 0$ and $k^2 > 0$ has a continuous spectrum of eigenvalues. If $k^2 < 0$ then spectrum is discrete.

We suppose that

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|\mathrm{d}x < \infty.$$

The equation (10) has a fundamental system of the solutions:

$$\psi_1(x,k) \cong \mathrm{e}^{-\mathrm{i}kx} + o(1), \qquad \psi_2(x,k) \cong \mathrm{e}^{\mathrm{i}kx} + o(1), \qquad x \to +\infty, \tag{11}$$

$$\varphi_1(x,k) \cong \mathrm{e}^{-\mathrm{i}kx} + o(1), \qquad \varphi_2(x,k) \cong \mathrm{e}^{\mathrm{i}kx} + o(1), \qquad x \to -\infty. \tag{12}$$

Note, that we have the following equations

$$\psi_1(x,k) = \psi_2^*(x,k) = \psi_2(x,-k), \tag{13}$$

$$\varphi_1(x,k) = \varphi_2^*(x,k) = \varphi_2^*(x,-k). \tag{14}$$

Here * is the complex conjugation.

The equation (10) has a two linear independent solutions, therefore each solution can be represented as a linear combination

$$\varphi(x,k) = a(k)\psi(x,k) + b(k)\psi^*(x,k), \tag{15}$$

$$\varphi^*(x,k) = c(k)\psi(x,k) + d(k)\psi^*(x,k). \tag{16}$$

Therefore we have, that

$$d(k) = a^*(k), \qquad c(k) = b^*(k).$$

It follows from (15) that

$$\frac{\varphi(x,k)}{a(k)} = \psi(x,k) + \frac{b(k)}{a(k)}\psi^*(x,k), \tag{17}$$

and using asymptotic for $x \to +\infty$ we obtain, that

$$\frac{\varphi(x,k)}{a(k)} = \mathrm{e}^{-\mathrm{i}kx} + r(k)\mathrm{e}^{\mathrm{i}kx} + o(1). \tag{18}$$

Here $r(k) = b(k)/a(k)$ is the reflection coefficient.

For $x \to -\infty$ we have

$$\varphi(x,k) \cong \mathrm{e}^{-\mathrm{i}kx}. \tag{19}$$

Then the last wave has the form

$$\frac{\varphi(x,k)}{a(k)} \cong t(k)\mathrm{e}^{-\mathrm{i}kx}.$$

Here $t(k) = 1/a(k)$ is the completion rate.

The discrete spectrum of Schrodinger operator is $k_n^2 = -\chi_n^2$ $(\chi_n > 0)$. If we consider the asymptotic for $x \to -\infty$ in the form

$$\varphi^{(n)}(x) = \mathrm{e}^{\chi_n x} + o(\mathrm{e}^{\chi_n x}), \tag{20}$$

then for $x \to +\infty$ we have the wave function in the form

$$\varphi^{(n)}(x) = b_n \mathrm{e}^{-\chi_n x} + o(\mathrm{e}^{-\chi_n x})$$

Eigenfunctions corresponding to the discrete spectrum and the eigenvalues are real-valued.

Let us enumerate eigenvalues

$$\chi_1^2 > \chi_2^2 > \ldots > \chi_n^2 > 0$$

and suppose that $\varphi^{(1)}(x)$ wave function corresponding to the χ_1^2 has no zeros for $x \in (-\infty,\infty)$. Then $\varphi^{(n)}(x)$ has $(n-1)$ zeros using oscillatory theorem [70] and we have that $b_n = (-1)^{n-1}|b_n|$.

The set of quantities

$$S = \{r(k),\ \chi_n,\ |b_n|,\ n = \overline{1,N}\}$$

are called scattering data.

The direct scattering problem is to find scattering data by given $q(x)$.

Let us consider the following functions

$$\chi_-(x,k) = \varphi(x,k)\mathrm{e}^{\mathrm{i}kx}, \tag{21}$$

$$\chi_+(x,k) = \psi(x,k)\mathrm{e}^{\mathrm{i}kx}, \tag{22}$$

and the following boundary conditions holds

$$\lim_{x\to-\infty}\chi_-(x,k)=1, \tag{23}$$
$$\lim_{x\to+\infty}\chi_+(x,k)=1. \tag{24}$$

Therefore, functions are the solutions of the following integral equations

$$\varphi(x,k)=\mathrm{e}^{-\mathrm{i}kx}+\int\limits_{-\infty}^{x}\frac{\sin k(x-\xi)}{k}q(\xi)\varphi(\xi,k)\mathrm{d}\xi, \tag{25}$$
$$\psi(x,k)=\mathrm{e}^{-\mathrm{i}kx}-\int\limits_{x}^{+\infty}\frac{\sin k(x-\xi)}{k}q(\xi)\psi(\xi,k)\mathrm{d}\xi. \tag{26}$$

Functions $\chi_-(x,k)$ and $\chi_+(x,k)$ solve the following equation

$$-\chi_{\pm xx}(x,k)+2\mathrm{i}k\chi_{\pm x}(x,k)+q(x)\chi_\pm(x,k)=0, \tag{27}$$

and instead of (25) and (26) we obtain

$$\chi_-(x,k)=1+\int\limits_{-\infty}^{x}\frac{\mathrm{e}^{2\mathrm{i}k(x-\xi)}-1}{2\mathrm{i}k}q(\xi)\chi_-(\xi,k)\mathrm{d}\xi. \tag{28}$$

$$\chi_+(x,k)=1-\int\limits_{x}^{\infty}\frac{\mathrm{e}^{2\mathrm{i}k(x-\xi)}-1}{2\mathrm{i}k}q(\xi)\chi_+(\xi,k)\mathrm{d}\xi. \tag{29}$$

For $k\to\infty$ we have

$$\chi_+(x,k)=1+\int\limits_{x}^{\infty}\frac{q(\xi)}{2\mathrm{i}k}\mathrm{d}\xi+o\Big(\frac{1}{k^2}\Big). \tag{30}$$

It follows from (15) that

$$\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}=\psi(x,k)\mathrm{e}^{\mathrm{i}ky}+r(k)\psi^*(x,k)\mathrm{e}^{\mathrm{i}ky}. \tag{31}$$

Let us integrate (31) with respect to k:

$$\int\limits_{-\infty}^{\infty}\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k=\int\limits_{-\infty}^{\infty}\psi(x,k)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}k+\int\limits_{-\infty}^{\infty}r(k)\psi^*(x,k)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}k. \tag{32}$$

In the left-hand side of (32)

$$\int\limits_{-\infty}^{\infty}\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k=2\pi\mathrm{i}\sum_{n=1}^{N}\frac{\varphi(x,\mathrm{i}\chi_n)}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n y}. \tag{33}$$

We have

$$\varphi(x,\mathrm{i}\chi_n)=b_n\psi^*(x,\mathrm{i}\chi_n)=b_n\psi(x,-\mathrm{i}\chi_n), \tag{34}$$

then (33) can be rewritten in the form

$$\int\limits_{-\infty}^{\infty}\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k=2\pi\mathrm{i}\sum_{n=1}^{N}\frac{b_n\psi(x,-\mathrm{i}\chi_n)}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n y}. \tag{35}$$

Let us introduce new function

$$\psi(x,k)=\mathrm{e}^{-\mathrm{i}kx}+\int\limits_{x}^{\infty}K(x,y)\mathrm{e}^{-\mathrm{i}ky}\mathrm{d}y, \tag{36}$$

then (35) is rewritten as follows

$$\int\limits_{-\infty}^{\infty}\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k=2\pi\mathrm{i}\sum_{n=1}^{N}\frac{b_n\mathrm{e}^{-\chi_n y}}{a_n(\mathrm{i}\chi_n)}\left[\mathrm{e}^{-\chi_n x}+\int\limits_{x}^{\infty}K(x,z)\mathrm{e}^{-\chi_n z}\mathrm{d}z\right]. \tag{37}$$

Otherwise, it follows from (32)

$$\int\limits_{-\infty}^{\infty}\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k=2\pi\int\limits_{x}^{\infty}K(x,z)\delta(y-z)\mathrm{d}z+$$
$$+\int\limits_{-\infty}^{\infty}r(k)\mathrm{e}^{\mathrm{i}k(y+x)}\mathrm{d}k+\int\limits_{x}^{\infty}K(x,z)\left[\int\limits_{-\infty}^{\infty}r(k)\mathrm{e}^{\mathrm{i}k(z+y)}\mathrm{d}k\right]\mathrm{d}z. \tag{38}$$

Therefore, we obtain

$$2\pi\mathrm{i}\sum_{n=1}^{N}\frac{b_n}{a_k(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n(x+y)}+2\pi\mathrm{i}\int\limits_{x}^{\infty}K(x,z)\sum_{n=1}^{\infty}\frac{b_n}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n(x+y)}\mathrm{d}z=$$
$$=2\pi K(x,y)+\int\limits_{-\infty}^{\infty}r(k)\mathrm{e}^{\mathrm{i}k(y+x)}\mathrm{d}k+\int\limits_{-\infty}^{\infty}K(x,z)\left[\int\limits_{-\infty}^{\infty}r(k)\mathrm{e}^{\mathrm{i}k(z+y)}\mathrm{d}k\right]\mathrm{d}z. \tag{39}$$

Define the function $F(x)$ which consists of scattering data

$$F(x)=\sum_{n=1}^{N}\frac{b_n\mathrm{e}^{-\chi_n x}}{\mathrm{i}a_n(\mathrm{i}\chi_n)}+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}r(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k, \tag{40}$$

then the equation (39) can be rewritten in the form

$$K(x,y)+F(x+y)+\int\limits_{x}^{\infty}K(x,s)F(s+y)\mathrm{d}s=0. \tag{41}$$

The equation (41) is the Marchenko integral equation.

Using (36) we obtain that

$$\chi_+(x,k) = 1 + \int\limits_x^\infty K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{d}y = 1 - \frac{1}{\mathrm{i}k}K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)}\Big|_{y=x}^{y=\infty} + o\Big(\frac{1}{k}\Big). \tag{42}$$

We have

$$\frac{1}{\mathrm{i}k}K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)} \to 0, \quad y \to \infty$$

then

$$\chi_+(x,k) = 1 + \frac{1}{\mathrm{i}k}K(x,x). \tag{43}$$

It follows from (30) that

$$K(x,x) = \frac{1}{2}\int\limits_x^\infty q(\xi)\mathrm{d}\xi. \tag{44}$$

Then the function $q(x)$ in the equation is reconstructed by the formula

$$q(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x).$$

Consider the inverse scattering problem, which is in studying some nonlinear equations of mathematical physics. The method, proposed by C.S. Gardner, J.M. Green, M.D. Kruskal, and R.M. Miura in 1967 [37], represents the nonlinear equation under study as a compatibility condition for a system of linear equations. An initial version of the method based on the theory of scattering for differential operators (hence, the name of the method) was applied to the Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{45}$$

can be represented in the form of the solution of linear equations

$$\psi_{xx} + (\lambda - u)\psi = 0, \tag{46}$$

$$\psi_t + \psi_{xxx} - 3(\lambda + u)\psi_x = C(t)\psi. \tag{47}$$

It is a compatibility condition of the overspecified linear system of equations

$$(L - \lambda)\psi = 0, \tag{48}$$

$$\psi_t + A\psi = 0,$$

where

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x,t), \quad A = \frac{\mathrm{d}^3}{\mathrm{d}x^3} - 3\left[u\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}u\right],$$

and is equivalent to the operator relation (Lax's representation)

$$\frac{\partial L}{\partial t} = [L, A]. \tag{49}$$

Let us consider the initial condition for the Korteweg–de Vries equation (45)

$$u(x,0) = f(x). \tag{50}$$

We suppose that

$$\int_{-\infty}^{\infty} (1+|x|)|f(x)|\mathrm{d}x < \infty.$$

Let for known $f(x)$ we find from the scattering data

$$S_n(0) = \{\lambda_n(0); r(k,0); b_n(0); n = \overline{1,N}\}. \tag{51}$$

Wave function in equations (46), (47) depends on the time variable t:

$$\varphi(x,k,t) = a(k,t)\psi(x,k,t) + b(k,t)\psi^*(x,k,t), \tag{52}$$

and we have the following asymptotics when $x \to +\infty$

$$\varphi(x,k,t) = a(k,t)\mathrm{e}^{-\mathrm{i}kx} + b(k,t)\mathrm{e}^{\mathrm{i}kx} + o(1). \tag{53}$$

Substituting (53) in (47) we obtain

$$\dot{a} + 4\mathrm{i}k^3 a - ca = 0, \tag{54}$$

$$\dot{b} - 4\mathrm{i}k^3 b - cb = 0. \tag{55}$$

Therefore solving (54) and (55) we obtain

$$r(k,t) = \frac{b(k,t)}{a(k,t)} = r(k,0)\mathrm{e}^{8\mathrm{i}k^3 t}. \tag{56}$$

$$b_n(t) = b_n(0)\mathrm{e}^{8\chi_n^3 t}, \qquad n = \overline{1,N}. \tag{57}$$

Therefore if by known initial data $f(x)$ we find $S(t=0)$, then $S(t)$ has the following form

$$S(t) = \Big\{ r(k,0)\mathrm{e}^{8\mathrm{i}k^3 t};\ \chi_n(0);\ b_n(0)\mathrm{e}^{8\chi_n^3 t}, \qquad n = \overline{1,N} \Big\}. \tag{58}$$

Let us denote

$$F(x,t) = \sum_{n=1}^{N} \frac{b_n(0)\mathrm{e}^{-\chi_n x + 8\chi_n^3 t}}{\mathrm{i}a_n(\mathrm{i}\chi_n)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k,0)\mathrm{e}^{\mathrm{i}kx+8\mathrm{i}k^3 t}\mathrm{d}k. \tag{59}$$

Then we solve the Marchenko integral equation for the function $K(x,y,t)$ to solve the inverse problem of scattering:

$$K(x,y,t) + F(x+y,t) + \int_{x}^{\infty} K(x,s,t)F(s+y,t)\mathrm{d}s = 0. \tag{60}$$

Then we find the solution of KdV equation by the formula

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t). \tag{61}$$

Therefore, on the first step we find the scattering data $S(t=0)$ for known $f(x)$. On the second step we find the scattering data $S(t)$. Then the function $F(x,t)$ is found by (59). On the fourth step we solve the Marchenko equation (60). On the last step we find the solution KdV equation by the formula (61).

To solve these problems efficiently, numerical calculations are required. An advantage of the inverse scattering method is that it allows advancing in time as far as possible without loss of accuracy.

$$\begin{tikzcd} \begin{array}{c}\text{The Cauchy problem}\\ u_t-6uu_x+u_{xxx}=0\\ u(0,x)=u_0(x)\end{array} \arrow[r] \arrow[d, "\text{Direct problem}"'] & \begin{array}{l}\text{The Cauchy problem}\\ \text{solution: } u(x,t)\end{array} \\ \begin{array}{l}\text{Inverse scattering}\\ \quad\text{data } t=0\text{: } S_0\end{array} \arrow[r, "\text{the evolution by } t", "\text{solved explicitly}"'] & \begin{array}{l}\text{Scattering data}\\ \text{for arbitrary } t\text{: } S(t)\end{array} \arrow[u, "\text{Inverse problem}"'] \end{tikzcd} \tag{62}$$

4 A two-dimensional analog of the Gelfand–Levitan equation

Consider the following sequence of direct problems ($k = 0, \pm 1, \pm 2, \ldots$):

$$u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)} - q(x,y)u^{(k)}, \qquad x \in \mathrm{R}, \quad y \in \mathrm{R}, \quad t > 0; \tag{63}$$

$$u^{(k)}|_{t=0} = 0, \quad u_t^{(k)}|_{t=0} = \delta(x)\mathrm{e}^{\mathrm{i}ky}. \tag{64}$$

We suppose that $q(x,y)$ is 2π-periodic function with respect to y.

Consider an **inverse problem**: determine the even function $q(x,y)$ from the additional information

$$u^{(k)}|_{x=0} = f^{(k)}(y,t), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{65}$$

Uniqueness of the solution to the inverse problem (63)—(65) can be proved using a technique proposed in [110, 103] in classes of analytic functions.

Consider the following auxiliary sequence of direct problems ($m = 0, \pm 1, \pm 2, \ldots$) [49, 54]:

$$w_{tt}^{(m)} = w_{xx}^{(m)} + w_{yy}^{(m)} - q(x,y)w^{(m)}, \qquad x > 0, \quad y \in \mathrm{R}, \quad t \in \mathrm{R}; \tag{66}$$

$$w^{(m)}|_{x=0} = \mathrm{e}^{\mathrm{i}my}\delta(t), \qquad w_x^{(m)}|_{x=0} = 0. \tag{67}$$

Now, using the d'Alembert formula for problem (66), (67), we can obtain [49, 54]:

$$w^{(m)}(x,y,t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}[\delta(t-x)+\delta(t+x)]+ \\ + \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \big[-w_{yy}^{(m)} + q(x,y)w^{(m)}\big](\xi,y,\tau)\mathrm{d}\xi\mathrm{d}\tau. \tag{68}$$

The following condition take place: $w^{(m)}(x,y,t) \equiv 0$, $0 < |x| < t$, $y \in \mathbb{R}$.

We denote

$$\tilde{w}^{(m)}(x,y,t) = w^{(m)}(x,y,t) - \frac{1}{2}\mathrm{e}^{\mathrm{i}my}\big[\delta(t-x)+\delta(t+x)\big]. \tag{69}$$

Using (103) in (68), we can obtain:

$$\tilde{w}^{(m)}(x,y,t) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\theta(x-|t|)\left[xm^2 + Q(x,y,t)\right] + \\ + \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \big[-\tilde{w}_{yy}^{(m)} + q(x,y)\tilde{w}^{(m)}\big](\xi,y,\tau)\mathrm{d}\xi\mathrm{d}\tau. \tag{70}$$

Here

$$Q(x,y,t) = \int_0^{\frac{x+t}{2}} q(\xi,y)\mathrm{d}\xi + \int_0^{\frac{x-t}{2}} q(\xi,y)\mathrm{d}\xi.$$

Then

$$\tilde{w}^{(m)}(x,y,x-0) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\left[xm^2 + \int_0^x q(\xi,y)\mathrm{d}\xi\right]. \tag{71}$$

The inverse problem (63)—(65) can be reduced formally to a system of integral equations ($k = 0, \pm1, \pm2, \ldots$) of the first kind

$$\int\limits_{-x}^{x} \sum_m f_m^{(k)}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = -\frac{1}{2}\left[f^{(k)}(y,t-x) + f^{(k)}(y,t+x)\right], \tag{72}$$

or the second kind

$$\tilde{w}^{(k)}(x,y,t) + \int\limits_{-x}^{x} \sum_m f_m^{(k)\prime}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = \\ = -\frac{1}{2}\left[f_t^{(k)}(y,t-x) + f_t^{(k)}(y,t+x)\right]. \tag{73}$$

Here $|t| < x$, $y \in \mathrm{R}$.

The systems of equations (72) and (73) are two-dimensional analogs of the Gelfand–Levitan equation.

Note that, according to formula (71), $q(x,y)$ can be calculated, for instance, by the formula

$$q(x,y) = 4\frac{\mathrm{d}}{\mathrm{d}x}\tilde{w}^{(0)}(x,y,x-0).$$

5 A two-dimensional analog of the Krein equation

Consider the following sequence of the direct problems ($k = 0, \pm 1, \pm 2, \dots$):

$$u^{(k)}_{tt} = u^{(k)}_{xx} + u^{(k)}_{yy} - \nabla \ln \rho(x,y)\, \nabla u^{(k)}, \quad x > 0,\ y \in \mathrm{R},\ t > 0; \tag{74}$$

$$u^{(k)}|_{t<0} \equiv 0, \qquad u^{(k)}_x(+0,y,t) = \mathrm{e}^{\mathrm{i}ky}\, \delta(t); \tag{75}$$

An **inverse problem** is to determine the function $\rho(x,y)$ from the additional information

$$u^{(k)}(+0,y,t) = f^{(k)}(y,t), \qquad k = 0, \pm 1, \pm 2, \dots \tag{76}$$

We suppose, that $\ln \rho(x,y)$ is 2π-periodic function.

The necessary condition of the inverse problem's solvability can be obtained [47, 48, 56]:

$$f^{(k)}(y,+0) = -\mathrm{e}^{\mathrm{i}ky}, y \in (-\pi,\pi), k \in \mathbb{Z}. \tag{77}$$

We also consider the following sequence of the auxiliary problem:

$$w^{(m)}_{tt} = w^{(m)}_{xx} + w^{(m)}_{yy} - \nabla \ln \rho(x,y)\, \nabla w^{(m)}, \qquad x > 0, y \in (-\pi,\pi), t \in \mathbb{R}, m \in \mathbb{Z}, \tag{78}$$

$$w^{(m)}|_{x=0} = \mathrm{e}^{\mathrm{i}my}\delta(t), \qquad w^{(m)}_x|_{x=0} = 0. \tag{79}$$

Then we have the following form of the solution of the problem (78)-(79):

$$w^{(m)}(x,y,t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}\sqrt{\frac{\rho(x,y)}{\rho(0,y)}}[\delta(x+t) + \delta(x-t)] + \widetilde{w}^{(m)}(x,y,t) \tag{80}$$

Here $\widetilde{w}^{(m)}(x,y,t)$ - piecewise-smooth function.

Solutions of the sequences (74), (75) and (78), (79) are connected:

$$u^{(k)}(x,y,t) = \int_R \sum_m f^{(k)}_m(t-s) w^{(m)}(x,y,s) ds, x > 0, y \in (-\pi,\pi), t \in \mathbb{R} \tag{81}$$

Here

$$f^{(k)}(y,t) = \sum_m f^{(k)}_m(t)\mathrm{e}^{\mathrm{i}my}.$$

Then we extends functions $f^{(k)}$ and $u^{(k)}$ for $t < 0$ as an odd continuation:

$$f^{(k)}(y,t) = -f^{(k)}(y,-t), t < 0$$
$$u^{(k)}(x,y,t) = -u^{(k)}(x,y,-t), t < 0$$

Now we apply the operator

$$\int_0^x \frac{(.)}{\rho(\xi,y)} d\xi$$

to the equality (81). Let us denote

$$V^{(m)}(x,y,t) = \int_0^x \frac{w^{(m)}(\xi,y,t)}{\rho(\xi,y)} d\xi, \tag{82}$$

Then we can obtain:

$$\frac{\partial}{\partial t}\int_0^x \frac{u^{(k)}(\xi,y,t)}{\rho(\xi,y)}d\xi = \frac{\partial}{\partial t}\int_R \sum_m V^{(m)}(x,y,s)f_m^{(k)}(t-s)ds =$$
$$= -2V^{(k)}(x,y,t) + \int_{-x}^{x}\sum_m V^{(m)}(x,y,s)f_m^{(k)\prime}(t-s)ds. \quad (83)$$

It was shown [48, 56], that the left part of the equation (83) does not depends on x, t, and satisfies to the following ratio:

$$\frac{\partial}{\partial t}\int_{-\pi}^{\pi}\int_0^x \frac{u^{(k)}(\xi,y,t)}{\rho(\xi,y)}d\xi dy = -\int_{-\pi}^{\pi}\frac{e^{iky}}{\rho(0,y)}\mathrm{d}y. \quad (84)$$

Let us denote

$$\Phi^{(m)}(x,t) = \int_{-\pi}^{\pi} V^{(m)}(x,y,t)\mathrm{d}y,$$

we can obtain from (84) and (83):

$$2\Phi^{(k)}(x,t) - \sum_m \int_{-x}^{x}(f_m^k)'(t-s)\Phi^{9m)}(x,s)ds = -\int_{-\pi}^{\pi}\frac{\mathrm{e}^{iky}}{\rho(0,y)}\mathrm{d}y, \quad k\in\mathbb{Z} \quad (85)$$

For every fix value of x the equation (85) is a linear Fredholm integral equation of the second kind. The set of equations (85) is the multi-dimensional analogue of the M.G. Krein equation [48, 54].

It was proved [49, 54], that

$$V^{(m)}(x,y,x-0) = \frac{\mathrm{e}^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}. \quad (86)$$

Therefore

$$\Phi^{(m)}(x,x-0) = \int_{-\pi}^{\pi}\frac{\mathrm{e}^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}\mathrm{d}y \quad (87)$$

The solution to the inverse problem $\rho(x,y)$ can be found by the formula

$$\rho(x,y) = \frac{\pi^2}{\rho(0,y)}\Big[\sum_m \Phi^m(x,x-0)\mathrm{e}^{-imy}\Big]^{-2} \quad (88)$$

To find the solution to the inverse problem $\rho(x,y)$ at a point $x_0>0$, we solve the system (85) setting $x=x_0$, and calculate $\rho(x_0,y)$ by the formula (88).

To numerically solve the two-dimensional analog of the Krein equation (see Figs. 1–4), we use the N th approximation [46, 55] of the Krein equation [54]. That is, in the system (85) we set $\Phi^k(x,t)\equiv 0$ for all $N<|k|$ [56].

Discrete analogs of the Gelfand–Levitan equation are investigated in [40, 94, 50, 51, 54].

6 A two-dimensional analog of the Marchenko equation

In this section, we consider two-dimensional generalizations of the Korteweg–de Vries equation and reduce the nonlinear equations to a two-dimensional analog of the Marchenko equation using the method of inverse scattering.

6.1 The Kadomtsev–Petviashvili equation

The Kadomtsev–Petviashvili equation (KP) is a nonlinear partial differential equation in two space and one time coordinates describing the evolution of long nonlinear waves of small amplitude as a slow function of the transverse coordinate. The Kadomtsev–Petviashvili equation may be written in normalized form as follows:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{89}$$

Here $u = u(x, y, t)$ is a function, and $\sigma^2 = \pm 1$.

There exist two different versions of the Kadomtsev–Petviashvili equation. At $\sigma = 1$ the equation is called KPII. It describes, for instance, waves on water with small surface tension.

At $\sigma = \mathrm{i}$, the equation is called KPI. It can be used to simulate waves in thin films with high surface tension. The equation often has different coefficients for different conditions. However, the specific values are insignificant, since they can be changed by scaling the variables.

This equation was first obtained in [58], as a model to study the evolution of long ionic-acoustic small-amplitude waves propagating in plasma under the action of long transverse perturbations. In the absence of transverse dynamics, this problem is described by the Korteweg–de Vries equation. The KP equation first came into wide use as a natural continuation of the classical KdV equation in two space dimensions and then, as a model for surface and inner waves on water [1], nonlinear optics [101], and other physical phenomena.

It was shown in papers [121, 43] that the Kadomtsev–Petviashvili equation

$$u_t + 6uu_x + u_{xxx} + 3\sigma^2 w_y = 0, \tag{90}$$

$$w_x = u_y, \tag{91}$$

on the half-plane $x \in \mathbb{R}$, $y > 0$, with the boundary condition

$$u_x + \sigma w|_{y=0} = 0, \tag{92}$$

is an integrable system.

The problem is reduced to the following Marchenko equation [121, 43]:

$$K(x, z, y, t) + F(x, z, y, t) + \int\limits_{-\infty}^{x} K(x, \xi, y, t) F(\xi, z, y, t) \mathrm{d}\xi = 0, \tag{93}$$

where the kernel $F(x,z,y,t)$ is a solution to the following system of partial differential equations:

$$\sigma F_y - F_{xx} + F_{zz} = 0,$$

$$F_t + 4(F_{xxx} + F_{zzz}) = 0.$$

Hence, the solution to the nonlinear wave equation can be found by the formula

$$u(x,y,t) = 2\frac{\partial}{\partial x}K(x,x,y,t).$$

6.2 The Veselov–Novikov equation

The Veselov–Novikov (VN) equation has the form

$$u_t = (uV)_z + (u\overline{V})_{\overline{z}} + u_{zzz} + u_{\overline{zzz}}, \qquad V_{\overline{z}} = -3u_z,$$

where $z = x + \mathrm{i}y$ was obtained in 1984 as a two-dimensional integrable extension of the KdV equation. The VN-equation is equivalent to a compatibility condition of the system of equations

$$\psi_{z\overline{z}} = u\psi, \qquad \psi_t = \psi_{zzz} + \psi_{\overline{zzz}} + V\psi_z + \overline{V}\psi_{\overline{z}}.$$

Applications of the Veselov–Novikov equation in differential geometry are well-known [35, 60]. It has been found that this equation describes light propagation in some class of nonlinear media in geometrical optics [61].

The VN equation generalizes the KdV equation

$$u_t = -6uu_x - u_{xxx}$$

in the sense that if $u(x,t)$ is a solution to the KdV equation and $\nu_x(x),t) = -3u_x(x,t)$, $u\,(x,t)$ is a solution to the VN equation.

The VN equation may be presented as a Manakov triplet [84]:

$$\dot{L} = [A, L] - BL \tag{94}$$

where

$$\begin{aligned} L &= -\Delta + q, \\ A &= 8\left(\partial^3 + \overline{\partial}^3\right) + 2\left(w\partial + \overline{w}\overline{\partial}\right), \\ B &= -2\left(\partial w + \overline{\partial}\overline{w}\right). \end{aligned}$$

Here B is the operator of multiplication by the function $2\left(\partial w + \overline{\partial}\overline{w}\right)$. Thus, the pair of functions (u, V) is a solution to the VN equation if and only if the relation (94) holds.

>From the above presentation as a Manakov triplet, one can conclude that formally the VN equation is a fully integrable equation.

The method of inverse scattering for the VN equation may be schematically shown as follows:

$$
\begin{tikzcd}
\mathbf{t}_0(k) \arrow[r, "\exp(i\tau(k^3+\overline{k}^3))(\cdot)"] & \mathbf{t}_\tau(k) \arrow[d, "\mathcal{Q}"] \\
q_0(z) \arrow[u, "\mathcal{T}"] \arrow[r, "\text{Veselov–Novikov equation}"] & q_\tau^{\mathrm{NV}}(z),
\end{tikzcd}
\tag{95}
$$

where $\mathcal{T}$ and $\mathcal{Q}$ are direct and inverse nonlinear Fourier transforms, respectively; $\mathbf{t}_\tau : \mathbb{C} \to \mathbb{C}$ is called the *scattering transformation* function. The method of inverse scattering was investigated by Boiti et al. [25], Tsai [106], Nachman [93], Bogdanov, Konopelchenko, Moro [24], Lassas-Mueller-Siltanen [72], Lassas-Mueller-Siltanen-Stahel [73, 71], Music [91], and Perry [102].

7 Numerical calculations

For numerical solution of the inverse problem (74)—(76), we used a regularization technique, based on a projection of the problem on N -dimensional subspace, produced by the basis $\{\mathrm{e}^{\mathrm{i}ky}\}_{k=0,\pm1,\dots,\pm N}$ [54, 55, 56]. This approach reduces the two-dimensional problem to a finite system of one-dimensional inverse problems [54].

We suppose that the solution of the problem (74), (75) can be represented as a series:

$$u^{(k)}(x,y,t) = \sum_m u_m^{(k)}(x,t)\mathrm{e}^{\mathrm{i}my}; \tag{96}$$

We also suppose that the function $\rho(x,y)$ also has the same representation:

$$\ln\rho(x,y) = \sum_m a_m(x)\mathrm{e}^{\mathrm{i}my};$$

In this case the problem (74)—(76) can be rewritten as follows

$$
\begin{gathered}
\frac{\partial^2 u_n^{(k)}}{\partial t^2} = \frac{\partial^2 u_n^{(k)}}{\partial x^2} - n^2 u_n^{(k)}(x,t) - \\
- \sum_{m\in\mathbb{Z}} \frac{\partial a_m}{\partial x}(x)\frac{\partial u_{n-m}^{(k)}}{\partial x}(x,t) + \sum_{m\in\mathbb{Z}} m(m-n)a_m(x)u_{n-m}^{(k)}(x,t) \\
x\in\mathbb{R},\ t>0,\ k,n\in\mathbb{Z};
\end{gathered}
\tag{97}
$$

$$u_n^{(k)}|_{t=0} = 0, \quad \frac{\partial u_n^{(k)}}{\partial t}|_{t=0} = \delta_{kn}\delta(x); \tag{98}$$

$$u_n^{(k)}|_{x=0} = f_n^{(k)}(t). \tag{99}$$

Here δ_{kn} is the Kronecker symbol:

$$\delta_{kn} = \begin{cases} 1, & k=n; \\ 0, & \text{else} \end{cases}$$

Now we suppose that all the Fourier coefficients with number greater than N vanish, and consider then the following problem:

$$\frac{\partial^2 \overline{V}_N^{(k)}}{\partial t^2} = \frac{\partial^2 \overline{V}_N^{(k)}}{\partial x^2} - K\overline{V}_N^{(k)} + A(x)\overline{V}_N^{(k)} - B(x)\frac{\partial \overline{V}_N^{(k)}}{\partial x}; \tag{100}$$

$$x > 0, \quad t > 0,$$

$$\overline{V}_N^{(k)}|_{t<0} \equiv 0, \qquad \frac{\partial \overline{V}_N^{(k)}}{\partial x}|_{x=0} = I_N^{(k)}\delta(t). \tag{101}$$

$$\overline{V}_N^{(k)}|_{x=0} = \overline{F}_N^{(k)} \tag{102}$$

The problem (100)-(102) is called an N-approximation of the inverse problem (74), (75). Here A, K, B are square matrices of size $2N+1$ with elements:

$$K_{km} = m^2\delta_{km};$$
$$A_{km}(x) = m(k-m)a_{k-m}(x), \qquad k, m = -N...N;$$
$$B_{km}(x) = \frac{\partial a_{k-m}}{\partial x}, \qquad k, m = -N...N.$$

Using the technique, proposed in [52], one can obtain that, as $N \to \infty$, the N-approximation converges to the solution of the system (74), (75).

The N-approximation of the Krein equation (85) can be also obtained:

$$\Phi^k(x,t) = \frac{1}{2}\int_{-x}^{x} \sum_{|m|<N} (f_m^k)'(t-s)\Phi^m(x,s)ds + G^k, k = -N, ..., N \tag{103}$$

Numerical calculations (see Figs. 1–4) are used to find an approximate solution to the inverse problem. The two-dimensional inverse problem is approximated by a finite system of one-dimensional inverse problems [54, 55, 56]. The problem is solved in the domain $x \in (0,1)$, $y \in (-\pi,\pi)$, $t \in (0,2)$. The number of Fourier harmonics is $N = 5$ in Fig. 2 and $N = 10$ in Figs. 3 and 4. Noisy data are taken in the following form:

$$f^\varepsilon(y,t) = f(y,t) + \varepsilon\alpha(y,t)(f_{\max} - f_{\min}).$$

Here ε is a noise level in the data, $\alpha(y,t)$ is a uniformly distributed random number on the interval $[-1,1]$ for a fixed y and t, and $f_{\max}$ and $f_{\min}$ are the maximum and minimum of the exact data, respectively. The spatial dimension of the grid is 100×100.

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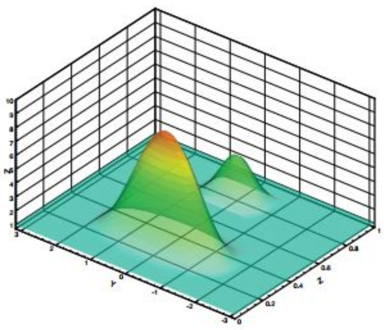

Figure 1: Exact solution of the inverse problem

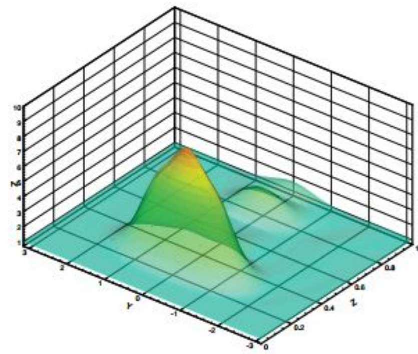

Figure 2: Approximate solution of the inverse problem, $N = 5$, $\varepsilon = 0$.

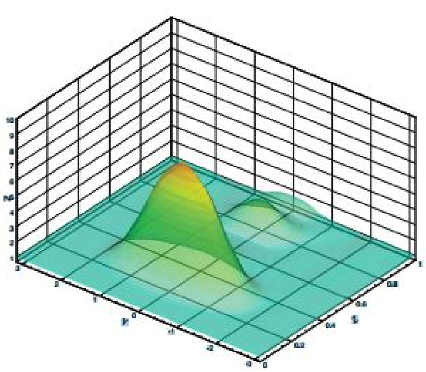

Figure 3: Approximate solution of the inverse problem, $N = 10$, $\varepsilon = 0$.

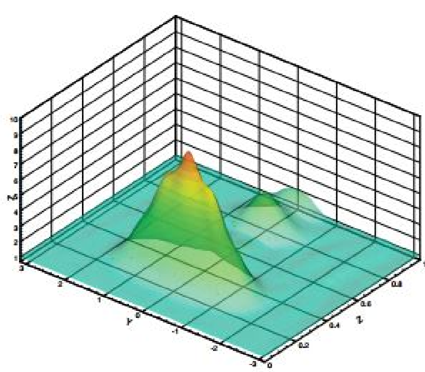

Figure 4: Approximate solution of the inverse problem, $N = 10$, $\varepsilon = 0.05$.

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