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TOMOGRAPHY OF TENSOR FIELDS IN THE PLAIN

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Abstract The problems of tomography and integral geometry of tensor fields in the plane are considered. Basic backgrounds on tensor fields, operators of ray transforms and back-projections are stated. The representations of symmetric tensor fields through potential that generates the field with usage of operators of differentiation and symmetrization are obtained. A detailed classification of symmetric m-tensor fields is suggested and decomposition theorems are obtained. Back-projection operators acting on ray transforms of tensor fields are introduced. The kernels and images of the operators of mixed and transverse ray transforms are described. The connections between ray transforms of the fields and Radon transforms of their potentials are established. The inversion formulas for the components of a field and for its potential are obtained.

Key words: tomography, tensor field, potential, ray transform, back-projection operator, inversion formula

AMS Mathematics Subject Classification: 44A12, 65R10, 65R32

1 Introduction

Mathematical backgrounds of computerized tomography were laid at the very beginning of twenty century in the papers of Radon [1] and Funk [2] on integral geometry which had pure theoretical sense. Thus in the paper of Radon the inversion formulas for Radon transform were suggested and proved. An essential extension of applications and formulations of tomography problems has led to an emergence of vector, tensor and refractive tomography as separate fields of investigations with their own problems, achievements and unsolved tasks. The formation and fast development of this new fields of tomography are also based on ideas, statements and results of integral geometry cultivated in framework of more general theory of inverse and ill-posed problems.

Integral geometry is certain category of inverse problems that are understood in wide sense and is connected closely with geometrical objects in n-dimensional space. The statements and investigation of integral geometry problems for the time being realized mainly in theoretical terms having rare and fragmentary applications in practice. Geophysical investigations of the inner structure of the Earth by means of natural (the earthquakes) or artificial sources of elastic and electromagnetic waves were a powerful stimulus for new settings in the integral geometry [3]. The emergence and fast formation of tomography as self-depended natural and practical subject led to the new intensive development of the integral geometry and the obtained results have been in demand for areas connected with investigations by nondestructive distant technique. Many Russian and foreign scientists explored the inverse and ill-posed problems, the tasks of integral geometry and tomography in which in particular a phenomenon of refraction was included. The Soviet and Russian scientists from Moscow (A. N. Tikhonov, I. M. Gelfand, V. Ya. Arsenin, A. V. Goncharsky, A. G. Yagola, etc.) and Novosibirsk (M. M. Lavrentiev, A. S. Alekseev, V. G. Romanov, S. I. Kabanikhin, Yu. E. Anikonov, etc.) scientific Schools have made a significant contribution to the theory of inverse and ill-posed problems. A large number of their great results are published in plenty papers, textbooks and monographs. We mention only mostly known of them, [4]–[30].

The integral geometry problem in traditional general setting [13] is the determination of an unknown function $u(x)$, $x \in \mathbb{R}^n$, by its known integrals

$$\int_{\mathcal{M}(\lambda)} u(x) d\sigma = v(\lambda) \tag{1}$$

over manifolds depending on a parameter $\lambda \in \mathbb{R}^k$ from a certain family $\{\mathcal{M}(\lambda)\}$. Complete mathematical study of the problem assumes to answer the listed below significant questions.

Uniqueness. What are the conditions when an assignment of a function $v(\lambda)$ determines $u(x)$ uniquely? A considerable part of theoretical investigations and results in the integral geometry answer this question.

Existence. What are the necessary and sufficient conditions for belonging $v(\lambda)$ to the set of functions that may be represented by the integrals of a form (1)? The question of existence usually is more complicated then a question of uniqueness. Fortunately in the tasks arising due to the practice requirements there exists a priori information that the solution exists because one has the date in which in one form or another the unknown quantity (solution) is "encrypted".

Method for solving. What is a way for determination a function u by the known v? The usual answer the question assumes obtaining u as a result of applications the explicit inversion formulas. Now the other approach got a wide spread in association with applications of computers. Namely it is numerical method of approximate solving to the problems of integral geometry. The approach uses as the universal tools of approximation theory so the special methods developed in framework of specific of concrete problems of the integral geometry.

Stability. What is a behavior of a solution u in dependence of the errors in data v? Problems of integral geometry with complete data are as a rule weakly ill-posed. But there exist an important statements of the problems with incomplete data, and they are ill-posed with strong degree just the same as well known problem of analytical continuity. The examples of the problems are the inverse kinematic problem of seismic and the tomography problem with limited angles of observation. In spite of pessimism among the researchers about future perspectives of this problems the attempts of working-out the effective and stability numerical methods are going on.

1.1 Practical statements of the vector tomography problems

Practical statements connected with reconstruction of vector properties of a medium by known information of tomographic type appeared at 70–80 years of previous century [31]–[36]. At present tomography directions focused on the investigation of vector or tensor features of media by nondestructive methods are developed rather intensively. It occurs primarily due to the fact that the areas of applications of tomographic methods to the non-scalar properties of objects are very broad. They are investigations of liquid or gas flows, physical and astrophysical experiments, studying of anisotropic characteristics of industrial products and terrestrial rocks, biological and medical applications, and many other areas of nature science, techniques, etc., [37]–[43]. Now we would like to formulate mostly characteristic situations that led to the problems of vector and tensor tomography.

Liquid or gas flow. Let $D \subset \mathbb{R}^2$ be bounded convex domain of the flow of liquid or gas with boundary ∂D. The boundary may be not "physical" boundary but be determined by a domain of investigation with sources and detectors in it. The flow is determined by a vector field of speed $w(x)$, $x = (x_1, x_2) \in D$. The vector field $w(x)$ shall be determined by measurements of time-of-flight of acoustic signal through the flow domain. Let $c(x)$ be a speed of sound in the domain D. We assume $|w| \ll c$ and c changes small enough in D so the propagation of the signal may be thought off as over straight lines. Then the time-of-flight between a source at a point P (or Q) and a receiver at a point Q (or P accordingly) is

$$t_{PQ} = \int_{L_{PQ}} \frac{dl}{c + \langle w, \eta \rangle}, \quad t_{QP} = \int_{L_{QP}} \frac{dl}{c - \langle w, \eta \rangle},$$

where η is a unit tangent vector of acoustic signal ray L_{PQ}, and the time t_{QP} is calculated taking into account the replacement of the direction vector η onto the direction vector $-\eta$.

By summing the time-of-flights

$$t_{PQ} + t_{QP} \approx 2 \int_{L_{PQ}} \frac{dl}{c(x)},$$

we obtain the traditional problem of (scalar) transmission computerized tomography of recovering of a sound speed $c(x)$ in the medium.

By subtracting

$$t_{PQ} - t_{QP} \approx -2 \int_{L_{PQ}} \frac{\langle w(x), \eta \rangle}{c^2(x)} dl = \int_{L_{PQ}} \langle \widetilde{w}(x), \eta \rangle dl,$$

with $\widetilde{w}(x) = -2w(x)/c^2(x)$, we have the statement of vector tomography problem consisting in determination of the flow speed $w(x)$ of the liquid or gas by the known longitudinal ray transform of the vector field $\widetilde{w}$.

Thus the information about time-of-flights is enough not only for reconstruction of scalar field (function) $c(x)$ corresponding to the sound speed in the medium, but also for the reconstruction of the vector field $\widetilde{w}$ that is connected with the field of speeds of the flow of liquid or gas. It should be mentioned that inverse ratio of $\widetilde{w}$ to the square $c^2(x)$ of a sound speed and natural assumption $|w| \ll c$ lead to the require of much more higher accuracy of measurement compared with the accuracy in computerized tomography.

Measurements of a temperature of gas on a base of Schlieren effect. A problem of temperature measurement in gas with usage of a distance methods is considered [33],

[44]. A propagation of light rays in an inhomogeneous medium with index of refraction $n = n(x)$, $x \in D \subset R^3$, which changes weakly in comparison with the length of wave, is described by the equation

$$\frac{d}{ds}(n\,\eta) = \nabla n,$$

where ds is the differential of arc length along the light ray. Then a vector η, tangent to the light ray that ran the distance l in a medium, may be expressed as

$$\eta = \frac{1}{n}\int_0^l \nabla n\, ds + \frac{n_0}{n}\eta_0,$$

where η_0 specify the initial direction of the light ray and n_0 is the index of refraction at $s = 0$. An optical Schlieren device transforms the difference in directions of propagation into an intensity of deviations

$$I = \frac{k}{n}\int_0^l \langle\theta, (\nabla n \times \eta_0)\rangle ds.$$

The unit vector θ along with the constant k describe a sensitivity of the device in dependence of the direction. If the vector θ is orthogonal to the plane of tomographic measurements, and $\theta \times \eta_0 = \xi$, where through ξ the unit normal vector to the ray belonging to the plane of investigations is designated, then

$$I = \frac{k}{n}\int_0^l \langle w, \xi\rangle ds,$$

where $w = \nabla n$. The last equation shows that the component $\langle\nabla n, \xi\rangle$ of potential field ∇n is orthogonal to the direction of propagation of the observed optical radiation. Hence the problem of vector tomography is formulated on determination of the potential vector field w by the known transverse ray transform. The process of solving this problem gives as potential vector field ∇n so the index of refraction n. In gas medium the index of refraction depends on a density, and if a pressure is constant, it means the dependence on temperature too so it can be estimated by the recovered n.

Doppler tomography. The Doppler effect is one of the physical phenomena on the base of which one can determine vector properties of a medium. The problem is to find a field of speeds $v = v(x, t)$ in a domain D, filled by liquid or gas, by the measurements obtained out of the domain.

This general statement has interesting concrete applications. Thus in physical experiment one can find the function of distribution of molecule by the speeds by means of the Doppler effect; the initial data are obtained by raying of the molecular ensemble by laser radiation [40], [41]. The problem of reconstruction of a vector field of mean speeds of ions in plasma was set and investigated, see [45] for example. The problem of speed distributions of the large masses of air is actual in hydrometeorology [37]. In hemodynamics it is necessary to determine the field of blood speed in vessels of a patient with usage of acoustic Doppler measurements [46]– [48]. As a rule the focused ultrasound beams or laser sources are applied in the problem of Doppler tomography as the probe signals.

An interaction of physical field with moving particles of a medium leads to an appearance of the reflected waves with a frequency ω which differs from the frequency ω_0 of the incident wave on the magnitude

$$\delta\omega = \frac{2c\omega_0 v_\eta}{c^2 - v_\eta^2},$$

where c is a speed of the incident wave, v_η is a projection of the speed of a particle, on which the wave is reflected, onto direction of the ray. Assuming $|v| \ll c$, which is actual practically always, we derive $\delta\omega \approx 2\omega_0 v_\eta / c$ good approximation of the Doppler shift. The shift is registered by receiving equipment and represents initial data for the problem of Doppler tomography.

The general problem of Doppler tomography consists in determination of the field of speeds of the ensemble of particles (molecule, ions, etc.) moving in a certain domain with different speeds and in different directions. This problem is very complicated and yields for solving with great difficulties. So as the date in investigations an integral moment of the first order is used much more often,

$$\mu(s') = \int_L v_\eta \, d\tau = \int_L \langle v \, , \, d\tau \rangle. \tag{2}$$

The moment (2) obviously is the longitudinal ray transform of the vector field v. Hence a more simpler statement is considered, namely consisting in changing of the complicated distribution by speeds onto one mean speed, and the problem is now to find a vector field of mean speeds.

We led references only on the main statements and directions of investigations on vector field tomography above. A review paper [49] contains essentially more detailed information on main environment and results of vector tomography. We ought mention a technique report [48] too. It contains a large bibliography references of the papers devoted to investigations of vector and tensor tomography problems in different physical statements.

1.2 Tensor tomography problems

It is well known fact that phenomenon of polarization arises during propagation of electromagnetic and elastic waves in anisotropic medium. An approximation of geometrical optics describes a behavior of ellipse of polarization by a system of differential equations connecting properties of the medium with the values of electromagnetic field that propagates along the ray [50]. Thus the problem of determination of the medium properties by a degree of polarization of the incident wave and the wave, passed through the medium, has tomographic nature (an information accumulated along the ray). If we have enough number of polarization measurements then the problem of polarization tomography can be stated. The rigorous general setting of the problem and its mathematical aspects are considered in [26], [51], [52]. An algorithm of solving the problem in linearized statement is developed in [53]. There are traditional applications of the problem to plasma diagnostic [54], [43], [55], [56], [39], photoelasticity and fiber optics, see [57]–[59] for example. In recent years applications and hence the statements of tensor tomography become much more wider. There arise absolutely new directions of development such as magneto-photoelasticity, [60], [61], the tomography of tensor fields of stress [62], tensor fields of residual stress [63], [64], the diffractive tomography of strains [65], the polarization tomography of quantum radiation [66]. New intensive developed directions for the methods in tensor tomography have especially successful applications in biology and medicine. This areas are the diffusion MRI-tomography allowing investigate the brain in more details, the cross-polarization optic coherent tomography applying in morphology, the vessel investigations, cancer diagnosis. An approaches of cross-polarization tomography allow to increase resolution nearly degree in comparison with the other tomographic methods [67]–[70], and certain other applications.

Described above examples show that, as in the considered earlier settings of vector tomography, instead of scalar variable it is required to recover vector (in particular electromagnetic) or tensor field. It's possible to speak about tensors of strains or stresses, a tensor of electromagnetic field. A complication of an object of investigation leads not only to the generalization of the Radon transform, the longitudinal and the transverse ray transforms, but to the appearance of quite new types of integral transforms of tomographic type. First of all it is the longitudinal ray transform of *symmetric m-tensor field.* This transform may be determined not only in n-dimensional Euclidean space but in the Riemannian manifold too. The Radon transform for functions also is generalized for the symmetric m-tensor fields depending on n variables. This type of integral transforms is called *the normal Radon transform. The truncated transverse ray transform* is simplified version of the transverse ray transform and directly related to the photoelasticity. Polarization tomography (in a narrow sense) has deal with measurements of subtraction of the phases between inductive and the transmitted wave. Sometimes it is more simple then the measurements of the phase on "entrance" and "exit". *The mixed (longitudinal-transverse) ray transform* arises as a result of application of the "ray approach" to the isotropic elastic medium which is described by a system of equations of the dynamic elasticity and further while consideration of simpler variant of the quasi-isotropic medium [25].

We would like to give an example of certain general setting of a problem of the integral geometry of tensor fields [25], consisting in determination of a symmetric m-tensor field $w = w_{i_1\dots i_m}(x)$, $x \in M$, given in the Riemannian manifold M, $\dim M = n$, by its known integrals (i.e. the longitudinal ray transform)

$$\int_{\gamma_{x,\xi}} w_{i_1\dots i_m}\big(\gamma_{x,\xi}(t)\big)\dot\gamma^{i_1}_{x,\xi}(t)\dots\dot\gamma^{i_m}_{x,\xi}(t)dt = (\mathcal{P}u)(x,\xi) \tag{3}$$

On the one hand the statement generalizes the traditional problem of integral geometry as the requirement is to recover tensor field but not a function. On the other hand we have more narrow and concrete manifolds of surfaces the integrals over which are known as initial information. Namely there are one-dimensional manifolds consisting of geodesics of certain given Riemannian metric. The purpose consists in determination of symmetric tensor field $w_{i_1\dots i_m}$ by its known integrals (3). It appears that as and for vector case and Euclidean metric this information is sufficient only for unique reconstruction of solenoidal part w^s of unknown field w. Here in the article more simple case of this transform is considered, i.e. only a case of Euclidean metric, but a number of other types of ray transforms are treated.

1.3 The algorithms of computerized tomography

Now we mention the main mathematical tools lying in a base of numerical methods and algorithms of computerized (scalar) tomography. Mathematically the inversion formulas are very attractive. The whole family of algorithms based on the inversion formulas are known, in particular, as algorithms of the convolution and back projection. Now they have wide applications but at first the idea was applied in radio-astronomy in $2D$ variant, see [71]. One of its forms was described at first in papers [72], [73] in application to medicine tasks. Later on the development of the approach led to a number of kinds of reconstruction algorithms, see [74]. They are algorithms of Davison-Grunbaum, Madikh-Nelson, ρ-filtration of back projection, Marr and others. The other class of algorithms based on projection theorems are known as *Fourier-algorithms*. They connect the transforms of Radon and Fourier. The first variants of these algorithms were suggested in [75] for solving the problems of radio-astronomy, and in [76] for electron microscopy. The algebraic algorithms, see [77], [78], are well known and used widely. The first step of the algorithm consists in construction of a system of linear algebraic equations, the second step devoted to solving of the system. Usually the iterative methods, and the main among them — the method of Karchmazha (the iterative orthogonal projections) — are used [79]–[81]. The probability models and mathematical-statistic tools [82], [83] have much less application. They are used usually in framework of more complicated tomographic models including high level of noise, or inner sources with unknown distribution, or phenomenon of multiple scattering, etc.

The most part of mentioned approaches and corresponding algorithms are developed in suggestion of absence of phenomenon of refraction in a medium. The possibility of the usage of them in mathematical models with refraction is inexplicit. A certain interest, especially in connection with possibility of the usage in the models of refractive tomography, represent two general approach of applied functional analysis. In particular they can be used for solving the integral equations of the first kind. The method of least squares (LSM) is the first of them. This approach is well known and employed to a great number of different problems of numerical calculations. LSM is exploited as within the framework of variational methods (in pairs of Hilbert or normed spaces) for solving of operator (in particular integral) equations [84], [85], so and for the solution of discrete and finite-dimensional approximations of these tasks [86]–[88]. This method is connected closely with the second approach named as *singular value decomposition (SVD)* method and show very powerful results in practice of numerical computations. The SVD-method has two main kinds of applications similar to the areas of applications of LSM-method. The first is the decomposition of operators acting from one Hilbert space to another; see, for example about singular value decompositions of tomographic operators [89]–[91], [74]. The second variant of the application of SVD-method is much more spreading and is used in a case of the operators acting from one finite-dimensional space to another finite-dimensional space, usually of another dimension, see [92] for example. The third method, similar to the described above two powerful methods, is so called *method of approximate inverse* developed by A. K. Louis and his pupils [93]–[96]

more then twenty years. Theoretical backgrounds of the method lays in functional analysis too. These are the Riesz theorem about representation of linear functional, a concept of fundamental solution and its properties, approximations of δ-function.

Currently there are published a great number of monographs containing as mathematical backgrounds, numerical methods and algorithms of tomography, so and descriptions, developments of tomography theory and practice in different subject areas especially in physics, biology and medicine, see [97]–[101] for example.

1.4 The goals and short content

The formulated above statements and characteristics of the vector and tensor tomography problems, even very short and schematic, lead to certain questions and conclusions. As we remember the measurements in transmission computerized tomography are based on the phenomenon of decreasing of a signal intensity while passing the continuous medium. The principles of measurements for creating of initial data in vector and tensor tomography are based on completely different of those used in transmission tomography. We would like to remind that in the base of measurements in mostly known problems of vector and tensor tomography problems lie such physical phenomena as different time-of-flights of direct and inverse signals, Doppler and Schlieren effects, the polarization of electromagnetic or elastic waves, etc.

The question arises on the possibility of application of methods and algorithms developed in the computerized tomography to the problems of vector and tensor tomography. Or researchers should elaborate absolutely new mathematical tools different from those have been exploited in the tomography of scalar fields with great success. Should new methods be so different from each other as different from each other physical effects? Do every physical effect needs in new own mathematical tools, the new methods and algorithms?

The answer to the first question is positive on the whole, and the answers two next questions are negative generally. The 2D-statements of vector and tensor tomography confirm this assertions most clearly. A variety of all physical effects, lying in a base of modern measurements of physical fields passed through an object, is aligned, and the ways of data obtaining are reduced to few types of ray transforms. More over absolutely different physical effects lead often to the same ray transforms.

The main purpose of the article consists in investigation of the questions formulated above, searching and justification the answers to this questions. We investigate the general case of symmetric m-tensor field for arbitrary m, not only for $m = 0, 1, 2$. We reduce the problem of reconstruction of a tensor field by its ray transforms to the problem of reconstruction of its components (as functions considered) by their known Radon transforms. Then it follows that the vast majority of the scalar tomography algorithms are useful and for the tasks of vector and tensor tomography.

Necessary notations, definitions and main material on tensor fields and tomographic operators are represented at Section 1 of the paper. The section is devoted to the properties of the Radon transform, ray and fan-beam transforms, the back projection operators. We pay special attention to the inversion formulas for the Radon and ray transforms. Important for the practice cases of recovering of vector (Section 1) and symmetric 2-tensor fields (Section 2) are considered in details for a possibility of direct construction of corresponding algorithms. The concepts of transverse and mixed ray transforms for the vector and symmetric 2-tensor fields are developed. The kernels and images of such operators are described.

In Section 3 a detailed classification of symmetric m-tensor fields, $m > 1$, is established. The classification lies at the base of proof of the decomposition theorems for the fields. An idea of representation of a symmetric tensor field through the potentials generating the field by superpositions of the operators of inner differentiation and inner orthogonal differentiation plays the main role for decomposition theorems. The concepts of transverse and mixed ray transforms for the vector and symmetric m-tensor fields are introduced. The operators of back projection acting on the transverse and mixed ray transforms for the fields, $m > 1$, are suggested. The kernels of the transverse and mixed ray transforms are described. The connections between ray transforms for the fields and Radon transforms for their potentials are established. In particular, this important connections allow us to establish the projections theorems for symmetric tensor fields as well as to obtain the inversion formulas for the components of tensor field and their potentials.

Generally the proposed article is a review but it contains a number of new results. Certain results of the authors concerning vector and tensor tomography problems including numerical methods, algorithms and their realizations are published in [102]–[109]. Stability estimates, on which investigations of the ill-posedness type of the problems of refractive, vector and tensor fields tomography, are obtained in [110], [108], [109].

2 Definitions and preliminary results

Let $B = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, (x^1)^2 + (x^2)^2 < 1\}$ be a unit disk with a boundary $\partial B = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, (x^1)^2 + (x^2)^2 = 1\}$. A notation $Z = \{(\alpha, s) \in \mathbb{R}^2 \,|\, \alpha \in [0, 2\pi], s \in [-1, 1]\}$ is used for the cylinder $[-1, 1] \times [0, 2\pi]$. Unit vectors $\xi \in \partial B, \xi = (\cos\alpha, \sin\alpha)$, $\eta := \xi^\perp \in \partial B$, $\eta = (-\sin\alpha, \cos\alpha)$ and a real number $s \in \mathbb{R}$ set a straight line $L_{\xi,s}$ by the normal equation $x^1 \cos\alpha + x^2 \sin\alpha - s = 0$ or by the parametrical equations $x^1 = s\cos\alpha - t\sin\alpha,\ x^2 = s\sin\alpha + t\cos\alpha$.

Functions (scalar fields) are denoted as $f(x), g(x), ...$ and through $\varphi(x), \psi(x), \chi(x), ...$ are designated potentials determining tensor fields. The functions and the potentials are given in unit disk B. Notations $x = (x^1, x^2)$, $y = (y^1, y^2)$,... are convenient for statement of the problems and formulation of necessary properties of tensor fields. But sometimes at description of numerical experiments the notations (x, y) for coordinates of the points in the plane are used. A set of symmetric m-tensor fields $w(x) = (w_{i_1...i_m}(x))$, $u(x) = (u_{i_1...i_m}(x))$, $v(x) = (v_{i_1...i_m}(x))$, ..., $i_1, \ldots, i_m = 1, 2$, given in B is designated by $S^m(B)$. The scalar product in $S^m(B)$ is defined by the formula

$$\langle u(x), v(x)\rangle = u_{i_1...i_m}(x) v^{i_1...i_m}(x), \tag{4}$$

where by repeating super- and subscripts in a monomial a summation from 1 to 2 is meant. We recall that in the Euclidean space with a rectangular Cartesian coordinate system there is no difference between contravariant and covariant components of tensors. Later the covariant components of tensors are exploited usually.

The notations for functional spaces should be remind now. Further we need in the spaces of square integrable functions $L_2(B)$ and symmetric m-tensor fields $L_2(S^m(B))$ as well as the L_2-space $L_2(Z)$. The inner product of symmetric m-tensor fields $u, v \in L_2(S^m(B))$ is defined as

$$(u, v)_{L_2(S^m(B))} = \int_B \langle u(x), v(x) \rangle dx. \tag{5}$$

The spaces of differentiable, with finite order k, symmetric m-tensor fields are designated as $C^k(S^m(B))$ and $C_0^k(S^m(B))$; the Sobolev spaces are denoted as $H^k(S^m(B))$, $H_0^k(S^m(B))$, $H^k(Z)$, $m = 0, 1, 2, \dots$, $k = 0, 1, \dots$ All these spaces are defined in usual manner.

Thus the inner product in the Sobolev space $H^k(B)$ is defined as

$$\big(f, g\big)_{H^k(B)} = \sum_{0 \le |a| \le k} \int_B D^a f D^a g \, dx^1 dx^2,$$

where a is the multi-index and an operator D^a is the operator of multi-index differentiation. The norm in $H^k(B)$ is generated by the inner product,

$$\big\| f \big\|_{H^k(B)} = \Big(\sum_{0 \le |a| \le k} \int_B |D^a f|^2 dx^1 dx^2 \Big)^{1/2}.$$

The inner product for vector fields $u, v \in H^1(S^1(B))$ is defined by the formula

$$\langle u, v \rangle_{H^1(S^1(B))} = \int_B \Big(\sum_{i=1}^{2} u_i v_i + \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x^j} \frac{\partial v_i}{\partial x^j} \Big) dx^1 dx^2,$$

the inner product for symmetric 2-tensor fields $u, v \in H^1(S^2(B))$ is defined as

$$\langle u, v \rangle_{H^1(S^2(B))} = \int_B \Big(\sum_{i,j=1}^{2} u_{ij} v_{ij} + \sum_{i,j,k=1}^{2} \frac{\partial u_{ij}}{\partial x^k} \frac{\partial v_{ij}}{\partial x^k} \Big) dx^1 dx^2,$$

and the inner product for functions $f, g \in H^1(\partial B \times \partial B)$ is defined by the formula

$$\langle f, g \rangle_{H^1(\partial B \times \partial B)} = \int_{\partial B \times \partial B} \Big(fg + \frac{\partial f}{\partial \alpha} \frac{\partial g}{\partial \alpha} + \frac{\partial f}{\partial \beta} \frac{\partial g}{\partial \beta} \Big) d\alpha d\beta.$$

The inner product in the space $L_2(Z)$ is defined by the relation

$$\big(f, g\big)_{L_2(Z)} = \int_Z f(x) g(x) \, dx,$$

so the norm should be defined as

$$\big\| f \big\|_{L_2(Z)} = \Big(\int_Z |f(x)|^2 \, dx \Big)^{1/2}.$$

Further sometimes the spaces $\mathcal{D}(B)$, $\mathcal{D}(\mathbb{R}^2)$ of test functions and the Schwartz spaces $\mathcal{S}$, $\mathcal{S}'$ are used too.

2.1 The Radon transform

Let a function $\varphi(x)$, $x = (x^1, x^2)$, be given in B, $\varphi(x) \in C_0^\infty(B)$. The Radon transform $\mathcal{R}\varphi$ of a function φ is defined as

$$(\mathcal{R}\varphi)(\xi, s) = \int_{-\infty}^{\infty} \varphi(s\xi + t\xi^\perp)dt, \tag{6}$$

The transform is the integrals along the lines $L_{\xi,s} = \{x \in \mathbb{R}^2 \mid \xi^1 x^1 + \xi^2 x^2 = s\}$ and maps the space $C_0^\infty(B)$ into the space $C_0^\infty(Z)$. The boundedness of functions given in Z is understood to the argument s.

The well known properties [97] of the Radon transform will be required further. Let A be a matrix of dimension 2×2, $\varphi \in \mathcal{D}(\mathbb{R}^2)$. The notation

$$\breve{\varphi}(\xi, s) := \int_{-\infty}^{\infty} \varphi(s\xi + t\xi^\perp)dt$$

is exploited.

1) The function $\breve{\varphi}(\xi, s)$ is a homogeneous of degree -1 function of its arguments,

$$\breve{\varphi}(t\xi, ts) = |t|^{-1}\breve{\varphi}(\xi, s).$$

2) The function $\breve{\varphi}(\xi, s)$ is an even function of the variables ξ, s,

$$\breve{\varphi}(-\xi, -s) = \breve{\varphi}(\xi, s).$$

3) Let in $\mathbb{R}^2$ a linear change of basis $y = Ax$, $\det A \neq 0$ are made. Then we have

$$\mathcal{R}\left(\varphi(Ax)\right) = |\det A^{-1}|\breve{\varphi}\left((A^{-1})^T\xi, s\right).$$

4) If in $\mathbb{R}^2$ a change of the origin is made, $y = x - a$ (shift of the Radon transform) then the relation

$$\mathcal{R}\left(\varphi(x - a)\right) = \breve{\varphi}(\xi, s - \langle\xi, a\rangle).$$

is valid.

2.2 Inversion formulas

A simple inversion formula can be derived with usage of the back projection operator and the Fourier transform. Let $g(\xi, s) \in C_0^\infty(Z)$, $g(\xi, s) = (\mathcal{R}\varphi)(\xi, s)$ be valid for some function $\varphi \in C_0^\infty(B)$. The back projection operator

$$f(x) = (\mathcal{R}^\# g)(x) = \frac{1}{2\pi}\int_0^{2\pi} (\mathcal{R}\varphi)(\xi(\alpha), x^1\cos\alpha + x^2\sin\alpha)\, d\alpha, \tag{7}$$

is an average of values of the Radon transform (applied to the function φ) calculated along all lines passing through the point x. The back projection operator allows to "return" from the space of functions depending on variables ξ, s to the space of functions depending on the variables x^1, x^2. Substituting the expression (6) to (7), we obtain

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{-\infty}^{\infty}\varphi(s\xi + t\xi^{\perp})\,dt\right)\,d\alpha.$$

Change of variable $y = s\xi + t\xi^{\perp}$ allows to break the inner integral in two integrals with limits from $-\infty$ to 0 and from 0 to ∞. Taking into account that $t\,dt\,d\alpha = dy^1\,dy^2 \equiv dy$ and $t = |x - y|$ we obtain the representation

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(y)}{|x-y|}\,dy \tag{8}$$

for the back projection $f(x)$ in the form of convolution, $f = \varphi * |x|^{-1}/\pi$. Applying the Fourier transform to both sides of (8) and using the convolution theorem, we arrive to the relation $F[f] \equiv \widehat{f} = \widehat{\varphi}\cdot h$, where $\widehat{\varphi} \equiv F[\varphi]$, $h = \big(|x|^{-1}\big)\widehat{\ }$. Hence $\widehat{\varphi} = \widehat{f}\,/\,h$. Applying to the resulting expression the inverse Fourier transform we get inversion formula

$$\varphi(x) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\widehat{f}(y)}{h(y)}\,\mathbf{e}^{\mathbf{i}\langle x,y\rangle}\,dy. \tag{9}$$

Algorithm based on (9) is very simple. Let the Radon transform $g(\xi, s) = \mathcal{R}\varphi$ of a function φ is given. The first step of the algorithm consists in calculation of back projection $f(x) = (\mathcal{R}^{\#}g)$. The second step is an application of two-dimensional Fourier transform $\widehat{f}(y)$. Then the Fourier transform $h(y)$ of the function $|x|^{-1}$ is computed and a relationship $\widehat{f}\,/\,h$ is obtained. At last the application of the two-dimensional inverse Fourier transform to the obtained expression gives the required function φ. The calculation of the back projection is reduced to the single integration with limits from from 0 to 2π. For the calculations we have at our disposal the numerous quadrature formulas. The Fourier transform, both direct and inverse, is replaced by a discrete Fourier transform. We would note that, by using the fast Fourier transform (FFT), we decrease a calculation time significantly.

It seems that the algorithm described above, due to its simplicity, has broad applications, but it is not so. Its usage is limited to the cases when one needs to get rough quality evaluation, especially without worrying about the good accuracy of calculations. The algorithm can be used for a comparative study of the effectiveness and accuracy characteristics of different reconstruction algorithms. And it is rarely used if it is necessary to determine the object with the maximum possible accuracy.

Main reasons for this situation are caused by two circumstances. The first is that a back projection $f(x) = (\mathcal{R}^{\#}g)$, where $g(\xi, s) = \mathcal{R}\varphi$ for $\varphi \in \mathcal{D}(\mathbb{R}^2)$ or $\varphi \in \mathcal{S}$, $(\varphi > 0)$, slowly decreases at the infinity. Generally speaking, $f \in \mathcal{S}'$, i.e. the function is an element of the space of slow-growing at the infinity functions. In particular $f \notin L_2(\mathbb{R}^2)$. The second circumstance is connected with the behavior of f at infinity too. Namely the discrete Fourier transform, apart from the continuous one, assumes the replacement both of it and its image by some periodic functions. Thus the approximation is the worse than the decrease of a transformed function is more slowly at the infinity.

Involving a terminology of the pseudodifferential operators theory, the inversion formula can be rewritten as

$$\varphi(x) = (-\Delta)^{1/2} f(x) = \big((-\Delta)^{1/2}(\mathcal{R}^{\#} g)\big)(x), \tag{10}$$

where $g(\xi, s) = \mathcal{R}\varphi$. Thus (10) is a composition of the back projection operator and the nonlocal pseudodifferential operator $(-\Delta)^{1/2}$ with a symbol $|y|$.

I. Radon used another approach for a derivation of his inversion formula. The derivation is based on the Abel integral equation and its inversion; the averaging operator over circles for a function $\varphi(x)$; the averaging of $(\mathcal{R}\varphi)(\alpha, s)$ over straight lines equidistant from the origin; commutative properties of the Radon transform with respect to shifts and rotations of the plane (properties 3 and 4 above). Inversion formulas are the best known, especially in proceedings of applied character, in the following forms,

$$\varphi(x^1, x^2) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{R}\varphi)'_s(\alpha, s + x^1\cos\alpha + x^2\sin\alpha)}{s}\,ds\,d\alpha, \tag{11}$$

where the integral over s is understood in a sense of the Cauchy principal value. The integration by parts of the inner integral leads to the inversion formulas of another types. Namely, in one of them the second derivative $(\mathcal{R}\varphi)''_{ss}$ of the Radon transform is used,

$$\varphi(x^1, x^2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}(\mathcal{R}\varphi)''_{ss}(\alpha, s + x^1\cos\alpha + x^2\sin\alpha)\ln|s|\,ds\,d\alpha, \tag{12}$$

while in the second the Radon transform $\mathcal{R}\varphi$ by itself,

$$\varphi(x^1, x^2) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{R}\varphi)(\alpha, s + x^1\cos\alpha + x^2\sin\alpha)}{s^2}\,ds\,d\alpha, \tag{13}$$

is exploited. It should be marked that the value s^2 in the denominator of the integrand appears (instead of s in the formula (11)).

The algorithms of recovering of a function φ by its known Radon transform can use any of the inversion formulas (11)–(13). It depends on a priori information about the function φ; an existence, level and nature of a noise in the data $\mathcal{R}\varphi$; a quality of mathematical tools for a numerical differentiation, smoothing and accounting the singularities of various orders.

Remark 2.1. *Looking ahead we note that the variants* (11), (12) *of the inversion formulas can be used directly for the recovery of potentials of vector or symmetric* 2*-tensor fields, respectively. In this case the derivatives with respect to s of the Radon transform of the corresponding potentials should be replaced by the longitudinal or transverse ray transforms of vector or tensor fields.*

We present now more general means that allow to obtain the whole family of the inversion formulas [74]. In the Euclidean space $\mathbb{R}^n$, $n \geq 2$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ the unit ball is denoted as B, ∂B is its boundary. The designation $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$ is used too. We establish the following sets: $\xi^\perp$ is a subspace $\{y \in \mathbb{R}^n \mid \langle \xi, y\rangle = 0\}$ orthogonal to a vector $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{S}^{n-1}$; Z is a cylinder $\mathbb{S}^{n-1} \times \mathbb{R} \subset \mathbb{R}^{n+1}$; T is a tangent bundle of the sphere $\mathbb{S}^{n-1} \times \xi^\perp = \{(\xi, x) \mid \xi \in \mathbb{S}^{n-1}, \langle x, \xi\rangle = 0\} \subset \mathbb{R}^{2n}$; $\mathbb{SR}^n$ is a sphere bundle $\{(x,\xi) \mid x \in \mathbb{R}^n, \xi \in \mathbb{S}^{n-1}$. Without loss of generality we give the definitions of the Radon transform, the ray and fan-beam transforms for the functions from Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

The (n-dimensional) Radon transform $\mathcal{R}$ performs the mapping of a function $\varphi(x)$ to a set of its integrals $(\mathcal{R}\varphi)(\xi, s)$ over hyperplanes in $\mathbb{R}^n$, $\mathcal{R} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(Z)$,

$$(\mathcal{R}\varphi)(\xi, s) = \int_{\langle \xi, x\rangle = s} \varphi(x)dx = \int_{\xi^\perp} \varphi(s\xi + y)dy.$$

The hyperplane is defined by the normal vector ξ and a distance s from the origin (taking a sign into account). The back projection operator $\mathcal{R}^\# : \mathcal{S}(Z) \to \mathcal{S}'(\mathbb{R}^n)$ (for the Radon transform) is defined by the formula

$$(\mathcal{R}^\# g)(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} g(\xi, \langle x, \xi\rangle)\, d\xi,$$

where $g(\xi, s) \in \mathcal{S}(Z)$, $|\mathbb{S}^{n-1}|$ is the square of the surface of the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

The ray transform $\mathcal{P}$ maps a function $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ into a set of its integrals along all lines in $\mathbb{R}^n$, $\mathcal{P} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(T)$,

$$(\mathcal{P}\varphi)(\xi, x) = \int_{-\infty}^{\infty} \varphi(x + t\xi)dt;$$

the line passes through a point $x \in \mathbb{R}^n$ towards a vector $\xi \in \mathbb{S}^{n-1}$. The back projection operator (for the ray transform) $\mathcal{P}^\# : \mathcal{S}(T) \to \mathcal{S}'(\mathbb{R}^n)$ looks as follows,

$$(\mathcal{R}^\# g)(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} g(\xi, E_\xi x)\, d\xi,$$

where $g(\xi, y) \in \mathcal{S}(T)$, E_ξ is an operator of orthogonal projection onto the subspace $\xi^\perp$.

The fan-beam transform $\mathcal{I} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{SR}^n)$,

$$(\mathcal{I}\varphi)(a, \eta) = \int_0^{\infty} \varphi(a + t\eta)dt$$

is an integral of a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ over a ray beginning at a point $a \in \mathbb{R}^n$ towards a vector $\eta \in \mathbb{S}^{n-1}$.

A whole class of inversion formulas and corresponding algorithms based on the projection theorems, which connect the Fourier transform of a required function with the Fourier transform of its Radon transform (ray transform). We use notations $(\mathcal{R}_\xi\varphi)(s) = (\mathcal{R}\varphi)(\xi, s)$, $s \in \mathbb{R}$; $(\mathcal{P}_\xi\varphi)(y) = (\mathcal{P}\varphi)(\xi, y)$, $y \in \xi^\perp$. An agreement works that the Fourier transform is realized by the second variable s for $(\mathcal{R}\varphi)(\xi, s)$ (by the second group of variables $y \in \xi^\perp$ for $(\mathcal{P}\varphi)(\xi, y)$). The projection theorems are formulated below.

Theorem 2.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function. Then*

$$(\mathcal{R}_\xi\varphi)\widehat{\ }(\sigma) = (2\pi)^{(n-1)/2}\widehat{\varphi}(\sigma\xi), \quad \sigma \in \mathbb{R},$$

$$(\mathcal{P}_\xi\varphi)\widehat{\ }(y) = (2\pi)^{1/2}\widehat{\varphi}(y), \quad y \in \xi^\perp.$$

The designation $\widehat{f}$ denotes an image of the Fourier transform applied to the function f.

Explicit formulas for a composition of the Radon transform or the ray transform operator and their back projection operators are of great importance. The following relations for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ hold

$$\mathcal{R}^\#\mathcal{R}\varphi = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\,|x|^{-1} * \varphi, \qquad \mathcal{P}^\#\mathcal{P}\varphi = \frac{2}{|\mathbb{S}^{n-1}|}\,|x|^{1-n} * \varphi.$$

Obviously these formulas give a complete impression of the behavior of functions $(\mathcal{R}^\#\mathcal{R}\varphi)(x)$ and $(\mathcal{P}^\#\mathcal{P}\varphi)(x)$ for $x \to \infty$, and at the origin, in dependence of the class of functions which owns the function $\varphi(x)$. There exists a whole family of inversion formulas and a set of algorithms based on this formulas, as for the Radon transform so and for the ray transform (for $n = 2, 3$) [74]. The above formulas use the Riesz potential which, for $\gamma < n$, is determined by means of the Fourier transform,

$$(\mathbf{I}^\gamma\varphi)\widehat{\ }(y) = |y|^{-\gamma}\widehat{\varphi}(y).$$

When the operator $\mathbf{I}^\gamma$ is applied to functions given on the set Z (or T), it is acting on the second variable (the second group of variables). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function, then $(\mathbf{I}^\gamma\varphi)\widehat{\ } \in L_1(\mathbb{R}^n)$, therefore the following equality holds, $\mathbf{I}^{-\gamma}(\mathbf{I}^\gamma\varphi) = \varphi$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function, then for any $\gamma < n$ we have the formulas

$$\varphi = \frac{1}{2}(2\pi)^{1-n}\mathbf{I}^{-\gamma}\mathcal{R}^\#\mathbf{I}^{\gamma-n+1}g, \quad g = \mathcal{R}\varphi. \tag{14}$$

$$\varphi = \frac{1}{|S^{n-2}|}(2\pi)^{-1}\mathbf{I}^{-\gamma}\mathcal{P}^\#\mathbf{I}^{\gamma-1}g, \quad g = \mathcal{P}\varphi.$$

It follows from these formulas that one of the Riesz operators is the identity operator if $n = 2$ and $\gamma = 0$ or $\gamma = n - 1 = 1$. Therefore unless there exist strong reasons for using both of the Riesz transforms in the inversion formula, the defined values for parameter γ should be chosen in the algorithms.

Assuming $n = 2$, $\gamma = n - 1 = 1$ in (14) we obtain

$$\varphi = \frac{1}{4\pi}\mathbf{I}^{-1}\mathcal{R}^\# g, \quad g = \mathcal{P}\varphi. \tag{15}$$

This variant of the inversion formula is useful because it allows to use the technique of the Fourier transform easily hence the fast Fourier transform too. Besides the well known algorithm of ρ-filtration of back projection based on this formula.

We would note that for odd $n = 2l + 1$ the operator $\mathbf{I}^{1-n}$ is an ordinary differential operator, $\mathbf{I}^{1-n} = \big(-\Delta\big)^l$, in particular when $n = 3$ we obtain the formula

$$\varphi(x) = -\frac{1}{8\pi^2}\Delta_x\int_{\mathbb{S}^2} g(\xi, \langle x, \xi\rangle)\,d\xi.$$

Let $\varphi \in L_2(B)$ be a function. In [74] the stability estimate

$$\|\varphi\|^2_{L_2(B)} \leq C \|\mathcal{R}\varphi\|^2_{H^1(Z)} \tag{16}$$

was proved with the constant C independent of φ.

3 Vector fields

We define the class of functions which are used for the description of not only the continuous or C^k-smooth, $k \in \mathbb{N}$, but the discontinuous vector and tensor fields also as well as the fields with discontinuous derivatives. Physical considerations allows to assume that we have deal with the discontinuities of the first kind only. Let a domain $D \subset \mathbb{R}^2$ be such that $\overline{D} \subset \overline{B}$. We assume it consisting of a finite number of disjoint subdomains $\{D_i\}$, $i = 1, \ldots, N$, such that their union $D_0 = \cup D_i$ is dense in $\overline{D}$ and their boundaries are smooth of class $\mathcal{C}^1$. At first we note that D may be a multiply connected and, secondly, $\overline{D}$ may coincide with $\overline{B}$. It can be marked easily that $\partial D \subset \partial D_0$ and the boundary ∂D_0 coincides with the union of the boundaries $\cup D_i$ of the subdomains D_i, $i = 1, \ldots, N$. An important requirement for the boundaries is that they must not contain straight segments.

Let a function $\varphi(x)$ of class $\mathcal{C}^k$ be given in B, it vanishes in the sets $\mathbb{R}^2 \backslash B$, $B \backslash \overline{D}$, and its support coincides with the closure D, $\operatorname{supp} \varphi = \overline{D}$. The function $\varphi(x)$ is infinitely differentiable at points $(x^1, x^2) \in D$. At points $(x^1, x^2) \in \partial D_0$ it is continuously differentiable up to k-th order and vanishes. The function φ has partial derivatives of any order due to its smoothness in D. But if the point belongs to ∂D_0 then all partial derivatives $\partial^l \varphi / \partial (x^1)^j \partial (x^2)^{l-j}$, $l = 0, \ldots, k$, $j \leq l$, up to the order k are continuous, and derivatives of order $k+1$ are discontinuous with discontinuities of the first kind. We say that the function is *a potential smoothness* $\mathcal{C}^k$, or $\mathcal{C}^k$*-potential* in $\mathbb{R}^2$. Further this notations will be fixed precisely for the described above class of potentials.

Alongside with the gradient operator $\mathrm{d} : H^k(B) \to H^{k-1}(S^1(B))$ and the operator $\delta : H^k(S^1(B)) \to H^{k-1}(B)$ of divergence,

$$\mathrm{d}\varphi = \big((\,\mathrm{d}\varphi)_1, \, (\,\mathrm{d}\varphi)_2 \big) = \Big(\frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^2} \Big), \quad (\delta u) = \frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2},$$

we define [29] *the orthogonal gradient operator* $\mathrm{d}^\perp : H^k(B) \to H^{k-1}(S^1(B))$, and *the operator of orthogonal divergence* $\delta^\perp : H^k(S^1(B)) \to H^{k-1}(B)$,

$$\mathrm{d}^\perp \varphi = \big((\,\mathrm{d}^\perp \varphi)_1, \, (\,\mathrm{d}^\perp \varphi)_2 \big) = \Big(-\frac{\partial \varphi}{\partial x^2}, \frac{\partial \varphi}{\partial x^1} \Big), \quad \big(\delta^\perp u\big) = \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}.$$

We recall that a vector field $u \in H^k(S^1(B))$ is called the potential vector field if there exists a function $\varphi \in H^{k+1}(B)$ (potential) such that $u = \mathrm{d}\varphi$. A field $v \in H^k(S^1(B))$ is called the solenoidal vector field if $\delta v \in H^{k-1}(B) = 0$. For every two-dimensional solenoidal vector field v there exists a potential $\psi \in H^{k+1}(B)$ such that $\mathrm{d}^\perp \psi \in H^k(S^1(B))$ [111]. A vector field $\mathrm{d}h \in \mathcal{C}^k(S^1(B))$ is called the harmonic vector field if h is the harmonic function in B. A decomposition of a vector field on a potential and a solenoidal parts is essential in the vector and tensor analysis, and in particular in vector tomography. The decomposition is not unique without a specifying the properties on a boundary (or at the infinity) for vector fields. This decomposition of vector fields is associated with the name of Helmholtz. There exists more detailed decomposition on three parts which is already unique [111], [112], [113]. This decomposition is called the Helmholtz-Hodge theorem. We formulate this result.

The unique decomposition for any vector field $w \in H^1(S^1(B))$

$$w = v + \mathrm{d}h + \mathrm{d}\psi, \quad \delta v = 0, \quad \psi\,|_{\partial B} = 0, \quad \langle v, \nu\rangle\,|_{\partial B} = 0, \tag{17}$$

holds with the potential vector field $\mathrm{d}\psi$, the solenoidal vector field v and the harmonic vector field $\mathrm{d}h$. A unit vector ν is the vector of outer to the boundary ∂B normal, $v \in H_0^1(S^1(B))$, $\psi \in H_0^2(B)$. We use the designation $w^s = v + \mathrm{d}h$. Obviously this field is the solenoidal field as h is a harmonic function and hence the equation (17) can be rewritten as

$$w = w^s + \mathrm{d}\psi, \quad \delta\, w^s = 0, \quad \psi\,|_{\partial B} = 0.$$

Combining $\mathrm{d}h$ and $\mathrm{d}\psi$, $u = \mathrm{d}h + \mathrm{d}\psi = \mathrm{d}(h+\psi) = \mathrm{d}\widetilde{\psi}$, we obtain a modification of the decomposition (17),

$$w = v + \mathrm{d}\widetilde{\psi}, \quad \delta\, v = 0, \quad \langle v, \nu\rangle\,|_{\partial B} = 0.$$

3.1 The ray transforms

We define two types of the ray transforms acting on vector fields given in the plane.

The longitudinal ray transform of a vector field $v(x) = (v_1(x), v_2(x))$, $x \in B$, $s \in [-1,1]$, $\eta \in \partial B$, is defined by a formula

$$(\mathcal{P}v)(\eta, s) = \int_{-\infty}^{\infty} \big\langle \eta, v(s\xi + t\eta)\big\rangle dt = \int_{-\infty}^{\infty} (\eta^1 v_1 + \eta^2 v_2) dt. \tag{18}$$

This transform is well known and often if one speaks about the vector tomography problem he means that the data are connected with is the values of this integral operator. The second type of ray transforms namely the transverse ray transform was defined previously for spaces of a dimension more than 2, and for symmetric tensor fields of any rank [26].

The transverse ray transform of a vector field $u(x) = (u_1(x), u_2(x))$, $x \in B$, $s \in [-1,1], \xi \in \partial B$, is defined similarly,

$$(\mathcal{P}^{\perp}u)(\xi, s) = \int_{-\infty}^{\infty} \big\langle \xi, u(s\xi + t\eta)\big\rangle dt = \int_{-\infty}^{\infty} (\xi^1 u_1 + \xi^2 u_2) dt. \tag{19}$$

Here $\xi = (\cos\alpha, \sin\alpha)$ is a normal vector of the straight line, $\eta = \xi^{\perp} = (-\sin\alpha, \cos\alpha)$ is a directional vector of the straight line along which the integration is carried out. The function $(\mathcal{P}w)(\alpha, s)$ is an image of the longitudinal ray transform which is the integral calculated along the straight line $L_{\xi,s}$ for the component of the vector field w parallel to $L_{\xi,s}$. The function $(\mathcal{P}^{\perp}w)(\alpha, s)$ (the image of the transverse ray transform) is the integral along the same straight line $L_{\xi,s}$ for the component of the vector field w orthogonal to $L_{\xi,s}$. The longitudinal and (or) transverse ray transforms are an input data for the vector tomography problem with a purpose to reconstruct the vector field w. In other terms it's required to solve the operator equation of the first kind (18) or (19).

Remark 3.1. *The definitions of the ray transforms do not contain the specifications of the vector fields and corresponding functional spaces. It turns out [26] that these spaces can be not only the spaces of the smooth vector fields but the Sobolev spaces $H^k(S^1(B))$ also, $k \geq 0$ integer, and even the space $L_2(S^1(B))$. In other words, the ray transform acting on a smooth field extended to a bounded operator acting from $H^k(S^1(B))$ to $H^k(Z)$. This result is general and applies to the ray transforms of symmetric m-tensor fields also, $m \geq 0$. Below we consider this fact as known and do not justify it or link it to the monograph [26].*

Further properties of the ray transforms as well as the inversion formulas are formulated for more narrow than $L_2(S^1(B))$ class of vector fields with potentials from $\mathcal{C}^k(B)$, $k \geq 0$.

Proposition 3.1. *Let $v \in \mathcal{C}^k(S^1(B))$ be a solenoidal and $u \in \mathcal{C}^k(S^1(B))$ a potential vector fields, k integer, $k \geq -1$. The following properties hold:*

1) *there exist φ, $\psi \in \mathcal{C}^{k+1}(B)$, such that $v = \mathrm{d}^\perp\varphi$, $u = \mathrm{d}\psi$;*

2) *if φ, $\psi \in \mathcal{C}^{k+1}(B)$, $k \geq 1$, then $\delta(\mathrm{d}\psi) = \triangle\psi$, $\delta(\mathrm{d}^\perp\varphi) = 0$, $\delta^\perp(\mathrm{d}\psi) = 0$, $\delta^\perp(\mathrm{d}^\perp\varphi) = \triangle\varphi$;*

3) $(\mathcal{P}u)(\xi^\perp, s) = 0, \quad (\mathcal{P}^\perp v)(\xi, s) = 0$;

4) *the transverse ray transform of the field $u \in \mathcal{C}^k(S^1(B))$, $u = \mathrm{d}\psi$, $\psi \in \mathcal{C}^{k+1}(B)$, is connected with the Radon transform of its potential ψ by the relation*

$$\big(\mathcal{P}^\perp u\big)(\xi, s) = \frac{\partial}{\partial s}\mathcal{R}\psi(\xi, s);$$

5) *the longitudinal ray transform of the field $v \in \mathcal{C}^k(S^1(B))$, $v = \mathrm{d}^\perp\varphi$, $\varphi \in \mathcal{C}^{k+1}(B)$, is connected with the Radon transform of its potential φ by the relation*

$$\big(\mathcal{P}v\big)(\xi^\perp, s) = \frac{\partial}{\partial s}\mathcal{R}\varphi(\xi, s);$$

6) *if $\varphi = \psi$, $\varphi, \psi \in \mathcal{C}^k(B)$, $k \geq 0$, $u = \mathrm{d}\varphi$, $v = \mathrm{d}^\perp\varphi$ then $\langle u\,, v\rangle = 0$. Besides the relations*

$$\big(\mathcal{P}^\perp u\big)(\xi, s) = \big(\mathcal{P}v\big)(\xi^\perp, s) = \frac{\partial}{\partial s}\mathcal{R}\varphi(\xi, s)$$

are valid.

The formulated above properties are either known or their proofs are simple and consist in a direct verification. We would emphasize the fact that vector fields themselves may be discontinuous but their potentials are continuous in $\mathbb{R}^2$.

Being limited ourselves by the class $\mathcal{C}^k(B)$ of potentials and taking into account the properties 1 and 2 we obtain the following version of the decomposition formula (17),

$$w = \mathrm{d}^\perp\varphi + \mathrm{d}\psi, \quad \psi\,|_{\partial B} = 0, \quad \varphi\,|_{\partial B} = 0, \tag{20}$$

without any harmonic vector field.

A vector field reconstruction. Conclusions.

1. The property 3 means that there are invisible vector fields. For the longitudinal ray transform this vector fields are potential with vanishing potential on the boundary. For the transverse ray transform this vector fields are solenoidal with a vanishing normal component on the boundary.

2. The properties 4, 5 imply that only the solenoidal part v of a vector field w is recovered from the longitudinal ray transform and only the potential part u of a vector field w is recovered from the transverse ray transform.

3. The longitudinal and transverse ray transforms allow to reconstruct the original vector field w entirely if they are known simultaneously.

4. The properties 4-6 connecting the ray transforms of vector fields with the Radon transform of their potentials allow to obtain the inversion formulas for the potentials ψ, φ, almost coinciding with the inversion formula for the Radon transform.

Thus the formulations of the vector tomography problems depend on the available input information.

The formulations of main vector tomography problems.

1. It is necessary to reconstruct the solenoidal part v of the unknown vector field w by its known longitudinal ray transform.

2. It is necessary to reconstruct the potential part u of the unknown vector field w by its known transverse ray transform.

3. It is necessary to reconstruct the original vector field w from its known ray transforms, i.e. by the longitudinal and transverse ray transforms.

3.2 Inversion formulas

The derivation of the inversion formulas for vector fields are described below. The formulated above properties of the ray transforms and the known inversion formulas for the Radon transform are taking in mind.

We obtain now the inversion formulas for components of a vector field under assumption that we have the input data of the type 3, i.e. we assume that the longitudinal (18) and transverse (19) ray transforms of the vector field w are known. The ray transforms can be written as

$$\begin{aligned}\mathcal{P}w &= \eta^1\big(\mathcal{R}w_1\big) + \eta^2\big(\mathcal{R}w_2\big) = -\sin\alpha\big(\mathcal{R}w_1\big) + \cos\alpha\big(\mathcal{R}w_2\big)\\ \mathcal{P}^{\perp}w &= \xi^1\big(\mathcal{R}w_1\big) + \xi^2\big(\mathcal{R}w_2\big) = \ \cos\alpha\big(\mathcal{R}w_1\big) + \sin\alpha\big(\mathcal{R}w_2\big),\end{aligned}$$

with unknown functions $\big(\mathcal{R}w_1\big)$, $\big(\mathcal{R}w_2\big)$. Solving this system we obtain the expressions

$$\begin{aligned}\mathcal{R}w_1 &= \eta^1\big(\mathcal{P}w\big) + \eta^2\big(\mathcal{P}^{\perp}w\big) = -\sin\alpha\big(\mathcal{P}w\big) + \cos\alpha\big(\mathcal{P}^{\perp}w\big)\\ \mathcal{R}w_2 &= \xi^1\big(\mathcal{P}w\big) + \xi^2\big(\mathcal{P}^{\perp}w\big) = \ \cos\alpha\big(\mathcal{P}w\big) + \sin\alpha\big(\mathcal{P}^{\perp}w\big).\end{aligned} \tag{21}$$

for the Radon transforms $\mathcal{R}w_1$, $\mathcal{R}w_2$ of the components of the unknown vector field w depending on the known ray transforms $\mathcal{P}w$ and $\mathcal{P}^{\perp}w$. Applying any of the numerous inversion formulas for the Radon transform to both sides of the obtained expressions, we obtain the components w_1, w_2 of the required field.

In practice often only one of the ray transforms is known: either longitudinal or transverse. Let the longitudinal ray transform of the vector field w of the class $\mathcal{C}^k(S^1(B))$ be known,

$$\mathcal{P}w = \eta^1\big(\mathcal{R}w_1\big) + \eta^2\big(\mathcal{R}w_2\big) = -\sin\alpha\big(\mathcal{R}w_1\big) + \cos\alpha\big(\mathcal{R}w_2\big).$$

First of all we would note that due to the decomposition (20) and the property 3 we have $\big(\mathcal{P}(\,\mathrm{d}\psi)\big) = 0$. Therefore $\mathcal{P}w = \mathcal{P}v$, where $v = \mathrm{d}^\perp\varphi$. Consequently we may reconstruct only the solenoidal part $v = (v_1, v_2)$ of the field. But due to the property 3 the solenoidal field v belongs to the kernel of the transverse ray transform, $\mathcal{P}^\perp v = 0$, ie $0 = \mathcal{P}^\perp v = \cos\alpha\big(\mathcal{R}v_1\big) + \sin\alpha\big(\mathcal{R}v_2\big)$. Hence it follows that

$$\mathcal{R}v_1 = \eta^1\mathcal{P}w \equiv -\sin\alpha\mathcal{P}w, \quad \mathcal{R}v_2 = \eta^2\mathcal{P}w \equiv \cos\alpha\mathcal{P}w.$$

We can use all the known inversion formulas again.

If the transverse ray transform $\mathcal{P}^\perp w$ is known then, analogically, we obtain the formulas

$$\mathcal{R}u_1 = \xi^1\mathcal{P}^\perp w \equiv \cos\alpha\mathcal{P}^\perp w, \quad \mathcal{R}u_2 = \xi^2\mathcal{P}^\perp w \equiv \sin\alpha\mathcal{P}^\perp w.$$

Further we apply any of the known inversion formulas for the Radon transform to the both sides of the system we obtain the components of the potential part u of the vector field w.

Analyzing the inversion formulas we can derive the definition of the back projection operator $\mathcal{P}^{\#}_{1tf}$ which is applied to the image of the longitudinal ray transform $g(\eta(\alpha), s(x,\alpha)) = (\mathcal{P}w)(\eta, s)$ of the vector field w. Here $s(x,\alpha) = x^1\cos\alpha + x^2\sin\alpha$. The image of this operator is a solenoidal vector field μ, $\delta\mu = 0$ (that is verified directly), independent of the field w,

$$\mu(x) = \mathcal{P}^{\#}_{1tf}\big((\mathcal{P}w)(\eta(\alpha), s(x,\alpha))\big)(x). \tag{22}$$

The components of the field (22) are

$$\mu_j(x) = \frac{1}{2\pi}\int_0^{2\pi} \eta^j(\mathcal{P}v)(\eta(\alpha), s(x,\alpha))d\alpha, \quad j = 1, 2.$$

The back projection operator, applying to the transverse ray transform $h(\xi(\alpha), s(x,\alpha)) = (\mathcal{P}^\perp w)(\xi, s)$ of the field w, gives a potential vector field λ, $\delta^\perp\lambda = 0$,

$$\lambda(x) = \big(\mathcal{P}^\perp\big)^{\#}_{1tf}\big((\mathcal{P}^\perp u)(\xi(\alpha), s(x,\alpha))\big)(x)$$

with the components

$$\lambda_j(x) = \frac{1}{2\pi}\int_0^{2\pi} \xi^j(\mathcal{P}^\perp w)(\xi(\alpha), s(x,\alpha))d\alpha, \quad j = 1, 2.$$

Here $s(x,\alpha) = x^1\cos\alpha + x^2\sin\alpha$.

In terms of vector fields μ, λ we can write the simple inversion formulas (the operator $(-\Delta)^{1/2}$ acts componentwise),

$$v = (-\Delta)^{1/2}\mu, \quad u = (-\Delta)^{1/2}\lambda,$$

where v is the solenoidal part of vector field w, u is the potential part of vector field w. This formulas are very similar to the inversion formulas (14), (15) for the functions, at $\gamma = 1$.

Using the properties 3–6 of vector fields and their potentials, we can obtain another type of the inversion formulas, which reconstruct potentials of vector fields by its known longitudinal or transverse ray transform. After the reconstruction of the potentials it is easy to construct the corresponding parts of the field, solenoidal or potential. We present the inversion formulas for the potential φ of the solenoidal part v of the field w, which, as already noted, completely repeat the formulas (11), (12) in the form

$$\varphi(x^1, x^2) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{P}w)(\alpha, s + x^1\cos\alpha + x^2\sin\alpha)}{s}ds\,d\alpha,$$

$$\varphi(x^1, x^2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}(\mathcal{P}w)'_s(\alpha, s + x^1\cos\alpha + x^2\sin\alpha)\ln(s)ds\,d\alpha.$$

The solenoidal field v is constructed on the base of recovered potential φ: $v = \mathrm{d}^\perp\varphi$. It is necessary to differentiate the potential φ numerically and thereby to find values of the components $v_1 = -\partial\varphi/\partial x^2$, $v_2 = \partial\varphi/\partial x^1$ of the field v. Analogous inversion formulas can be used for reconstruction of the potential ψ of the potential field $u = \mathrm{d}\psi$, and then we find a values of components $u_1 = \partial\psi/\partial x^1$, $u_2 = \partial\psi/\partial x^2$ of the field u.

We formulate the projection theorems for the ray transforms of vector fields. The notations $(\mathcal{P}_\xi w)(s) = (\mathcal{P}w)(\xi, s)$, $(\mathcal{P}_\xi^\perp w)(s) = (\mathcal{P}^\perp w)(\xi, s)$, $|s| \le 1$ are used.

Theorem 3.1. *Let $u = \mathrm{d}\varphi$ and $v = \mathrm{d}^\perp\varphi$ be vector fields, $\varphi \in \mathcal{C}^k(B)$, $k \ge 0$. Then*

$$(\mathcal{P}_\xi v)\widehat{\ }(\sigma) = (\mathcal{P}_\xi^\perp u)\widehat{\ }(\sigma) = i\,\sigma\sqrt{2\pi}\widehat{\varphi}(\sigma\xi), \qquad \sigma \in \mathbb{R}.$$

The proof consists in usage of the projection theorem for the Radon transform, the property 6 of the ray transforms of vector fields and the properties of the Fourier transform.

Let v be a solenoidal part of a symmetric m-tensor field w. In [26] for a case of a Riemannian manifold there was proved the conditional stability estimate

$$\|v\|^2_{L_2(S^m(B))} \le C_t\left(\|w\|_{H^1(S^m(B))}\|\mathcal{P}w\|_{L_2(Z)} + \|\mathcal{P}w\|^2_{H^1(Z)}\right) \tag{23}$$

with a constant C_t independent of w.

In the case of the Euclidian metric it is possible to obtain the estimates for the ray transforms of vector fields, which are stronger then (23) and similar to the estimate for the Radon transform applying to functions (16).

Theorem 3.2. *Let $u = \mathrm{d}^\perp\varphi + \mathrm{d}\psi$ be a vector field with potentials $\psi, \varphi \in H_0^1(B)$. The stability estimates*

$$\begin{aligned}\|\mathrm{d}^\perp\varphi\|^2_{L_2(S^1(B))} &\le C_1\|\mathcal{P}w\|^2_{H^1(Z)},\\ \|\mathrm{d}\psi\|^2_{L_2(S^1(B))} &\le C_1^\perp\|\mathcal{P}^\perp w\|^2_{H^1(Z)},\end{aligned} \tag{24}$$

hold with constants C_1 and $C_1^\perp$ independent of w.

Here are the main steps of proof of this theorem (for details, see [108]). With usage of the property 3 for the ray transforms we deduce $\left(\mathcal{P}^\perp\mathrm{d}^\perp\varphi\right)(\alpha,s) = 0$, $\left(\mathcal{P}\mathrm{d}^\perp\varphi\right)(\alpha,s) = (\mathcal{P}w)(\alpha,s)$. Using (21) we have

$$\left(\mathcal{R}(\mathrm{d}^\perp\varphi)_1\right)(\alpha,s) = -\sin\alpha(\mathcal{P}w)(\alpha,s), \qquad \left(\mathcal{R}(\mathrm{d}^\perp\varphi)_2\right)(\alpha,s) = \cos\alpha(\mathcal{P}w)(\alpha,s).$$

Thus the norm $\|\mathcal{R}(\mathrm{d}^\perp\varphi)_1\|^2_{H^1(Z)}$ can be estimated as

$$\begin{aligned}\|\mathcal{R}(\mathrm{d}^\perp\varphi)_1\|^2_{H^1(Z)} &= \int_Z \Big(\left(\mathcal{R}(\mathrm{d}^\perp\varphi)_1\right)^2 + \Big(\left(\mathcal{R}(\mathrm{d}^\perp\varphi)_1\right)'_s\Big)^2 + \Big(\left(\mathcal{R}(\mathrm{d}^\perp\varphi)_1\right)'_\alpha\Big)^2 \Big) d\alpha\, ds\\ &= \int_Z \Big(((\mathcal{P}\,w)\sin\alpha)^2 + ((\mathcal{P}\,w)'_s\sin\alpha)^2 + ((\mathcal{P}\,w)'_\alpha\sin\alpha + (\mathcal{P}\,w)\cos\alpha)^2\Big) d\alpha\, ds\\ &\leqslant \int_Z \Big((\mathcal{P}\,w)^2(1+\cos^2\alpha) + ((\mathcal{P}\,w)'_s\sin\alpha)^2 + 2((\mathcal{P}\,w)'_\alpha\sin\alpha)^2\Big) d\alpha\, ds\\ &\leqslant 2\|\mathcal{P}\,w\|^2_{H^1(Z)}.\end{aligned}$$

The inequality $\|\mathcal{R}(\mathrm{d}^\perp\varphi)_2\|^2_{H^1(Z)} \leqslant 2\|\mathcal{P}\,w\|^2_{H^1(Z)}$ is deduced in the same way. The estimate (16) leads to the formula

$$\begin{aligned}\|\mathrm{d}^\perp\varphi\|^2_{L_2(S^1(B))} &= \|(\mathrm{d}^\perp\varphi)_1\|^2_{L_2(B)} + \|(\mathrm{d}^\perp\varphi)_2\|^2_{L_2(B)}\\ &\leqslant C\|\mathcal{R}(\mathrm{d}^\perp\varphi)_1\|^2_{H^1(Z)} + C\|\mathcal{R}(\mathrm{d}^\perp\varphi)_2\|^2_{H^1(Z)}\\ &\leqslant 4C\|\mathcal{P}\,w\|^2_{H^1(Z)} = C_1\|\mathcal{P}\,w\|^2_{H^1(Z)}.\end{aligned}$$

The estimate (24) can be obtained analogously.

4 Symmetric 2-tensor fields

The operators of gradient, orthogonal gradient, divergence and orthogonal divergence d, $\mathrm{d}^\perp$, δ, $\delta^\perp$ defined above can be generalized naturally. Namely, the operator d of inner derivation and the operator $\mathrm{d}^\perp$ of inner orthogonal derivation are compositions of the operators of differentiation and symmetrization, d, $\mathrm{d}^\perp : H^k(S^1(B)) \to H^{k-1}(S^2(B))$, $k \ge 1$. They act on vector fields according to the rules

$$\begin{aligned}u_{ij} :=&\ \ (\mathrm{d}w)_{ij} = \frac{1}{2}\Big(\frac{\partial w_i}{\partial x^j} + \frac{\partial w_j}{\partial x^i}\Big),\\ v_{ij} :=&\ \ (\mathrm{d}^\perp w)_{ij} = \frac{1}{2}\Big((-1)^j\frac{\partial w_i}{\partial x^{3-j}} + (-1)^i\frac{\partial w_j}{\partial x^{3-i}}\Big),\end{aligned}$$

and give symmetric 2-tensor fields as a result, $w \in H^k(S^1(B))$, $u, v \in H^{k-1}(S^2(B))$. The operators of divergence δ and orthogonal divergence $\delta^\perp$, $\delta, \delta^\perp : H^k(S^2(B)) \to H^{k-1}(S^1(B))$, act on symmetric 2-tensor field w by the formulas,

$$u_i := (\delta w)_i = \frac{\partial w_{ij}}{\partial x^j} \equiv \frac{\partial w_{i1}}{\partial x^1} + \frac{\partial w_{i2}}{\partial x^2}, \quad v_i := (\delta^\perp w)_i = -\frac{\partial w_{i1}}{\partial x^2} + \frac{\partial w_{i2}}{\partial x^1},$$

and give vector fields u, v.

Three types of symmetric 2-tensor fields can be constructed with usage of potentials from the space $\mathcal{C}^k$, $k \geq 1$, and the operators d, $\mathrm{d}^\perp$. The potential symmetric 2-tensor field $u \in \mathcal{C}^{k-2}(S^2(B))$,

$$u_{ij} := \mathrm{d}^2\psi = \frac{1}{2}\Big(\frac{\partial(\mathrm{d}\psi)_i}{\partial x^j} + \frac{\partial(\mathrm{d}\psi)_j}{\partial x^i}\Big) = \frac{\partial^2\psi}{\partial x^i \partial x^j} \tag{25}$$

is generated by potential $\psi \in \mathcal{C}^k(B)$. The potential symmetric 2-tensor field $\widetilde{u} \in \mathcal{C}^{k-2}(S^2(B))$ has the form

$$\begin{aligned}\widetilde{u}_{ij} \ &:= \mathrm{d}(\mathrm{d}^\perp\chi) = \mathrm{d}^\perp(\mathrm{d}\chi) = \frac{1}{2}\Big(\frac{\partial(\mathrm{d}^\perp\chi)_i}{\partial x^j} + \frac{\partial(\mathrm{d}^\perp\chi)_j}{\partial x^i}\Big)\\ &= \frac{1}{2}\Big((-1)^j\frac{\partial(\mathrm{d}\chi)_i}{\partial x^{3-j}} + (-1)^i\frac{\partial(\mathrm{d}\chi)_j}{\partial x^{3-i}}\Big)\\ &= \frac{1}{2}\Big((-1)^i\frac{\partial^2\chi}{\partial x^{3-i}\partial x^j} + (-1)^j\frac{\partial^2\chi}{\partial x^{3-j}\partial x^i}\Big).\end{aligned} \tag{26}$$

It is defined by potential $\chi \in \mathcal{C}^k(B)$. The solenoidal symmetric 2-tensor field $v \in \mathcal{C}^{k-2}(S^2(B))$,

$$\begin{aligned}v_{ij} := (\mathrm{d}^\perp)^2\varphi = \ & \frac{1}{2}\Big((-1)^j\frac{\partial(\mathrm{d}^\perp\varphi)_i}{\partial x^{3-j}} + (-1)^i\frac{\partial(\mathrm{d}^\perp\varphi)_j}{\partial x^{3-i}}\Big)\\ = \ & (-1)^{i+j}\frac{\partial^2\varphi}{\partial x^{3-i}\partial x^{3-j}}\end{aligned} \tag{27}$$

is defined by potential $\varphi \in \mathcal{C}^k(B)$.

The following properties of commutativity and the connections between scalar products (4) of symmetric 2-tensor fields defined by the formulas (25)–(27) are verified directly easily.

Lemma 4.1. *Let the potentials φ, $\psi \in \mathcal{C}^k(B)$, $k \geq 1$, and tensor field $w_{ij} \in \mathcal{C}^l(S^2(B))$, $l \geq 1$ be given. Then the following properties and relations are valid:*
1) *the operators δ and $\delta^\perp$ are commutative,*

$$\delta\,(\delta^\perp w_{ij}) = \delta^\perp\,(\delta w_{ij}) = \Big(\frac{\partial^2}{\partial(x^1)^2} - \frac{\partial^2}{\partial(x^2)^2}\Big)w_{12} - \frac{\partial^2}{\partial x^1\partial x^2}\big(w_{11} - w_{22}\big);$$

2) *the operators* d *and* $\mathrm{d}^\perp$ *are commutative,*

$$\mathrm{d}(\mathrm{d}^\perp\varphi) = \mathrm{d}^\perp(\mathrm{d}\varphi),$$

and the components of a field $(\mathrm{d}\,\mathrm{d}^\perp)\varphi$ *are calculated by the formulas* (26)*;*
3) *the following equalities are fulfilled,*

$$\langle \mathrm{d}^2\varphi\,,\,\mathrm{d}^2\psi\rangle = \langle (\mathrm{d}^\perp)^2\varphi\,,\,(\mathrm{d}^\perp)^2\psi\rangle;$$
$$\langle \mathrm{d}^2\varphi\,,\,(\mathrm{d}^\perp)^2\psi\rangle = \langle (\mathrm{d}^\perp)^2\varphi\,,\,\mathrm{d}^2\psi\rangle;$$
$$\langle \mathrm{d}^2\varphi\,,\,\mathrm{d}^2\psi\rangle = \Delta\varphi\Delta\psi - \langle \mathrm{d}^2\varphi\,,\,(\mathrm{d}^\perp)^2\psi\rangle;$$
$$\langle (\mathrm{d}^\perp\mathrm{d})\varphi\,,\,(\mathrm{d}^\perp\mathrm{d})\psi\rangle = \frac{1}{2}\Delta\varphi\Delta\psi - \langle \mathrm{d}^2\varphi\,,\,(\mathrm{d}^\perp)^2\psi\rangle.$$

The results of the action of operators δ and $\delta^\perp$ on symmetric 2-tensor fields of various types are collected in the next proposition. All of them can be easily verified directly.

Proposition 4.1. *Let the potentials* φ, ψ $\chi \in \mathcal{C}^k(B)$, $k \geq 3$, *and tensor fields* $u_{ij} = d^2\psi$, $\widetilde{u}_{ij} = (\mathrm{dd}^\perp)\chi$, $v_{ij} = (d^\perp)^2\varphi \in \mathcal{C}^l(S^2(B))$, $l \geq 1$ *be given. Then:*
1)

$$\delta(\mathrm{d}^2\psi) = \mathrm{d}(\triangle\psi) = \triangle(\mathrm{d}\psi),$$
$$\delta^\perp(\mathrm{d}^2\psi) = 0;$$
$$\delta^2(\mathrm{d}^2\psi) = \delta(\mathrm{d}\triangle\psi) = \triangle^2\psi;$$

2)

$$\delta(\mathrm{dd}^\perp\chi) = \frac{1}{2}\mathrm{d}^\perp\triangle\chi = \frac{1}{2}\triangle(\mathrm{d}^\perp\chi),$$
$$\delta^\perp(\mathrm{dd}^\perp\chi) = \frac{1}{2}\mathrm{d}\triangle\chi = \frac{1}{2}\triangle(\mathrm{d}\chi),$$
$$\delta^2(\mathrm{dd}^\perp)\chi = 0,$$
$$(\delta^\perp\delta)(\mathrm{dd}^\perp\chi) = \frac{1}{2}\triangle^2\chi,$$
$$(\delta^\perp)^2(\mathrm{dd}^\perp\chi) = 0;$$

3)

$$\delta^\perp((\mathrm{d}^\perp)^2\varphi) = \mathrm{d}^\perp(\triangle\varphi) = \triangle(\mathrm{d}^\perp\varphi),$$
$$\delta((\mathrm{d}^\perp)^2\varphi) = 0;$$
$$(\delta^\perp)^2((\mathrm{d}^\perp)^2\varphi) = \triangle^2\varphi.$$

Theorems of decomposition of a tensor field on a potential and solenoidal part are important for studying of tensor fields. The general theorem of decomposition of a symmetric m-tensor field given in a Riemannian manifold of dimension n is proved in [26]. We formulate it in terms of our statements.

Theorem 4.1. *For every field* $w \in H^{k-m}(S^m(B))$, $k \geq m$, *there exist a potential* $u \in H_0^{k-m+1}(S^{m-1}(B))$ *and solenoidal* $v \in H^{k-m}(S^m(B))$ *symmetric* m*-tensor fields such that*

$$w = v + \mathrm{d}u, \quad \delta v = 0.$$

This decomposition is unique.

Applying this theorem for $m = 1$, $m = 2$, the decomposition theorem of a vector field [111], [112] and representation of a solenoidal field by potential [114], we obtain one of the variants of the decomposition theorem for the symmetric 2-tensor field (see also [103]). A property of orthogonality of the corresponding subspaces is checked by means of Gauss theorem.

Theorem 4.2. *For any field $w \in H^{k-2}(S^2(B))$, $k \geq 2$, there exists the unique decomposition onto potential $u, \widetilde{u} \in H^{k-2}(S^2(B))$ and solenoidal $v \in H^{k-2}(S^2(B))$ symmetric 2-tensor fields generated by potentials $\varphi, \chi, \psi \in H_0^k(B)$,*

$$w = v + \widetilde{u} + u \equiv (\mathrm{d}^{\perp})^2\varphi + (\mathrm{d}\mathrm{d}^{\perp})\chi + \mathrm{d}^2\psi, \quad \delta v = 0, \quad \delta^{\perp} u = 0. \tag{28}$$

The decomposition (28) is orthogonal in $L_2(S^2(B))$ in terms of the scalar product (5),

$$\big(\mathrm{d}^2\psi\,,\,(\mathrm{d}^{\perp})^2\varphi\big) = 0; \quad \big(\mathrm{d}^2\psi\,,\,(\mathrm{d}\mathrm{d}^{\perp})\chi\big) = 0; \quad \big((\mathrm{d}^{\perp})^2\varphi\,,\,(\mathrm{d}\mathrm{d}^{\perp})\chi\big) = 0.$$

Remark 4.1. *The decomposition of symmetric 2-tensor field (28) is valid if the following conditions on the potentials are carried out,*

$$\psi\mid_{\partial B} = 0, \quad \chi\mid_{\partial B} = 0, \quad \varphi\mid_{\partial B} = 0,$$

$$\mathrm{d}\psi\mid_{\partial B} = 0, \quad \mathrm{d}\chi\mid_{\partial B} = 0, \quad \mathrm{d}\varphi\mid_{\partial B} = 0,$$

Particularly they are valid automatically with potentials which belong to the space $\mathcal{D}(B)$ of test functions. Hence the fields (w_{ij}) belong to the space $\mathcal{D}(S^2(B))$. We can choose the other spaces of potentials. For example, we can choose potentials ψ, χ, φ from the classes $\mathcal{C}^k(B)$, $k > 0$. Then $\mathrm{d}^2 : \mathcal{C}^k(B) \to \mathcal{C}^{k-2}(S^2(B))$, $(\mathrm{d}^{\perp})^2 : \mathcal{C}^k(B) \to \mathcal{C}^{k-2}(S^2(B))$, $\mathrm{d}(\mathrm{d}^{\perp}) : \mathcal{C}^k(B) \to \mathcal{C}^{k-2}(S^2(B))$. In this case the above properties of vanishing of the potentials and their first derivatives saved, but generated by them 2-tensor fields may be discontinuous with discontinuities of the first kind. This property is differs from those for the fields $w \in \mathcal{D}(S^2(B))$ vanishing on ∂B together with all their derivatives.

4.1 The ray transforms

We define the ray transforms acting on symmetric 2-tensor fields below.

The longitudinal ray transform of a symmetric 2-tensor field $w = (w_{ij})$, being a direct generalization (the definition (18)) of the longitudinal ray transform for vector fields, takes the form

$$(\mathcal{P}w)(\eta, s) = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\eta^i\eta^j dt. \tag{29}$$

The transverse ray transform of a symmetric 2-tensor field (w_{ij}) is defined by the formula

$$(\mathcal{P}^{\perp}w)(\xi, s) = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\xi^i\xi^j dt \tag{30}$$

and generalize the transverse ray transform for vector fields.

With increasing of the rank $m > 1$ of a symmetric tensor field it becomes possible to define new types of the ray transforms, namely the mixed ray transforms. When $m = 2$, the only one type of such transforms is possible.

The mixed ray transform of a symmetric 2-tensor field (w_{ij}) is defined by the formula

$$(\mathcal{P}^{\dagger}w)(\xi, s) = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\eta^i\xi^j dt \equiv \int_{-\infty}^{\infty} w_{ij}\xi^i\eta^j dt. \tag{31}$$

The last identity means that the integrand, by virtue of the symmetry properties $w_{12} = w_{21}$, is invariant under interchange of vectors $\xi \leftrightarrow \eta$, i.e. the definition is correct.

Descriptions of images and kernels of the ray transforms, as well as relationships between them and the Radon transform of the corresponding potentials, are given in the following proposition.

Proposition 4.2. *Let $v \in \mathcal{C}^k(S^2(B))$ be a solenoidal, $u \in \mathcal{C}^k(S^2(B))$, $\widetilde{u} \in \mathcal{C}^k(S^2(B))$ — potential 2-tensor fields, k is integer, $k \geq -1$, and $w = v + \widetilde{u} + u$. Then the following properties are valid.*

1) $$(\mathcal{P}\widetilde{u})(\eta, s) = (\mathcal{P}u)(\eta, s) = 0, \quad (\mathcal{P}^{\perp}v)(\xi, s) = (\mathcal{P}^{\perp}\widetilde{u})(\xi, s) = 0,$$
$$(\mathcal{P}^{\dagger}v)(\xi, s) = (\mathcal{P}^{\dagger}u)(\xi, s) = 0.$$

2) $$\mathcal{P}w = \mathcal{P}\big((\mathrm{d}^{\perp})^2\varphi\big) = \mathcal{P}v, \quad \mathcal{P}^{\dagger}w = \mathcal{P}^{\dagger}\big((\mathrm{d}\mathrm{d}^{\perp})\chi\big) = \mathcal{P}^{\dagger}\widetilde{u},$$
$$\mathcal{P}^{\perp}w = \mathcal{P}^{\perp}(\mathrm{d}^2\psi) = \mathcal{P}^{\perp}u.$$

3) *The transverse ray transform of a potential 2-tensor field $u \in \mathcal{C}^k(S^2(B))$, $u = \mathrm{d}^2\psi$, $\psi \in \mathcal{C}^{k+2}(B)$, is connected with the Radon transform of its potential ψ by the relation*

$$\big(\mathcal{P}^{\perp}u\big)(\xi, s) = \frac{\partial^2}{\partial s^2}\mathcal{R}\psi(\xi, s).$$

4) *The longitudinal ray transform of a solenoidal 2-tensor field $v \in \mathcal{C}^k(S^2(B))$, $v = (\mathrm{d}^{\perp})^2\varphi$, $\varphi \in \mathcal{C}^{k+2}(B)$, is connected with the Radon transform of its potential φ by the relation*

$$\big(\mathcal{P}v\big)(\xi^{\perp}, s) = \frac{\partial^2}{\partial s^2}\mathcal{R}\varphi(\xi, s).$$

5) *The mixed ray transform of a potential 2-tensor field $\widetilde{u} \in \mathcal{C}^k(S^2(B))$, $\widetilde{u} = (\underset{\cdot}{}\mathrm{d}^{\perp})\chi$, $\chi \in \mathcal{C}^{k+2}(B)$, is connected with the Radon transform of its potential χ by the relation*

$$\big(\mathcal{P}^{\dagger}w\big)(\xi^{\perp}, s) = \big(\mathcal{P}^{\dagger}\widetilde{u}\big)(\xi^{\perp}, s) = \frac{1}{2}\frac{\partial^2}{\partial s^2}\mathcal{R}\chi(\xi, s).$$

6) *Let $\varphi, \psi, \chi \in \mathcal{C}^k(B)$, $k \geq 1$, be potentials. The fields $u = \mathrm{d}^2\varphi$, $v = (\mathrm{d}^{\perp})^2\varphi$, $\widetilde{u} = (\underset{\cdot}{}\mathrm{d}^{\perp})\chi$ are the corresponding symmetric 2-tensor fields. If $\varphi = \psi = \chi$ then*

$$\big(\mathcal{P}^{\perp}u\big)(\xi, s) = \big(\mathcal{P}v\big)(\xi^{\perp}, s) = 2\big(\mathcal{P}^{\dagger}\widetilde{u}\big)(\xi^{\perp}, s) = \frac{\partial^2}{\partial s^2}\mathcal{R}\varphi(\xi, s).$$

The proofs of these statements are simple and consist in applications of the properties of the Radon transform, which were given above, and direct verification. We emphasize that the 2-tensor fields on the boundary ∂B may be discontinuous, despite the fact that the potentials and their first derivatives are continuous in $\mathbb{R}^2$.

The reconstruction of symmetric 2-tensor fields.

We draw conclusions from the properties formulated in Proposition 4.2.

The property 1 means that there exist invisible 2-tensor fields. That's potential fields of the type $\mathrm{d}^2\psi$ with vanishing on the boundary potential ψ and its first derivatives — for the longitudinal and mixed ray transforms. That's potential fields of the type $(\mathrm{d}\mathrm{d}^{\perp})\chi$ with vanishing on the boundary potential χ and its first derivatives — for the longitudinal and transverse ray transforms. That,s solenoidal fields with vanishing on the boundary potential φ and its first derivatives — for the mixed and transverse ray transforms.

It follows from the property 2 that only the solenoidal part $v = (\mathrm{d}^{\perp})^2\varphi$ of a symmetric 2-tensor field w is recovered from the longitudinal ray transform of w. The potential part $u = \mathrm{d}^2\psi$ of 2-tensor field w is recovered from the transverse ray transform of w. The potential part $\widetilde{u} = (\mathrm{d}\mathrm{d}^{\perp})\chi$ of 2-tensor field w is recovered from the mixed ray transform of w.

The longitudinal, mixed and transverse ray transforms to be known simultaneously allow to reconstruct the original symmetric 2-tensor field w entirely.

The relations 3–5 are the base for derivation of inversion formulas for potentials ψ, φ, χ, which are easily derived from the inversion formula for the Radon transforms of the corresponding potentials.

Statements of the main tomography problems for symmetric 2-tensor fields.

Clearly the statements of the problems depend on the type and completeness (in terms of presence of the information about the values of all three types of ray transforms) of the input data. The only one of formulated below statements was known earlier, the others are formulated for the first time.

1. It is necessary to reconstruct the solenoidal part v of the unknown symmetric 2-tensor field w by its known longitudinal ray transform. It was exactly this statement that was known within the framework of the tensor tomography and the integral geometry of tensor fields.

2. It is necessary to reconstruct the potential part u of the unknown 2-tensor field w by its known transverse ray transform.

3. It is necessary to reconstruct the potential part $\widetilde{u}$ of the unknown 2-tensor field w by its known mixed ray transform.

4. It is necessary to reconstruct the whole original symmetric 2-tensor field w by its known three ray transforms, i.e. from the longitudinal, mixed and transverse ray transforms.

5. It is necessary to reconstruct two certain parts (of three) of the original 2-tensor field by its known two corresponding ray transforms.

4.2 Inversion formulas

We proceed to derivation of the inversion formulas. First of all with usage of the properties 3–5 of symmetric 2-tensor fields and their potentials, we present the inversion formula, which reconstructs a potential of corresponding part of a field w by its known longitudinal, or mixed, or transverse ray transform. After the reconstruction of the potential the corresponding part of the field, solenoidal or one of the potentials can be constructed by double differentiation. As for vector fields the inversion formula for the potential φ is similar to the formula (12),

$$\varphi(x^1, x^2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} (\widetilde{\mathcal{P}}w)(\alpha, s + x^1 \cos\alpha + x^2 \sin\alpha) \ln|s| ds\, d\alpha,$$

where $\widetilde{\mathcal{P}}w$ is the longitudinal (29), the transverse (30) or, multiplied by 2, the mixed (31) ray transform of symmetric 2-tensor field of the type (28). The solenoidal field v of the field w constructed on the base of recovered potential φ, $v = (\mathrm{d}^\perp)^2\varphi$, if the longitudinal ray transform of the field w was known as input data. One of potential parts $u = \mathrm{d}^2\varphi$, $\widetilde{u} = (\mathrm{d}\mathrm{d}^\perp)\varphi$ of the field w is reconstructed on the base of recovered potential φ if the mixed or the transverse ray transform was known. Further it is necessary twice differentiation of the potential numerically and thereby to find the values of components of corresponding parts of the field.

We turn to the construction of inversion formulas that give the components of a field. Let w be a symmetric 2-tensor field of the type (28). We assume that all the ray transforms are known. Namely the longitudinal (29), the transverse (30) and the mixed (31) ray transforms of the field w are known. We use the same approach that was used for obtaining the inversion formulas in the case of vector fields. For this purpose we express the ray transforms of the field w through the Radon transform of its components,

$$\begin{aligned}
\mathcal{P}w &= \sin^2\alpha\big(\mathcal{R}w_{11}\big) - \sin 2\alpha\big(\mathcal{R}w_{12}\big) + \cos^2\alpha\big(\mathcal{R}w_{22}\big) \\
\mathcal{P}^\dagger w &= -\frac{1}{2}\sin 2\alpha\big(\mathcal{R}w_{11}\big) + \cos 2\alpha\big(\mathcal{R}w_{12}\big) + \frac{1}{2}\sin 2\alpha\big(\mathcal{R}w_{22}\big) \\
\mathcal{P}^\perp w &= \cos^2\alpha\big(\mathcal{R}w_{11}\big) + \sin 2\alpha\big(\mathcal{R}w_{12}\big) + \sin^2\alpha\big(\mathcal{R}w_{22}\big).
\end{aligned} \tag{32}$$

The matrix of this system of equations has rank 3 and its determinant is equal to -1. The system (32) has unique solution,

$$\begin{aligned}
\mathcal{R}w_{11} &= \sin^2\alpha\big(\mathcal{P}w\big) - \sin 2\alpha\big(\mathcal{P}^\dagger w\big) + \cos^2\alpha\big(\mathcal{P}^\perp w\big) \\
\mathcal{R}w_{12} &= -\frac{1}{2}\sin 2\alpha\big(\mathcal{P}w\big) + \cos 2\alpha\big(\mathcal{P}^\dagger w\big) + \frac{1}{2}\sin 2\alpha\big(\mathcal{P}^\perp w\big) \\
\mathcal{R}w_{22} &= \cos^2\alpha\big(\mathcal{P}w\big) + \sin 2\alpha\big(\mathcal{P}^\dagger w\big) + \sin^2\alpha\big(\mathcal{P}^\perp w\big),
\end{aligned} \tag{33}$$

and gives a dependence of the Radon transforms $\mathcal{R}w_{11}$, $\mathcal{R}w_{12}$, $\mathcal{R}w_{22}$ of components of the field w on the known ray transforms $\mathcal{P}w$, $\mathcal{P}^\dagger w$ and $\mathcal{P}^\perp w$. Applying any of a numerous inversion formulas for the Radon transform to both sides of the obtained expressions, we get the components of the required field.

Suppose now that two of three ray transforms are given as input data. For example the longitudinal (29) and transverse (30) ray transforms. By means of the property 1 of Proposition 4.2, we obtain a system of equations of the form (32), in which we assume $\mathcal{P}^\dagger w = 0$. Solving this system we obtain the relations

$$\begin{aligned}\mathcal{R}\widetilde{w}_{11} &= \sin^2\alpha\big(\mathcal{P}w\big) + \cos^2\alpha\big(\mathcal{P}^\perp w\big)\\ \mathcal{R}\widetilde{w}_{12} &= -\frac{1}{2}\sin 2\alpha\big(\mathcal{P}w\big) + \frac{1}{2}\sin 2\alpha\big(\mathcal{P}^\perp w\big)\\ \mathcal{R}\widetilde{w}_{22} &= \cos^2\alpha\big(\mathcal{P}w\big) + \sin^2\alpha\big(\mathcal{P}^\perp w\big).\end{aligned}$$

for the Radon transforms $\mathcal{R}\widetilde{w}_{11}$, $\mathcal{R}\widetilde{w}_{12}$, $\mathcal{R}\widetilde{w}_{22}$ of components of the field $\widetilde{w} = (\mathrm{d}^\perp)^2\varphi + \mathrm{d}^2\psi$ depending on the known ray transforms $\mathcal{P}w$ and $\mathcal{P}^\perp w$. It is enough now to apply any of numerous inversion formulas, known for the Radon transform, to both sides of the obtained expressions, and we get the corresponding components of the required field as a result.

Finally we assume that only one of three ray transforms is given as input data. Let it be the longitudinal ray transform. In this case it is possible to reconstruct only the solenoidal part $v = (\,\mathrm{d}^\perp)^2\varphi$ of the field w. It is necessary for this to use the relations

$$\mathcal{R}v_{11} = \sin^2\alpha\big(\mathcal{P}w\big), \quad \mathcal{R}v_{12} = -\sin\alpha\cos\alpha\big(\mathcal{P}w\big), \quad \mathcal{R}v_{22} = \cos^2\alpha\big(\mathcal{P}w\big).$$

If the transverse ray transform of the field w is known only, then it is possible to reconstruct the potential part $u = \mathrm{d}^2\psi$ of the field w, which components can be found from the relations

$$\mathcal{R}u_{11} = \cos^2\alpha\big(\mathcal{P}^\perp w\big), \quad \mathcal{R}u_{12} = \sin\alpha\cos\alpha\big(\mathcal{P}^\perp w\big), \quad \mathcal{R}u_{22} = \sin^2\alpha\big(\mathcal{P}^\perp w\big).$$

Once again we can use all the inversion formulas, which are suitable for a reconstruction of functions.

It is convenient to present the inversion formulas in operator form (15) or (14) for $\gamma = 1$. At first we need in expressions for back projection operators.

Suppose that the longitudinal ray transform of the field w is known. We should reconstruct the solenoidal part v of the field w, $w = v + \widetilde{u} + u$,

$$f(\eta, s) = \mathcal{P}w = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\eta^i\eta^j\, dt.$$

A symmetric 2-tensor field

$$\mu(x) = \mathcal{P}^{\#}_{2tf}\big(f(\eta, s(x,\alpha))\big)(x)$$

is the result of action of the back projection operator for the longitudinal ray transform. It has the components

$$\mu_{ij}(x^1, x^2) = \frac{1}{2\pi}\int_0^{2\pi} \eta^i\,\eta^j\,(\mathcal{P}w)(\eta, s(x,\alpha))d\alpha, \quad i,j = 1,2, \tag{34}$$

where $\eta = (-\sin\alpha, \cos\alpha)$, $s(x,\alpha) = x^1\cos\alpha + x^2\sin\alpha$. It may be checked in direct way that the field (34) is solenoidal. The inversion formulas

$$v_{11} = (-\Delta)^{1/2}\mu_{11}, \quad v_{12} = (-\Delta)^{1/2}\mu_{12}, \quad v_{22} = (-\Delta)^{1/2}\mu_{22},$$

give the solenoidal part v of the field w.

Based on the transverse ray transform as on input data, by which we can reconstruct only the potential part u of the field w,

$$g(\xi, s) = \mathcal{P}^{\perp} w = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\xi^i \xi^j \, dt,$$

we construct a symmetric 2-tensor field

$$\lambda(x) = \big(\mathcal{P}^{\perp}\big)^{\#}_{2tf}\big(g(\xi, s(x, \alpha))\big)(x),$$

which is the result of action of the back projection operator for the transverse ray transform. This field has the components

$$\lambda_{ij}(x^1, x^2) = \frac{1}{2\pi} \int_0^{2\pi} \xi^i \, \xi^j \, (\mathcal{P}^{\perp} w)(\xi, s(x, \alpha)) d\alpha, \quad i, j = 1, 2,$$

$\xi = (\cos\alpha, \sin\alpha)$, $s(x, \alpha) = x^1 \cos\alpha + x^2 \sin\alpha$. The field is potential, $\delta^{\perp}\lambda = 0$. The inversion formulas

$$u_{11} = (-\Delta)^{1/2}\lambda_{11}, \quad u_{12} = (-\Delta)^{1/2}\lambda_{12}, \quad u_{22} = (-\Delta)^{1/2}\lambda_{22},$$

give the potential part u of the field w.

It is possible to reconstruction the potential part $\widetilde{u} = (\mathrm{d}\mathrm{d}^{\perp})\chi$ of the field w by the mixed ray transform

$$h(\xi, s) = \mathcal{P}^{\dagger} w = \int_{-\infty}^{\infty} w_{ij}(s\xi + t\eta)\eta^i \xi^j \, dt$$

The back projection operator maps the function $h(\xi, s)$ into a symmetric 2-tensor field by the rule

$$\nu(x) = \big(\mathcal{P}^{\dagger}\big)^{\#}_{2tf}\big(h(\xi, s(x, \alpha))\big)(x).$$

Components of the field are

$$\nu_{ij}(x^1, x^2) = \frac{1}{2\pi} \int_0^{2\pi} \eta^i \, \xi^j \, (\mathcal{P}^{\dagger} w)(\xi, s(x, \alpha)) d\alpha, \quad i, j = 1, 2,$$

Inversion formulas look similarly. They give the components $\widetilde{u}_{11}$, $\widetilde{u}_{12}$, $\widetilde{u}_{22}$ of the potential field $\widetilde{u}$.

The projection theorems for the ray transforms of 2-tensor fields are formulated below. The notations $(\mathcal{P}_{\xi} w)(s) = (\mathcal{P} w)(\xi, s)$, $(\mathcal{P}^{\dagger}_{\xi} w)(s) = (\mathcal{P}^{\dagger} w)(\xi, s)$, $(\mathcal{P}^{\perp}_{\xi} w)(s) = (\mathcal{P}^{\perp} w)(\xi, s)$, $|s| \le 1$, are used.

Theorem 4.3. *If $u = \mathrm{d}^2\varphi$, $\widetilde{u} = \mathrm{d}\mathrm{d}^{\perp}\varphi$, $v = (\mathrm{d}^{\perp})^2\varphi$, where $\varphi \in \mathcal{C}^k(B)$, $k \ge 1$, then*

$$(\mathcal{P}_{\xi} v)\hat{\ }(\sigma) = 2(\mathcal{P}^{\dagger}_{\xi} \widetilde{u})\hat{\ }(\sigma) = (\mathcal{P}^{\perp}_{\xi} u)\hat{\ }(\sigma) = -\sigma^2 \sqrt{2\pi}\, \widehat{\varphi}(\sigma\xi), \qquad \sigma \in \mathbb{R}.$$

The proof consists in usage of the projection theorem 2.1 for the Radon transform, property 6 of the ray transforms of symmetric 2-tensor fields and properties of the Fourier transform.

We wish now to obtain stability estimates for the ray transforms of symmetric 2-tensor fields.

Theorem 4.4. *Let $w = (\mathrm{d}^\perp)^2\varphi + \mathrm{d}\mathrm{d}^\perp\chi + \mathrm{d}^2\psi$ be a symmetric 2-tensor field with potentials $\psi, \chi, \varphi \in H_0^2(B)$. Then the stability estimates*

$$\|(\mathrm{d}^\perp)^2\varphi\|^2_{L_2(S^2(B))} \le C_2\|\mathcal{P}w\|^2_{H^1(Z)},$$

$$\|\mathrm{d}^2\psi\|^2_{L_2(S^2(B))} \le C_2^\perp\|\mathcal{P}^\perp w\|^2_{H^1(Z)}, \tag{35}$$

$$\|\mathrm{d}\mathrm{d}^\perp\chi\|^2_{L_2(S^2(B))} \le C_2^\dagger\|\mathcal{P}^\dagger w\|^2_{H^1(Z)}, \tag{36}$$

are valid with constants C_2, $C_2^\perp$ and $C_2^\dagger$ independent of w.

The proof of this theorem is similar to the proof of theorem 3.2 (for details, see [109]). The properties 1–2 of the ray transforms imply

$$\big(\mathcal{P}^\perp(\mathrm{d}^\perp)^2\varphi\big)(\alpha,s) = \big(\mathcal{P}^\dagger(\mathrm{d}^\perp)^2\varphi\big)(\alpha,s) = 0, \qquad \big(\mathcal{P}(\mathrm{d}^\perp)^2\varphi\big)(\alpha,s) = (\mathcal{P}w)(\alpha,s).$$

By means of (33) we have

$$\begin{aligned}
\big(\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11}\big)(\alpha,s) &= \sin^2\alpha(\mathcal{P}w)(\alpha,s),\\
\big(\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{12}\big)(\alpha,s) &= -\sin\alpha\cos\alpha(\mathcal{P}w)(\alpha,s),\\
\big(\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{22}\big)(\alpha,s) &= \cos^2\alpha(\mathcal{P}w)(\alpha,s).
\end{aligned}$$

Thus the norm $\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11}\|^2_{H^1(Z)}$ can be estimated as

$$\begin{aligned}
&\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11}\|^2_{H^1(Z)}\\
&\quad = \int_Z \Big(\big(\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11}\big)^2 + \big((\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11})'_s\big)^2 + \big((\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11})'_\alpha\big)^2 \Big) d\alpha\, ds\\
&\quad = \int_Z \Big((\mathcal{P}\, w)^2\sin^4\alpha + ((\mathcal{P}\, w)'_s)^2\sin^4\alpha + \big((\mathcal{P}\, w)'_\alpha\sin^2\alpha + 2(\mathcal{P}\, w)\sin\alpha\,\cos\alpha\big)^2 \Big) d\alpha\, ds\\
&\quad \leqslant \int_Z \Big((\mathcal{P}\, w)^2\sin^2\alpha(1+7\cos^2\alpha) + ((\mathcal{P}\, w)'_s)^2\sin^4\alpha + 2\,((\mathcal{P}\, w)'_\alpha)^2\sin^4\alpha \Big) d\alpha\, ds\\
&\quad \leqslant \frac{16}{7}\|\mathcal{P}w\|^2_{H^1(Z)}.
\end{aligned}$$

The inequalities

$$\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{12}\|^2_{H^1(Z)} \leqslant 2\|\mathcal{P}w\|^2_{H^1(Z)}, \qquad \|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{22}\|^2_{H^1(Z)} \leqslant \frac{16}{7}\|\mathcal{P}w\|^2_{H^1(Z)}$$

are deduced in the same way. The estimate (16) leads to the formula

$$\begin{aligned}
\|(\mathrm{d}^\perp)^2\varphi\|^2_{L_2(S^2(B))} &= \|((\mathrm{d}^\perp)^2\varphi)_{11}\|^2_{L_2(B)} + 2\|((\mathrm{d}^\perp)^2\varphi)_{12}\|^2_{L_2(B)} + \|((\mathrm{d}^\perp)^2\varphi)_{22}\|^2_{L_2(B)}\\
&\leqslant C\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{11}\|^2_{H^1(Z)} + 2C\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{12}\|^2_{H^1(Z)} + C\|\mathcal{R}((\mathrm{d}^\perp)^2\varphi)_{22}\|^2_{H^1(Z)}\\
&\leqslant \frac{60}{7}C\|\mathcal{P}\, w\|^2_{H^1(Z)} = C_2\|\mathcal{P}\, w\|^2_{H^1(Z)}.
\end{aligned}$$

The estimates (35) and (36) can be derived in the similar way.

5 Symmetric m-tensor fields

A construction of symmetric m-tensor fields is carried out with usage of operators d and $\mathrm{d}^\perp$ also. This operators are compositions of the operators of gradient and orthogonal gradient, acting on tensor fields, with the operators of symmetrization. Below we present the direct formulas of action of these operators on symmetric tensor fields.

The operator of inner differentiation $\mathrm{d} : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B))$, $k \geq 1$, acts on a symmetric m-tensor field w and gives a symmetric $(m+1)$-tensor field u by the rule

$$u_{i_1\dots i_m j} := (\mathrm{d}w)_{i_1\dots i_m j} = \frac{1}{m+1}\Big(\frac{\partial w_{i_1\dots i_m}}{\partial x^j} + \sum_{k=1}^{m} \frac{\partial w_{i_1\dots i_{k-1} j i_{k+1}\dots i_m}}{\partial x^{i_k}}\Big).$$

The operator of inner orthogonal differentiation $\mathrm{d}^\perp : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B))$,

$$(\mathrm{d}^\perp w)_{i_1\dots i_m j} = \frac{1}{m+1}\Big((-1)^j \frac{\partial w_{i_1\dots i_m}}{\partial x^{3-j}} + \sum_{k=1}^{m} (-1)^{i_k} \frac{\partial w_{i_1\dots i_{k-1} j i_{k+1}\dots i_m}}{\partial x^{3-i_k}}\Big),$$

acts similarly to the operator d. Here $w \in H^k(S^m(B))$, $u \in H^{k-1}(S^{m+1}(B))$, $k \geq 1$. The operators of divergence and orthogonal divergence $\delta, \delta^\perp : H^k(S^m(B)) \to H^{k-1}(S^{m-1}(B))$, $k \geq 1$, act on symmetric m-tensor field w,

$$u_{i_1\dots i_{m-1}} := (\delta w)_{i_1\dots i_{m-1}} = \frac{\partial w_{i_1\dots i_{m-1} j}}{\partial x^j} \equiv \frac{\partial w_{i_1\dots i_{m-1} 1}}{\partial x^1} + \frac{\partial w_{i_1\dots \text{ß}_{m-1} 2}}{\partial x^2},$$

$$v_{i_1\dots i_{m-1}} := (\delta^\perp w)_{i_1\dots i_{m-1}} = (-1)^j \frac{\partial w_{i_1\dots i_{m-1} j}}{\partial x^{3-j}} \equiv -\frac{\partial w_{i_1\dots i_{m-1} 1}}{\partial x^2} + \frac{\partial w_{i_1\dots i_{m-1} 2}}{\partial x^1},$$

and give symmetric $(m-1)$-tensor fields u, v.

One of the statements of Lemma 4.1 consists in the fact that the operators d and $\mathrm{d}^\perp$ are commutative if they act on the potentials. The question arises whether this property be saved if the operators act on a symmetric m-tensor field? The answer to this question is positive.

Lemma 5.1. *Let $w \in H^k(S^m(B))$, $k \geq 2$ be a symmetric m-tensor field. Then*

$$\mathrm{d}^\perp(\mathrm{d}w) = \mathrm{d}(\mathrm{d}^\perp w).$$

The proof consists in a direct verification. Thus it is necessary to use the property of independence of the mixed second derivative of the order of differentiation.

Remark 5.1. *It should be noted that the operators* d *and* $\mathrm{d}^\perp$ *commute and when they act on the arbitrary tensor field, not necessarily symmetric. The proof of this property is much more difficult but it requires the purely technical additions only.*

We give a classification of symmetric m-tensor fields. Let a potential $\varphi \in \mathcal{C}^k(B)$, $k \geq m-1$, be given. It is easy to see that there are $m+1$ different symmetric m-tensor fields

$$(\mathrm{d}^\perp)^m\varphi,\ (\mathrm{d}^\perp)^{m-1}\mathrm{d}\,\varphi,\ \dots,\ (\mathrm{d}^\perp)^{m-j}\mathrm{d}^j\varphi,\ \dots,\ \mathrm{d}^m\varphi,$$

generated by the potential φ. While constructing these fields we make essential use of the commutative property of the operators d and $\mathrm{d}^\perp$. The field $\big(\mathrm{d}^\perp\big)^m\varphi$ is solenoidal, $\delta\big(\mathrm{d}^\perp\big)^m\varphi = 0$, the remaining m fields are potential, and $\delta^\perp\big(\mathrm{d}^j(\mathrm{d}^\perp)^{m-j}\big)\varphi = 0$, $j > 0$. As for vector and symmetric 2-tensor fields, using Theorem 4.1 and the method of mathematical induction, it is easy to obtain a decomposition theorem for a symmetric m-tensor field.

Theorem 5.1. *For every field $w \in H^{k-m}(S^m(B))$ there exist potential $u^{(1)}, \dots, u^{(m)} \in H^{k-m}(S^m(B))$ and solenoidal $v \in H^{k-m}(S^m(B))$ symmetric m-tensor fields, $k \ge m$, and potentials $\varphi, \psi^{(1)}, \dots, \psi^{(m)} \in H^k_0(B)$ such that*

$$w = v + \sum_{j=1}^{m} u^{(j)} \equiv \big(\mathrm{d}^\perp\big)^m \varphi + \sum_{j=1}^{m} \big(\mathrm{d}^\perp\big)^{m-j}\big(\mathrm{d}\big)^j \psi^{(j)}. \tag{37}$$

5.1 The ray transforms

The ray transform operators $\mathcal{P}^{(j)} : H^k(S^m) \to H^k(Z)$, $j = 0, \dots, m$, $k \ge 0$, acting on symmetric m-tensor fields and transforming it into functions $g^{(j)}(\xi(\alpha), s)$, given in the cylinder Z, are defined by the formula

$$\big(\mathcal{P}^{(j)} w\big)(\xi, s) = \int_{-\infty}^{\infty} w_{i_1 \dots i_m} \xi^{i_1} \dots \xi^{i_j} \eta^{i_{j+1}} \dots \eta^{i_m} dt, \tag{38}$$

where $\xi = (\cos\alpha, \sin\alpha)$, $\eta = (-\sin\alpha, \cos\alpha)$. The transform for $j = 0$ is the longitudinal ray transform,

$$\big(\mathcal{P} w\big)(\xi, s) := \big(\mathcal{P}^{(0)} w\big)(\xi, s) = \int_{-\infty}^{\infty} w_{i_1 \dots i_m} \eta^{i_1} \dots \eta^{i_m} dt.$$

It is the transverse ray transform for $j = m$,

$$\big(\mathcal{P}^\perp w\big)(\xi, s) := \big(\mathcal{P}^{(m)} w\big)(\xi, s) = \int_{-\infty}^{\infty} w_{i_1 \dots i_m} \xi^{i_1} \dots \xi^{i_m} dt.$$

For other j, $0 < j < m$, the transform $\mathcal{P}^{(j)}$ is called mixed ray transform.

We give a description of kernels and images of the ray transformations, as well as relationships between the ray transforms and the Radon transforms of corresponding potentials. The proof of this proposition is similar to the analogous for vector and symmetric 2-tensor fields. It needs in more laborious calculations of a purely technical nature and therefore is not given.

Proposition 5.1. *Let a symmetric m-tensor field $w \in H^{k-m}(S^m(B))$ be given,*

$$w = \sum_{j=0}^{m} u^{(j)} \equiv \sum_{j=0}^{m} \big(\mathrm{d}^\perp\big)^{m-j}\big(\mathrm{d}\big)^j \psi^{(j)},$$

and $u^{(j)} = \big(\mathrm{d}^\perp\big)^{m-j}\big(\mathrm{d}\big)^j \psi^{(j)} \in H^{k-m}(S^m(B))$, $k \ge m$, for potentials $\psi^{(0)}, \dots, \psi^{(m)} \in H^k_0(B)$. Then

1) $\quad (\mathcal{P}^{(j)}w)(\xi,s) = (\mathcal{P}^{(j)}u^{(j)})(\xi,s), \quad j=0,\dots,m.$

2) *The ray transform* $\mathcal{P}^{(j)}$ *of a symmetric* m*-tensor field* $u^{(j)} \in H^k(S^m(B))$, $u^{(j)} = \left(\mathrm{d}^{\perp}\right)^{m-j}\left(\mathrm{d}\right)^j \psi^{(j)}$, $\psi^{(j)} \in H_0^{k+m}(B)$, $j=0,\dots,m$, *associated with the Radon transform of its potential* $\psi^{(j)}$ *by the ratio*

$$\left(\mathcal{P}^{(j)}u^{(j)}\right)(\xi,s) = \frac{1}{C_m^j}\frac{\partial^m}{\partial s^m}\mathcal{R}\psi^{(j)}(\xi,s), \quad j=0,\dots,m,$$

where $C_m^j = m!/(j!\,(m-j)!)$.

3) *If* $\psi^{(j)} = \varphi \in H_0^k(B)$ *for all* $j=0,\dots,m$, $k \geq m$. *Then*

$$C_m^j\left(\mathcal{P}^{(j)}u^{(j)}\right)(\xi,s) = C_m^l\left(\mathcal{P}^{(l)}u^{(l)}\right)(\xi,s) = \frac{\partial^m}{\partial s^m}\mathcal{R}\varphi(\xi,s).$$

5.2 Inversion formulas

We pass to a system of equations connecting the ray transforms $\mathcal{P}^{(j)}w$, $j=0,\dots,m$ of a symmetric m-tensor field $w = (w_{i_1\dots i_m})$ (37) with the Radon transform of its components $w_{1\dots11}$, $w_{1\dots12}$, $\dots$, $w_{2\dots22}$,

$$\begin{cases} \mathcal{P}^1 w(\xi,s) = \eta^{i_1}\dots\eta^{i_{m-1}}\eta^{i_m}\mathcal{R}w_{i_1\dots i_m}(\xi,s), \\ \mathcal{P}^2 w(\xi,s) = \eta^{i_1}\dots\eta^{i_{m-1}}\xi^{i_m}\mathcal{R}w_{i_1\dots i_m}(\xi,s), \\ \dots \\ \mathcal{P}^{m+1} w(\xi,s) = \xi^{i_1}\dots\xi^{i_{m-1}}\xi^{i_m}\mathcal{R}w_{i_1\dots i_m}(\xi,s), \end{cases} \tag{39}$$

where the notations $\mathcal{P}^{j+1}w = \mathcal{P}^{(j)}w$ are used. We would recall that the rule of Einstein's summation is meant. The system (39) has a unique solution, which gives the dependence of the Radon transforms $\mathcal{R}w_{i_1\dots i_m}$ of the field w components from the known ray transforms $\mathcal{P}^j w$. Let's find this solution. We use the notations $e_1 = (1,0)$, $e_2 = (0,1)$, and $f_1 = \eta$, $f_2 = \xi$. The vectors f_1, f_2 and e_1, e_2 are associated between themselves as follows,

$$\begin{cases} f_1 = -\sin\alpha\, e_1 + \cos\alpha\, e_2 = \mathbf{f}_1^j e_j, \\ f_2 = \cos\alpha\, e_1 + \sin\alpha\, e_2 = \mathbf{f}_2^j e_j, \end{cases} \tag{40}$$

$$\begin{cases} e_1 = -\sin\alpha\, f_1 + \cos\alpha\, f_2 = \mathbf{e}_1^l f_l, \\ e_2 = \cos\alpha\, f_1 + \sin\alpha\, f_2 = \mathbf{e}_2^l f_l. \end{cases} \tag{41}$$

The definitions of the matrices $\left(\mathbf{e}_j^k\right)$, $\left(\mathbf{f}_j^k\right)$, $j,k=1,2$ imply that $\left(\mathbf{e}_j^k\right) = \left(\mathbf{f}_j^k\right)^{-1}$. Furthermore, we have $\mathbf{f}_j^l = \mathbf{e}_j^l$, $j,l=1,2$. It follows that $\left(\mathbf{f}_j^k\right) = \left(\mathbf{e}_j^k\right) = \left(\mathbf{f}_j^k\right)^{-1} = \left(\mathbf{e}_j^k\right)^{-1}$ and $\left(\mathbf{e}_j^k\right)^2 = \left(\mathbf{f}_j^k\right)^2 = E$, where E is the identity matrix. Essentially, exactly these properties of matrices $\left(\mathbf{e}_j^k\right)$, $\left(\mathbf{f}_j^k\right)$ allow to obtain inversion formulas.

We define a multilinear form $\mathbf{R}$ by the rule

$$\mathcal{R}w_{i_1\dots i_m} = \mathbf{R}\langle e_{i_1},\dots,e_{i_m}\rangle. \tag{42}$$

A tensor field w is symmetric, therefore the form $\mathbf{R}$ is symmetric too,

$$\mathbf{R}\langle e_{i_1},\dots,e_{i_j},\,\dots,\,e_{i_l},\dots,e_{i_m}\rangle = \mathbf{R}\langle e_{i_1},\dots,e_{i_l},\,\dots,\,e_{i_j},\dots,\,e_{i_m}\rangle.$$

Here $j,l = 1,\dots,m,\ j\neq l$. Symmetry of the form $\mathbf{R}$ allows to construct a decomposition of the form $\mathcal{R}w_{i_1\dots i_m}$ values, depending on the set of indices $\{i_1,\dots,i_m\}$, into $m+1$ subsets $\mathcal{R}^k$, $k=1,\dots,m+1$,

$$\bigcup_{k=1}^{m+1}\mathcal{R}^k w = \mathbf{R}\langle e_{i_1},\dots,e_{i_m}\rangle, \tag{43}$$

where k is such that among the indices $i_1,\dots,i_m$ exactly $m-(k-1)$ indices are equal to 1, and the others $k-1$ are equal to 2. A multilinear form $\mathbf{P}$ (or $\mathcal{P}w_{i_1\dots i_m}$) by the set $\mathcal{P}^k w$ is defined by analogy,

$$\mathcal{P}w_{i_1\dots i_m} = \mathbf{P}\langle e_{i_1},\dots,e_{i_m}\rangle = \bigcup_{k=1}^{m+1}\mathcal{P}^k w, \tag{44}$$

where k, as well as earlier, is such that among the indices $i_1,\dots,i_m$ exactly $m-(k-1)$ indices are equal to 1, and the others $k-1$ are equal to 2. It should be note that the new notations and definitions are introduced only for more convenient description of the system (39).

Taking into account new notations and definitions described above, the system (39) takes the following form,

$$\mathbf{P}\langle e_{i_1},\dots,e_{i_m}\rangle = \mathbf{R}\langle f_{i_1},\dots,f_{i_m}\rangle$$

for all sets of indices $i_1,\dots,i_m\in\{1,2\}$. In a more detailed record we have

$$\begin{pmatrix} \mathbf{P}\langle e_1,\dots,e_1,e_1\rangle \\ \mathbf{P}\langle e_1,\dots,e_1,e_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle e_2,\dots,e_2,e_2\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{R}\langle f_1,\dots,f_1,f_1\rangle \\ \mathbf{R}\langle f_1,\dots,f_1,f_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{R}\langle f_2,\dots,f_2,f_2\rangle \end{pmatrix}.$$

We shall check it now. Actually, for fixed k, for which among the indices $i_1,\dots,i_m$ exactly $m-(k-1)$ indices are equal to 1 and the others $k-1$ are equal to 2, with usage of (44), (39) and (43) we obtain

$$\begin{aligned} \mathcal{P}^k w &= \mathbf{f}_1^{j_1}\dots\mathbf{f}_1^{j_{m-(k-1)}}\mathbf{f}_2^{j_{m-(k-2)}}\dots\mathbf{f}_2^{j_m}\mathcal{R}w_{j_1\dots j_m} \\ &= \mathbf{f}_1^{j_1}\dots\mathbf{f}_1^{j_{m-(k-1)}}\mathbf{f}_2^{j_{m-(k-2)}}\dots\mathbf{f}_2^{j_m}\mathbf{R}\langle e_{j_1},\dots,e_{j_m}\rangle \\ &= \mathbf{R}\langle \mathbf{f}_1^{j_1}e_{j_1},\dots,\mathbf{f}_1^{j_{m-(k-1)}}e_{j_{m-(k-1)}},\mathbf{f}_2^{j_{m-(k-2)}}e_{j_{m-(k-2)}},\dots,\mathbf{f}_2^{j_m}e_{j_m}\rangle \\ &= \mathbf{R}\langle\underbrace{\mathbf{f}_1^{j}e_j,\dots,\mathbf{f}_1^{j}e_j}_{m-(k-1)},\underbrace{\mathbf{f}_2^{j}e_j,\dots,\mathbf{f}_2^{j}e_j}_{k-1}\rangle = \mathcal{R}^k w. \end{aligned}$$

It turns out, that the matrix $\mathbf{T}$ (of a size $(m+1)\times(m+1)$) of the system of equations (39), consisting of sums of products of the vectors η, ξ components, can be determined by the equality (taking into account the notations $f_1 = \eta,\ f_2 = \xi$)

$$\mathbf{T}\begin{pmatrix} \mathbf{P}\langle e_1,\dots,e_1,e_1\rangle \\ \mathbf{P}\langle e_1,\dots,e_1,e_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle e_2,\dots,e_2,e_2\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{P}\langle f_1,\dots,f_1,f_1\rangle \\ \mathbf{P}\langle f_1,\dots,f_1,f_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle f_2,\dots,f_2,f_2\rangle \end{pmatrix}. \tag{45}$$

We are ready now to show that $\mathbf{T}^2 = E$. The matrix $\mathbf{T}$ can be constructed from row vectors $t_k = (t_{k1},\dots,t_{k(m+1)})$, $k = 1,\dots,m+1$, as follows. We fix k. Taking into account (40) we obtain

$$\mathbf{P}^k := \mathbf{P}\langle\underbrace{f_1,\dots,f_1}_{m-(k-1)},\underbrace{f_2,\dots,f_2}_{k-1}\rangle = \mathbf{P}\langle\underbrace{\mathbf{f}_1^j e_j,\dots,\mathbf{f}_1^j e_j}_{m-(k-1)},\underbrace{\mathbf{f}_2^j e_j,\dots,\mathbf{f}_2^j e_j}_{k-1}\rangle.$$

By means of the rules of working with multilinear form we come to the conclusion

$$\mathbf{P}^k = t_k\begin{pmatrix} \mathbf{P}\langle e_1,\dots,e_1,e_1\rangle \\ \mathbf{P}\langle e_1,\dots,e_1,e_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle e_2,\dots,e_2,e_2\rangle \end{pmatrix},$$

and thus $\mathbf{T} = \begin{pmatrix} t_1 \\ t_2 \\ \cdots \\ t_{m+1}\end{pmatrix}$. The relations (40), (41) imply that

$$\begin{pmatrix} \mathbf{P}\langle f_1,\dots,f_1,f_1\rangle \\ \mathbf{P}\langle f_1,\dots,f_1,f_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle f_2,\dots,f_2,f_2\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{P}\langle \mathbf{f}_1^j e_j,\dots,\mathbf{f}_1^j e_j,\mathbf{f}_1^j e_j\rangle \\ \mathbf{P}\langle \mathbf{f}_1^j e_j,\dots,\mathbf{f}_1^j e_j,\mathbf{f}_2^j e_j\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle \mathbf{f}_2^j e_j,\dots,\mathbf{f}_2^j e_j,\mathbf{f}_2^j e_j\rangle \end{pmatrix} = \mathbf{T}\begin{pmatrix} \mathbf{P}\langle e_1,\dots,e_1,e_1\rangle \\ \mathbf{P}\langle e_1,\dots,e_1,e_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle e_2,\dots,e_2,e_2\rangle \end{pmatrix}$$

$$= \mathbf{T}\begin{pmatrix} \mathbf{P}\langle \mathbf{e}_1^j f_j,\dots,\mathbf{e}_1^j f_j,\mathbf{e}_1^j f_j\rangle \\ \mathbf{P}\langle \mathbf{e}_1^j f_j,\dots,\mathbf{e}_1^j f_j,\mathbf{e}_2^j f_j\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle \mathbf{e}_2^j f_j,\dots,\mathbf{e}_2^j f_j,\mathbf{e}_2^j f_j\rangle \end{pmatrix} = \mathbf{T}^2\begin{pmatrix} \mathbf{P}\langle f_1,\dots,f_1,f_1\rangle \\ \mathbf{P}\langle f_1,\dots,f_1,f_2\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle f_2,\dots,f_2,f_2\rangle \end{pmatrix}.$$

As the column vector $\begin{pmatrix} \mathbf{P}\langle f_1,\dots,f_1,f_1\rangle \\ \cdots\quad\cdots\quad\cdots \\ \mathbf{P}\langle f_2,\dots,f_2,f_2\rangle \end{pmatrix}$ (a column of the ray transforms) is arbitrary so we conclude that $\mathbf{T}^2 = E$.

Theorem 5.2. *Let a system of linear equations*

$$\mathbf{T}\,\mathbf{X} = \mathbf{B}$$

be given. The matrix $\mathbf{T}$ *is defined by the relation* (45)*,* $\mathbf{X}$ *is a vector of unknowns with components*

$$x_1 = \mathcal{R}w_{1\dots 11},\ x_2 = \mathcal{R}w_{1\dots 12},\ \dots,\ x_{m+1} = \mathcal{R}w_{2\dots 22};$$

$\mathbf{B}$ *is a vector of right part,*

$$b_1 = \mathcal{P}^{(0)}w,\ b_2 = \mathcal{P}^{(1)}w,\ \dots,\ b_{m+1} = \mathcal{P}^{(m)}w,$$

consisting of the values of the ray transforms, defined by (38)*. Then the relation* $\mathbf{X} = \mathbf{T}\,\mathbf{B}$ *is valid. In more details the following equalities hold*

$$\begin{cases} \mathcal{R}w_{1\dots 11} = \eta^{i_1}\dots\eta^{i_{m-1}}\eta^{i_m}\mathcal{P}w_{i_1\dots i_m}, \\ \mathcal{R}w_{1\dots 12} = \eta^{i_1}\dots\eta^{i_{m-1}}\xi^{i_m}\mathcal{P}w_{i_1\dots i_m}, \\ \qquad\dots \qquad \dots \qquad \dots \\ \mathcal{R}w_{2\dots 22} = \xi^{i_1}\dots\xi^{i_{m-1}}\xi^{i_m}\mathcal{P}w_{i_1\dots i_m}, \end{cases} \tag{46}$$

where $\mathcal{P}w_{i_1\dots i_m} = \mathcal{P}^{(k-1)}w \equiv b_k$*,* $k = 1,\dots,m+1$*, if and only if, when among the indices* $i_1,\dots,i_m$ *exactly* $m-(k-1)$ *indices are equal to* 1*, and the others* $k-1$ *are equal to* 2*.*

As in the cases of vector and symmetric 2-tensor fields described above, the formulas (46) give the solution to the problem, as knowing of the Radon transform of the components of required symmetric m-tensor field, we can apply any of the numerous approaches to its inversion.

The projection theorems for the ray transforms of m-tensor fields are formulated below. We use the notations $(\mathcal{P}^{(j)}_\xi w)(s) = (\mathcal{P}^{(j)}w)(\xi,s)$, $|s| \le 1$, here.

Theorem 5.3. *If* $u^{(j)} = \big(\mathrm{d}^\perp\big)^{m-j}\big(\mathrm{d}\big)^j\varphi$*,* $j = 0,\dots,m$*, where* $\varphi \in \mathcal{C}^k(B)$*,* $k \ge (m-1)$*, then*

$$(\mathcal{P}^{(j)}_\xi u^{(j)})\widehat{\ }(\sigma) = (i\sigma)^m \frac{\sqrt{2\pi}}{C^j_m}\,\widehat{\varphi}(\sigma\xi), \qquad \sigma\in\mathbb{R}.$$

The proof consists in usage of the projection theorem 2.1 for the Radon transform, property 3 of ray transforms of m-tensor fields and properties of the Fourier transform.

We pass to obtaining the stability estimates for the ray transforms of symmetric m-tensor fields.

Theorem 5.4. *Let* $u = \sum\limits_{j=0}^{m} u^{(j)}$ *is a symmetric* m*-tensor field, where*

$$u^{(j)} = (\mathrm{d}^\perp)^{m-j}\mathrm{d}^j\psi^{(j)},$$

with potentials $\psi^{(j)} \in H^m_0(B)$*,* $j = 0,\dots,m$*. Then the stability estimates*

$$\|u^{(j)}\|^2_{L_2(S^m(B))} \le C^{(j)}_m \|\mathcal{P}^{(j)}w\|^2_{H^1(Z)}, \qquad j = 0,\dots,m, \tag{47}$$

hold with constants $C^{(j)}_m$ *independent of* w*.*

The proof of this theorem is similar to the proofs of theorems 3.2 and 4.4. The property 1 of the ray transforms implies $\left(\mathcal{P}^{(l)}u^{(0)}\right)(\alpha,s)=0$ for $l\neq 0$ and $\left(\mathcal{P}^{(0)}u^{(0)}\right)(\alpha,s)=\left(\mathcal{P}^{(0)}w\right)(\alpha,s)$. Applying (46) we get

$$\left(\mathcal{R}u^{(0)}_{i_1\ldots i_m}\right)(\alpha,s)=(-1)^k(\sin\alpha)^k(\cos\alpha)^{m-k}(\mathcal{P}^{(0)}w)(\alpha,s),$$

where among the indices $i_1,\ldots,i_m$ exactly k indices equal 1, and the others $m-k$ equal 2. Thus the norm $\|\mathcal{R}u^{(0)}_{1\ldots 1}\|^2_{H^1(Z)}$ can be estimated as

$$\begin{aligned}\|\mathcal{R}u^{(0)}_{1\ldots 1}\|^2_{H^1(Z)} &= \int_Z \Big((\mathcal{R}u^{(0)}_{1\ldots 1})^2+((\mathcal{R}u^{(0)}_{1\ldots 1})'_s)^2+((\mathcal{R}u^{(0)}_{1\ldots 1})'_\alpha)^2\Big)d\alpha\,ds\\ &= \int_Z \Big((\mathcal{P}^{(0)}\,w)^2(\sin\alpha)^{2m}+\left((\mathcal{P}^{(0)}\,w)'_s\right)^2(\sin\alpha)^{2m}\\ &\qquad +\left((\mathcal{P}^{(0)}\,w)'_\alpha(\sin\alpha)^m+m(\mathcal{P}^{(0)}\,w)(\sin\alpha)^{m-1}\cos\alpha\right)^2\Big)d\alpha\,ds\\ &\leqslant \int_Z \Big((\mathcal{P}^{(0)}\,w)^2\left((\sin\alpha)^{2m}+2m^2(\sin\alpha)^{m-2}\cos^2\alpha\right)\\ &\qquad +\left((\mathcal{P}^{(0)}\,w)'_s\right)^2(\sin\alpha)^{2m}+2\left((\mathcal{P}^{(0)}\,w)'_\alpha\right)^2(\sin\alpha)^{2m}\Big)d\alpha\,ds\\ &\leqslant (2m^2+1)\|\mathcal{P}^{(0)}w\|^2_{H^1(Z)}.\end{aligned}$$

The inequalities $\|\mathcal{R}u^{(0)}_{i_1\ldots i_m}\|^2_{H^1(Z)}\leqslant(2m^2+1)\|\mathcal{P}^{(0)}w\|^2_{H^1(Z)}$ for any set of indices $i_1,\ldots,i_m$ are deduced in the same way. The estimate (16) leads to the formula

$$\begin{aligned}\|u^{(0)}\|^2_{L_2(S^m(B))} &= \sum_{i_1,\ldots,i_m=1}^{2}\|u^{(0)}_{i_1\ldots i_m}\|^2_{L_2(B)}\leqslant\sum_{i_1,\ldots,i_m=1}^{2}C\|\mathcal{R}u^{(0)}_{i_1\ldots i_m}\|^2_{H^1(Z)}\\ &\leqslant 2^m(2m^2+1)C\|\mathcal{P}^{(0)}\,w\|^2_{H^1(Z)}=C^{(0)}_m\|\mathcal{P}^{(0)}\,w\|^2_{H^1(Z)}.\end{aligned}$$

The estimates (47) for $j>0$ follow analogously.

Conclusion and acknowledgement

Symmetric tensor fields given in the plane were investigated in the article. The problem of reconstruction of a tensor field by its ray transforms was reduced to the problem of reconstruction of its components by their known Radon transforms. Important for the practice cases of recovering of vector and symmetric 2-tensor fields are considered in details. The detailed classification of symmetric tensor fields is established. The decomposition theorems for the fields are proved. The concepts of transverse and mixed ray transforms for the vector and symmetric m-tensor fields are introduced. The operators of back projection acting on the transverse and mixed ray transforms are suggested. The kernels and images of the transverse and mixed ray transforms are described. The connections between ray transforms for the fields and Radon transforms for their potentials are established. The connections allow to establish the projections theorems for symmetric tensor fields as well as to obtain the inversion formulas for the components of tensor field and their potentials.

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