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EXPLICIT SOLUTION OF 3-D PHASELESS INVERSE SCATTERING PROBLEM FOR THE SCHRÖDINGER EQUATION: THE PLANE WAVE CASE

M.V. Klibanov, V.G. Romanov

Abstract The inverse scattering problem of the reconstruction of the unknown potential in the 3-d Schrödinger equation is considered. Unlike the conventional case of an inverse scattering problem, here only the modulus of the complex valued scattered wave field is assumed to be known outside of the scatterer. The phase is unknown. This problem arises in inverse quantum scattering. Unlike the previous work of the authors [8], where the case of the point source was studied, here the incident plane wave is considered. An explicit solution via the inverse Radon transform is obtained.

Key words: phaseless inverse scattering, incident plane wave, Schrödinger equation, reconstruction formula, Radon transform

AMS Mathematics Subject Classification: 35R30.

1 Introduction

The current paper is a continuation of our recent works [8, 9]. In [8] we have derived the reconstruction formula for the unknown potential for the phaseless inverse scattering problem (ISP) for the 3-d Schrödinger equation. This was done for the case when the wave field is generated by the point source. We have considered the tomographic measurement scheme and came up with the reconstruction formula via the inversion of the Radon transform. It is worthy to note that a long standing problems posed in 1977 in Chapter 10 of the book [8] was addressed in [8] *for the first time* (see some details in this section below). In [9] a similar inversion formula was obtained for the case of the Born approximation for the wave-like equation $\Delta u+k^2 (1+\beta (x))\, u=-\delta (x-x_0)\,, x\in \mathbb{R}^3$. The difference between this equation and the Schrödinger equation (4) is that in (4) the unknown potential $q(x)$ is not multiplied by k^2. Unlike [8, 9], in the current paper we consider the case of the incident plane wave. Again, we consider the tomographic measurement scheme and again end up with the inversion of the Radon transform. However, a modification of the technique of [8] is used here.

The Radon transform is well understood by now and has wide applications. Probably the most spectacular application is in medicine under the name Computerized Tomography (CT). CT results in very high quality images, see, e.g. [28]. This leads us to believe that images resulting from our formula of this paper as well as from [8, 9] will also have a high quality.

The term “phaseless" means here that we assume that only the modulus of the scattered complex valued wave field is measured outside of the support of the scatterer.

However, the phase is not measured. This situation is typical in quantum scattering, where only the so-called "differential scattering cross section" is measured, see page 8 of [19]. On the other hand, the entire theory of inverse scattering problems, including quantum inverse scattering is based on the assumption that both the modulus and the phase are measured [2, 19, 20, 21]. The latter has prompted the authors of the book [2] to pose in Chapter 10 the question of the reconstruction of the potential of the Schrödinger equation for the case of the phaseless data. However, this problem was not addressed in [2]. The first complete solution of this problem was published in our work [8], for the case of the point source. Below we do the same for the case of the incident plane wave.

The phase reconstruction problem is the central one in many applied inverse problems, especially in the case when nano structures are probed by X-rays, see, e.g. [3, 7, 24]. In addition, we refer to, e.g. [4, 6, 15, 18, 25] for some other phase reconstruction techniques. However, the common drawback of all available methods of the phase reconstruction is that there is no rigorous guarantee that the correct phase will indeed be reconstructed. We refer to [1, 22] for rigorous approaches to the phase reconstruction problem for the case when some known objects, in addition to the unknown one, are involved in experiments. In [23] a rigorous phase reconstruction algorithm is considered in a quite general situation for the case when the modulus of the total wave field is measured on at least two spheres in the far zone. In both types of works [1, 22] and [23] the modulus of the superposition of two wave fields, one of which is known, is considered. Unlike these, we study here the case when the modulus of the scattered wave field only is known, i.e. a superposition is not in place, and the same was in [8, 9]. Uniqueness theorems for the 1-d case and 3-d cases were proven in [10, 18] and [11, 12, 13, 14] respectively. However, those proofs are non-constructive ones. Reconstruction procedures of [1, 22, 23] lead to corresponding uniqueness theorems.

2 The Main Result

Let $B > 0$ be a number and $\Omega = \{|x| < B\} \subset \mathbb{R}^3$ be the ball of the radius B with the center at $\{0\}$. Denote the corresponding sphere $S = \{|x| = B\}$. Let the potential $q(x), x \in \mathbb{R}^3$ be a real valued function such that

$$q(x) \in C^4(\mathbb{R}^3), \tag{1}$$

$$q(x) \geq 0, \forall x \in \Omega, \tag{2}$$

$$q(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega. \tag{3}$$

We consider the following equation

$$\Delta u + k^2 u - q(x)u = 0, \quad x \in \mathbb{R}^3. \tag{4}$$

Let

$$u(x, k, \nu) = \exp(ikx \cdot \nu) + u_{sc}(x, k, \nu), \tag{5}$$

where $k > 0$ is the frequency and $u_0 = \exp(ikx \cdot \nu)$ is the incident plane wave propagating in the direction of the unit vector $\nu \in S^2$, where $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. In

(5) $u_{sc}(x,k,\nu)$ is the scattering wave field. It satisfies the equation

$$\Delta u_{sc}+k^2u_{sc}-q(x)u_{sc}=q(x)\exp\left(ikx\cdot\nu\right),\quad x\in\mathbb{R}^3 \tag{6}$$

with the usual radiation conditions

$$u_{sc}(x,k,\nu)=O\left(\frac{1}{r}\right),r\to\infty, \tag{7}$$

$$\frac{\partial u_{sc}}{\partial r}-iku_{sc}=o(r^{-1})\,,|r|\to\infty, \tag{8}$$

where $r=|x|$.

Theorem 3.3 of the paper of [26], Theorem 6 of Chapter 9 of [27] as well as Theorem 6.17 of [5] guarantee that for each pair $(k,\nu)\in(0,\infty)\times S^2$ there exists a unique solution $u\left(x,x^0,k\right)$ of the problem (6)-(8) such that $u_{sc}\in C^4\left(\mathbb{R}^3\right)$.

For any number $a\in\mathbb{R}$ consider the plane $P_a=\{x_3=a\}$. Consider the disk $Q_a=\overline{\Omega}\cap P_a$ and let $S_a=S\cap P_a$ be its boundary. Clearly $Q_a\neq\varnothing$ for $a\in(-B,B)$ and $Q_a=\varnothing$ for $|a|\geq B$. Denote $0_a=(0,0,a)\in Q_a$ the orthogonal projection of the origin on the plane P_a. We have

$$\Omega=\bigcup_{a=-B}^{B}Q_a,\quad \partial\Omega:=S=\bigcup_{a=-B}^{B}S_a.$$

Let

$$\begin{aligned} S_0^2 &= \left\{\nu=(\nu_1,\nu_2,\nu_3)\in S^2:\nu_3=0\right\}, \\ S_a(\nu) &= \{x\in S_a:\ x\cdot\nu>0\}. \end{aligned}$$

In our inverse problem we assume that the modulus $|u_{sc}|$ of the scattered wave is measured for all pairs $(x\in S_a(\nu),\nu\in S_0^2)$, for every $a\in(-B,B)$ and for all frequencies $k>0$.

Phaseless Inverse Scattering Problem. *Suppose that the potential $q\left(x\right)$ satisfies conditions (1)-(3). Determine the function $q\left(x\right)$ for $x\in\Omega$, assuming that the following function $f\left(x,k,\nu\right)$ is known*

$$f\left(x,k,\nu\right)=\left|u_{sc}\left(x,k,\nu\right)\right|,\forall x\in S_a(\nu),\forall\nu\in S_0^2,\forall a\in\left(-B,B\right),\forall k\in\left(0,\infty\right). \tag{9}$$

Remark 1. As to the issue of collecting experimental data, it follows from (9) and Theorem 1 that if one wants to image only one 2-d cross-section Q_a of the potential q, then it is sufficient to run independently detectors x only around the semicircle $S_a(\nu)$ and the vector ν along the circle S_0^2. This is more economical than running them independently around $S\times S^2$.

To formulate our inversion formula, we introduce now some notations of the conventional Radon transform [17]. For an arbitrary $a\in(-B,B)$ and for any pair $(x\in S_a(\nu),\nu\in S_0^2)$ let $\widetilde{L}\left(x,\nu\right)$ be the straight passing through the point x and parallel to the vector ν. Therefore, $\widetilde{L}(x,\nu)$ is lying in the plane $P_a, \widetilde{L}(x,\nu) \subset P_a$. Next, let $L(x,\nu)$ be the part of $\widetilde{L}(x,\nu)$, which is inside of the disk Q_a, i.e. $L(x,\nu) = \widetilde{L}(x,\nu) \cap Q_a$.

Since our reconstruction formula is based on the inversion of the two-dimensional Radon transform, we now parameterize $L(x,\nu)$ in the conventional way of the parametrization of the Radon transform [17]. For $(x \in S_a(\nu), \nu \in S_0^2)$ let n be the unit normal vector to the line $\widetilde{L}(x,\nu)$ lying in the plane P_a and pointing outside of the point 0_a. Let $\alpha \in (0, 2\pi]$ be the angle between n and the x_1-axis. Then $n = n(\alpha) = (\cos\alpha, \sin\alpha)$ (it is convenient here to discount the third coordinate of n, which is zero). Let s be the signed distance between $L(x,\nu)$ and the point 0_a (page 9 of [17]). It is clear that there exists a one-to-one correspondence between pairs (x,ν) and $(n(\alpha), s)$,

$$(x,\nu) \Leftrightarrow (n(\alpha), s); (x \in S_a(\nu), \nu \in S_0^2), \alpha = \alpha(x,\nu) \in (0, 2\pi], s = s(x,\nu) \in (-B_a, B_a), \tag{10}$$

where $B_a = \sqrt{B^2 - a^2}$ is the radius of the circle S_a. Hence, we can write

$$L(x,\nu) = \{y_a = (y_1, y_2, a) \in Q_a : y \cdot n(\alpha) = s\}, \tag{11}$$

where $y = (y_1, y_2) \in \mathbb{R}^2$ and parameters $\alpha = \alpha(x,\nu)$ and $s = s(x,\nu)$ are defined as in (10).

Consider an arbitrary function $g = g(y) \in C^4(P_a)$ such that $g(y) = 0$ for $y \in P_a \diagdown Q_a$. Hence,

$$\int\limits_{L(x,\nu)} g(y)\, d\sigma = \int\limits_{y \cdot n(\alpha) = s} g(y)\, d\sigma,\ \forall x \in S_a(\nu),\ \forall \nu \in S_0^2, \tag{12}$$

where $\alpha = \alpha(x,\nu), s = s(x,\nu)$ are as in (10). In (12) σ is the arc length and the parametrization of $L(x,\nu)$ is given in (11). Therefore, using (10)-(12), we can define the 2-d Radon transform Rg of the function g as

$$(Rg)(x,\nu) = (Rg)(\alpha, s) = \int\limits_{y \cdot n(\alpha) = s} g(y)\, d\sigma. \tag{13}$$

We are ready now to formulate Theorem 1, which is our main result.

Theorem 1. *Suppose that the potential $q(x)$ satisfies conditions (1)-(3). Let $u_{sc}(x,k,\nu)$ be the function defined in (6)-(8). Then for each $a \in (-B, B)$ and for each pair $(x \in S_a(\nu), \nu \in S_0^2)$ the asymptotic behavior of $u_{sc}(x,k,\nu)$ is*

$$|u_{sc}(x,k,\nu)| = \frac{1}{2k} \int\limits_{L(x,\nu)} q(\xi) d\sigma + O\left(\frac{1}{k^2}\right),\ k \to \infty. \tag{14}$$

Hence, the asymptotic behavior of the function $f(x,k,\nu)$ defined in (9) is

$$f(x,k,\nu) = \frac{1}{2k}\left[(Rq)(x,\nu) + O\Big(\frac{1}{k}\Big)\right],\ k \to \infty; \forall x \in S_a(\nu),\ \forall \nu \in S_0^2, \tag{15}$$

for all $a \in (-B, B)$. Thus, for $y \in Q_a, a \in (-B, B)$ the reconstruction formula for the function $q(y,a)$ is

$$q(y,a) = 2R^{-1}\{\lim_{k\to\infty} [kf(x,k,\nu)]\}(y,a). \tag{16}$$

Since the inversion formula (16) follows immediately from (12), (13), (15) and the results of the book [17], we focus below on the proof of (14). Since the explicit form of the operator R^{-1} is well known [17], we are not citing it here for brevity. We assume everywhere below that conditions of Theorem 1 are satisfied. The goal of the rest of this paper is to prove this theorem.

3 Smoothness of the solution of a Cauchy problem for a hyperbolic equation

We prove (14) using connection between the problem (6)-(8) and the solution of a certain Cauchy problem for a hyperbolic PDE via the Fourier transform. To do this, we investigate first the smoothness property of this solution.

Consider the hyperbolic equation

$$v_{tt} - \Delta v + q\left(x\right)v = -q(x)\delta(t - x\cdot\nu), \quad (x,t) \in \mathbb{R}^4 \tag{17}$$

with the initial condition

$$v|_{t<x\cdot\nu} \equiv 0. \tag{18}$$

For any $T > 0$ and for any $\nu \in S^2$ denote

$$G(T,B,\nu) = \{(x,t) :\ x\cdot\nu < t < T - |x|, t + x\cdot\nu > -2B\}$$

and by $\overline{G}(T,B,\nu)$ its closure. Theorem 2 establishes a smoothness property of the solution $v\left(x,t,\nu\right)$ of the Cauchy problem (17), (18). Below $C = C\left(T,B\right)$ denotes different positive constants depending on listed parameters.

Theorem 2. *The solution of the problem (17), (18) $v(x,t,\nu)$ vanishes for $t+x\cdot\nu < -2B$ and for $k = 0,1,2$ $\partial_t^k v(x,t,\nu) \in C(\overline{G}(T,B,\nu))$, $\forall T > 0, \forall\ \nu \in S^2$. Moreover, the following limit is valid*

$$v|_{t=x\cdot\nu+0} = a(x,\nu) = -\frac{1}{2}\int\limits_{L_-(x,\nu)} q(\xi)d\sigma = -\frac{1}{2}\int\limits_0^\infty q(x - s\nu)ds, \tag{19}$$

where $L_-(x,\nu) = \{\xi : \xi = x - s\nu, s \in [0,\infty)\}$ is the ray that is going from point x in the direction $-\nu$.

Corollary. *The function $\Delta_x v(x,t,\nu) \in C(\overline{G}(T,B,\nu))$.*

Proof of Corollary. By (17)

$$\Delta v = v_{tt} + q\left(x\right)v, t > x\cdot\nu.$$

By Theorem 2 the function in the right hand side belongs to $C(\overline{G}(T,B,\nu))$. Hence, the function $\Delta v \in C(\overline{G}(T,B,\nu))$. □

Proof of Theorem 2. The function $v(x,t,\nu)$ for $t > x\cdot\nu$ satisfies the following integral equation

$$v(x,t,\nu) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} q(y)[\delta(t - y\cdot\nu - |x-y|) + v(y, t - |x-y|, \nu)]\frac{dy}{|x-y|}. \tag{20}$$

We represent the solution to this equation as

$$v(x,t,\nu) = \sum_{n=0}^{\infty} v_n(x,t,\nu), \quad t \geq x \cdot \nu, \tag{21}$$

where

$$v_0(x,t,\nu) = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{q(y)\delta(t - y\cdot\nu - |x-y|)}{|x-y|} dy, \tag{22}$$

$$\begin{aligned} v_n(x,t,\nu) &= -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{q(y)v_{n-1}(y, t-|x-y|, \nu)}{|x-y|} dy \\ &= -\frac{1}{4\pi} \int\limits_{y\cdot\nu + |x-y| \leq t} \frac{q(y)v_{n-1}(y, t-|x-y|, \nu)}{|x-y|} dy, \quad n \geq 1. \end{aligned} \tag{23}$$

The latter equality valid since $v_n(y, t-|x-y|, \nu) \equiv 0$ for all $t - |x-y| < y\cdot\nu$ and for all $n = 1, 2, \ldots$.

For $t > x\cdot\nu$ denote by $S(x,t,\nu)$ the following paraboloid

$$S(x,t,\nu) = \{y \in \mathbb{R}^3 : \; y\cdot\nu + |x-y| = t\}. \tag{24}$$

If $t \to (x\cdot\nu)^+$, then $S(x,t,\nu)$ degenerates into the ray $L_-(x,\nu)$, which goes from the point x in the direction $-\nu$.

Consider the orthogonal coordinate system ξ_1, ξ_2, ξ_3 with the origin at the point x. Let orthogonal unite vectors e_1, e_2 and e_3 be directed along axis ξ_1, ξ_2 and ξ_3 respectively. We choose the system ξ_1, ξ_2, ξ_3 such that $e_1 = \nu$ and e_2 and e_3 as the orthogonal to e_1 and to each other. Let $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$ be angles in the spherical coordinate system associated with the system ξ_1, ξ_2, ξ_3. Then for the sake of definiteness, we set

$$\begin{aligned} e_1 &= \nu = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ e_2 &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta). \\ e_3 &= (-\sin\phi, \cos\phi, 0). \end{aligned} \tag{25}$$

Moreover, we consider the cylindrical coordinates z, r, ψ associated with the system ξ_1, ξ_2, ξ_3. More precisely, we set $\xi_1 = z$, $\xi_2 = r\cos\psi$, $\xi_3 = r\sin\psi$, $\psi \in [0, 2\pi]$. Then

$$y = x + \xi, \quad \xi = e_1 z + e_2 r\cos\psi + e_3 r\sin\psi, \tag{26}$$

Equation (24), which defines $S(x,t,\nu)$ can be written in the form

$$S(x,t,\nu) = \left\{(r,z) : z + (z^2 + r^2)^{1/2} = t - x\cdot\nu\right\},$$

or

$$r = r(t - x\cdot\nu, z) = \sqrt{(t - x\cdot\nu)(t - x\cdot\nu - 2z)}. \tag{27}$$

Change variables $(y_1, y_2, y_3) \leftrightarrow (t, z, \psi)$ using formulas (26), (27) which connect y and (t, z, ψ). Then

$$\frac{dy}{|x-y|} = \frac{d\xi}{|\xi|} = \frac{rr_t}{|\xi|} dtdzd\psi = dtdzd\psi.$$

Hence,

$$\begin{aligned} v_0(x,t,\nu) &= -\frac{1}{4\pi} \int\limits_{S(x,t,\nu)} q(y) dzd\psi = -\int\limits_0^{2\pi} \int\limits_{-\infty}^{(t-x\cdot\nu)/2} q(y) dzd\psi \\ &= J(x, t - x\cdot\nu, \nu), \quad t > x\cdot\nu. \end{aligned} \tag{28}$$

Here

$$J(x,\tau,\nu) = -\frac{1}{4\pi} \int\limits_0^{2\pi} \int\limits_{-\infty}^{\tau/2} q(x + e_1 z + \sqrt{\tau(\tau - 2z)} p) dzd\psi, \tag{29}$$

$$p = e_2 \cos\psi + e_3 \sin\psi. \tag{30}$$

Note that

$$J(x, 0^+, \nu) = a(x,\nu) = -\frac{1}{2} \int\limits_{-\infty}^{0} q(x + z\nu) dz. \tag{31}$$

Indeed, if $\tau = 0$ then (29) implies that the square root in the integrand in (29) vanishes and the resulting function $q(x + z\nu)$ does not depend on ψ. It follows from (31) that

$$2\nabla a(x,\nu)\cdot\nu = -q(x), \tag{32}$$

$$a(x,\nu) = 0 \ \text{ if } x\cdot\nu < -B. \tag{33}$$

The equality (33) is true because if $x \cdot \nu < -B$ and $z \le 0$, then $(x + z\nu)\cdot\nu < -B$. This means that the point $(x + z\nu)$ as well as the point x lie outside of Ω and therefore $q(x + z\nu) = 0$ for all $z \in (-\infty, 0]$. Hence, $a(x,\nu) = 0$ in this case. Let $\hat{D}(x,t,\nu)$ be the interior of the paraboloid $S(x,t,\nu)$,

$$\hat{D}(x,t,\nu) = \left\{ y \in \mathbb{R}^3 : y\cdot\nu + |x-y| < t \right\}.$$

Then for each $x \in \mathbb{R}^3$ and $\nu \in S^2$ there exists a number $\tau^*(x,\nu) > 0$ such that for all $t > x\cdot\nu + \tau^*(x,\nu)$ the domain Ω is a subset of $\hat{D}(x,t,\nu)$. Since $q(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$ and $S(x,t,\nu)$ is the boundary of $\hat{D}(x,t,\nu)$, then

$$J(x,\tau,\nu) = 0 \ \text{ for all } \ \tau \ge \tau^*(x,\nu).$$

Hence, for each pair x, ν the function $J(x,\tau,\nu)$ is a finite support with respect to τ. Hence, $v_0(x,t,\nu)$ also has a finite support with respect to t for each pair x, ν.

Let $T > 0$ be a fixed number. It follows from (28), (29) that $v_0(x,t,\nu)$ is a continuous function for all $(x,t) \in D(T,\nu) = \{(x,t) : x\cdot\nu \le t \le T - |x|\}$. Consider the projection of the ball Ω on the straight line $y = x + z\nu, z \in (-\infty, +\infty)$ and denote by $[z_1, z_2]$ the segment of this line containing the projection of Ω. Simple calculations lead to the following equalities $z_1 = z_1(x\cdot\nu) = -B - x\cdot\nu$, $z_2 = z_2(x\cdot\nu) = B - x\cdot\nu$. The integrand in (29) vanishes if $\tau/2 \le z_1(x\cdot\nu)$, i.e., if $(t-x\cdot\nu)/2 \le -x\cdot\nu - B$. Indeed, in this case the paraboloid $S(x,t,\nu)$ does not intersect Ω, because the integration segment with respect to z in (29) has no intersection with $[z_1, z_2]$. Hence, $v_0(x,t,\nu) = 0$ for $t \le -x\cdot\nu - 2B$. Hence, we can consider $v_0(x,t,\nu)$ only inside of the domain

$$G(T,B,\nu) = D(T,\nu) \cap \{(x,t) :\ t + x\cdot\nu \ge -2B\}. \tag{34}$$

If $\tau/2 \ge z_1$, then the intersection $[z_1, z_2] \cap (-\infty, \tau/2] \ne \varnothing$ and the length of this intersection does not exceed $2B$. Hence, the following estimate holds

$$|v_0(x,t,\nu)| = |J(x, t - x\cdot\nu, \nu)| \le q_0 B, \quad (x,t) \in G(T,B,\nu), \tag{35}$$

where $q_0 = \|q\|_{C(\overline{\Omega})}$.

We now calculate the derivative $\partial_\tau J(x,\tau,\nu)$. We have

$$\partial_\tau J(x,\nu,\tau) = -\frac{1}{4} q\Big(x + e_1\frac{\tau}{2}\Big) - \frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_{-\infty}^{\tau/2} \nabla q(y)\cdot(e_2\cos\psi + e_3\sin\psi)\frac{(\tau - z)dzd\psi}{\sqrt{\tau(\tau-2z)}}, \tag{36}$$

where

$$y = x + e_1 z + \sqrt{\tau(\tau-2z)}(e_2\cos\psi + e_3\sin\psi). \tag{37}$$

Integrating by parts with respect to ψ, we obtain

$$\partial_\tau J(x,\nu,\tau) = -\frac{1}{4} q\Big(x + e_1\frac{\tau}{2}\Big) \tag{38}$$

$$-\frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_{-\infty}^{\tau/2} \nabla(\nabla q(y)\cdot(e_2\sin\psi - e_3\cos\psi))\cdot(e_2\sin\psi - e_3\cos\psi)(\tau - z)dzd\psi.$$

Hence,

$$\begin{aligned}\partial_\tau J(x,0^+,\nu) &= -\frac{1}{4}q(x) + \frac{1}{4}\int\limits_{-\infty}^{0}[\nabla(\nabla q(y)\cdot e_2)\cdot e_2 + \nabla(\nabla q(y)\cdot e_3)\cdot e_3]_{y=x+z\nu} z dz\\ &= \frac{1}{4}\int\limits_{-\infty}^{0}[\nabla(\nabla q(y)\cdot e_1)\cdot e_1 + \nabla(\nabla q(y)\cdot e_2)\cdot e_2 + \nabla(\nabla q(y)\cdot e_3)\cdot e_3]_{y=x+z\nu} z dz\\ &= \frac{1}{4}\int\limits_{-\infty}^{0}\Delta q(x+z\nu) z dz = \frac{1}{4}\Delta\int\limits_{-\infty}^{0} q(x+z\nu) z dz \qquad (39)\\ &= -\frac{1}{2}\Delta\int\limits_{-\infty}^{0}\nabla a(x+z\nu,\nu)\cdot\nu z dz = \frac{1}{2}\Delta\int\limits_{-\infty}^{0} a(x+z\nu,\nu)dz\\ &= \frac{1}{2}\int\limits_{-\infty}^{0}\Delta a(x+z\nu,\nu)dz.\end{aligned}$$

Function $\partial_t v_0(x,t,\nu)$ is continuous for all $(x,t)\in D(T,\nu)$. Since in (39) function $q(y)$ does not vanish only for $z\in[z_1,z_2]$, we obtain the estimate

$$|\partial_t v_0(x,t,\nu)|\le Cq_2,\ (x,t)\in G(T,B,\nu), \tag{40}$$

where $q_2=\|q\|_{C^2(\overline{\Omega})}$.

Estimate now the second derivative $\partial_\tau^2 J(x,\tau,\nu)$. Using (38), we find

$$\partial_\tau^2 J(x,\tau,\nu)=-\frac{1}{8}\nabla q\Big(x+e_1\frac{\tau}{2}\Big)\cdot e_1$$

$$-\frac{\tau}{8}[\nabla(\nabla q(y)\cdot e_2)\cdot e_2+\nabla(\nabla q(y)\cdot e_3)\cdot e_3]_{y=x+e_1\tau/2}$$

$$-\frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{\tau/2}\nabla(\nabla q(y)\cdot(e_2\sin\psi-e_3\cos\psi))\cdot(e_2\sin\psi-e_3\cos\psi)dzd\psi+I(x,\tau,\nu), \tag{41}$$

where

$$\begin{aligned}I(x,\tau,\nu)\ &=\ \frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{\tau/2}\nabla[\nabla(\nabla q(y)\cdot(e_2\sin\psi-e_3\cos\psi))\\ &\cdot(e_2\sin\psi-e_3\cos\psi)]\cdot(e_2\sin\psi-e_3\cos\psi)\frac{(\tau-z)^2}{\sqrt{\tau(\tau-2z)}}dzd\psi.\end{aligned}$$

Integrate by parts in this integral with respect to ψ. To do this, we first represent the integrand in the form

$$\nabla[\nabla(\nabla q(y)\cdot(e_2\sin\psi-e_3\cos\psi))\cdot(e_2\sin\psi-e_3\cos\psi)]\cdot(e_2\sin\psi-e_3\cos\psi)$$

$$=\frac{1}{4}\{\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_2]\cdot e_2(3\sin\psi-\sin(3\psi))+3\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_2]\cdot e_3(\cos(3\psi)-\cos\psi)$$

$$+3\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_3]\cdot e_3(\sin\psi+\sin(3\psi))+\nabla[\nabla(\nabla q(y)\cdot e_3)\cdot e_3]\cdot e_3(3\cos\psi-\cos(3\psi))\}.$$

Then

$$\begin{aligned}I(x,\tau,\nu)\ &=\ -\frac{1}{16\pi}\int_0^{2\pi}\int_{-\infty}^{\tau/2}\nabla\{\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_2]\cdot e_2(-3\cos\psi+\cos(3\psi)/3)\\ &+3\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_2]\cdot e_3(\sin(3\psi)/3-\sin\psi)\\ &+3\nabla[\nabla(\nabla q(y)\cdot e_2)\cdot e_3]\cdot e_3(-cos\psi-\cos(3\psi)/3)\\ &+\nabla[\nabla(\nabla q(y)\cdot e_3)\cdot e_3]\cdot e_3(3\sin\psi-\sin(3\psi)/3)\}\\ &\cdot(e_2\sin\psi-e_3\cos\psi)(\tau-z)^2dzd\psi.\end{aligned} \tag{42}$$

It follows from (41) and (42) that the function $\partial_t^2 v_0(x,t,\nu)\in C\left(\overline{G}(T,B,\nu)\right)$ and

$$\left|\partial_t^2 v_0(x,t,\nu)\right|\le Cq_4,\quad (x,t)\in G(T,B,\nu), \tag{43}$$

where $q_4 = \|q\|_{C^4(\overline{\Omega})}$.

For $n \geq 1$ consider now functions $v_n(x,t,\nu)$ in (23). Since the function $v_0(x,t,\nu) \equiv 0$ for all $t + x\cdot\nu \leq -2B$, we conclude that all $v_n(x,t,\nu)$ have the same property:

$$v_n(x,t,\nu) = 0 \text{ if } \; t + x\cdot\nu \leq -2B, \quad n \geq 1.$$

Hence, we can consider these functions only inside the domain $G(T,B,\nu)$ defined in (34). In this domain $t \geq -B + |x\cdot\nu + B| = t_0 = t_0(x\cdot\nu)$. For $(x,t) \in G(T,B,\nu)$ we represent the domain of the integration in (23) as the union of paraboloids $S(x,t',\nu)$ for $t' \in [x\cdot\nu, t]$. On the paraboloid $S(x,t',\nu)$ we have $t - |x-y| = t - t' + y\cdot\nu$, where z belongs to the interval $(-\infty, (t' - x\cdot\nu)/2]$. Then equation (23) for the function $v_n(x,t,\nu)$ can be written as

$$\begin{aligned} v_n(x,t,\nu) &= -\frac{1}{4\pi}\int\limits_{x\cdot\nu}^{t}\int\limits_{S(x,t',\nu)} q(y)v_{n-1}(y, t-t'+y\cdot\nu, \nu)\,dz d\psi dt' \\ &= -\frac{1}{4\pi}\int\limits_{x\cdot\nu}^{t}\int\limits_{0}^{2\pi}\int\limits_{-\infty}^{(t'-x\cdot\nu)/2} q(y)v_{n-1}(y, t-t'+y\cdot\nu, \nu)\,dz d\psi dt', \end{aligned} \tag{44}$$

where

$$y = x + e_1 z + \sqrt{(t'-x\cdot\nu)(t'-x\cdot\nu-2z)}(e_2\cos\psi + e_3\sin\psi).$$

It follows from (44) that functions $v_n(x,t,\nu) \in C\left(\overline{G}(T,B,\nu)\right), n \geq 1$. Using (36) and noting that $q(y)$ vanishes for $z \notin [z_1(x\cdot\nu), z_2(x\cdot\nu)]$, we get

$$|v_1(x,t,\nu)| \leq (Bq_0)^2(t - x\cdot\nu),$$

$$|v_2(x,t,\nu)| \leq (Bq_0)^2\frac{q_0}{2}\int\limits_{x\cdot\nu}^{t}\int\limits_{z_1(x\cdot\nu)}^{z_2(x\cdot\nu)}(t-t')\,dz dt' = (Bq_0)^3\frac{(t-x\cdot\nu)^2}{2!}.$$

Using the method of the mathematical induction, it easy to prove that for all $n \geq 1$ the following estimates hold

$$|v_n(x,t,\nu)| \leq (Bq_0)^{n+1}\frac{(t-x\cdot\nu)^n}{n!} \leq (Bq_0)^{n+1}\frac{T^n}{n!}, \quad (x,t) \in \overline{G}(T,B,\nu). \tag{45}$$

Hence, the series (21) is uniformly converges in the space $C\left(\overline{G}(T,B,\nu)\right)$. Moreover, the following estimate is valid for the sum $v(x,t,\nu)$ of this series

$$|v(x,t,\nu)| \leq Bq_0\exp(Bq_0T), \quad (x,t) \in G(T,B,\nu).$$

We prove now that the first derivative of $v(x,t,\nu)$ with respect to t exists and it is a continuous function in $\overline{G}(T,B,\nu)$. To do this we prove first the existence of $\partial_t v_n(x,t,\nu)$ for $n \geq 1$ and obtain estimates for them. Using (44) we have

$$\begin{aligned} \partial_t v_1(x,t,\nu) &= -\frac{1}{4\pi}\int\limits_{0}^{2\pi}\int\limits_{-\infty}^{(t-x\cdot\nu)/2} q(y)v_0(y, y\cdot\nu, \nu)\,dz d\psi \\ &\quad -\frac{1}{4\pi}\int\limits_{x\cdot\nu}^{t}\int\limits_{0}^{2\pi}\int\limits_{-\infty}^{(t'-x\cdot\nu)/2} q(y)\partial_t v_0(y, t-t'+y\cdot\nu, \nu)\,dz d\psi dt'. \end{aligned} \tag{46}$$

Since the function $v_0(x,t,\nu)$ and its derivative $\partial_t v_0(x,t,\nu)$ are continuous in $\overline{G}(T,B,\nu)$, then (46) implies that $\partial_t v_1(x,t,\nu)$ is also a continuous function in $\overline{G}(T,B,\nu)$. Note that integration intervals with respect to z in the integral terms do not exceed $2B$. Using (35), (40), we obtain the estimate

$$|\partial_t v_1(x,t,\nu)| \leq Bq_0^2 + CBq_2q_0(t - x\cdot\nu) \leq Cq_2q_0, \quad (x,t) \in \overline{G}(T,B,\nu). \tag{47}$$

We now estimate functions $\partial_t v_n(x,t,\nu)$ for $n \geq 2$. The equation for $\partial_t v_n(x,t,\nu)$, $n \geq 2$, has the form

$$\begin{aligned}\partial_t v_n(x,t,\nu) &= -\frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_{-\infty}^{(t-x\cdot\nu)/2} q(y)v_{n-1}(y, y\cdot\nu,\nu)dzd\psi \\ &\quad -\frac{1}{4\pi}\int\limits_{t-x\cdot\nu}^{t}\int\limits_0^{2\pi}\int\limits_{-\infty}^{(t'-x\cdot\nu)/2} q(y)\partial_t v_{n-1}(y, t-t'+y\cdot\nu,\nu)dzd\psi dt'. \end{aligned} \tag{48}$$

Note that, by (45), $v_{n-1}(y, y\cdot\nu, \nu) = 0$ for all $n \geq 2$. Hence, the equation for $\partial_t v_n(x,t,\nu)$, $n \geq 2$, can be written as

$$\partial_t v_n(x,t,\nu) = -\frac{1}{4\pi}\int\limits_{t-x\cdot\nu}^{t}\int\limits_0^{2\pi}\int\limits_{-\infty}^{(t'-x\cdot\nu)/2} q(y)\partial_t v_{n-1}(y, t-t'+y\cdot\nu,\nu)dzd\psi dt'. \tag{49}$$

Again, since the function $\partial_t v_1(x,t,\nu)$ are continuous in $C\left(\overline{G}(T,B,\nu)\right)$, then (49) implies that all functions $\partial_t v_n(x,t,\nu) \in C\left(\overline{G}(T,B,\nu)\right)$ for $n \geq 2$. Using estimate (47), we obtain

$$|\partial_t v_2(x,t,\nu)| \leq \frac{Cq_0^2q_2}{4\pi}\int\limits_{x\cdot\nu}^{t}\int\limits_0^{2\pi}\int\limits_{z_1(x\cdot\nu)}^{z_2(x\cdot\nu)} dzd\psi dt' \leq Cq_2Bq_0^2(t - x\cdot\nu).$$

Then

$$|\partial_t v_3(x,t,\nu)| \leq \frac{Cq_2Bq_0^3}{4\pi}\int\limits_{x\cdot\nu}^{t}\int\limits_0^{2\pi}\int\limits_{z_1(x\cdot\nu)}^{z_2(x\cdot\nu)} (t - t'' \leq Cq_2B^2q_0^3\frac{(t-x\cdot\nu)^2}{2!}.$$

Using the method of the mathematical induction, we similarly prove that the following estimate holds for all $(x,t) \in \overline{G}(T,B,\nu)$

$$|\partial_t v_n(x,t,\nu)| \leq Cq_0q_2(Bq_0)^{n-1}\frac{(t-x\cdot\nu)^{n-1}}{(n-1)!} \leq Cq_0q_2(Bq_0)^{n-1}\frac{T^{n-1}}{(n-1)!},\ n \geq 2. \tag{50}$$

Hence, the series

$$\sum_{n=0}^{\infty} \partial_t v_n(x,t,\nu)$$

converges uniformly in the space $C\left(\overline{G}(T,B,\nu)\right)$ and its sum coincides with $\partial_t v(x,t,\nu) \in C\left(\overline{G}(T,B,\nu)\right)$.

Next, we use the similar ideas to prove that the second derivatives $\partial_t^2 v_n(x,t,\nu)$ are continuous in $\overline{G}(T,B,\nu)$ and the series from the second derivatives uniformly converges in the space $C\left(\overline{G}(T,B,\nu)\right)$. Differentiating (48) with respect to t, we obtain

$$\begin{aligned}\partial_t^2 v_n(x,t,\nu) &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_{-\infty}^{(t-x\cdot\nu)/2} q(y)v_{n-1}(y,y\cdot\nu,\nu)dzd\psi \\ &\quad -\frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{(t-x\cdot\nu)/2} q(y)\partial_t v_{n-1}(y,y\cdot\nu,\nu)dzd\psi \qquad (51) \\ &\quad -\frac{1}{4\pi}\int_{t-x\cdot\nu}^{t}\int_0^{2\pi}\int_{-\infty}^{(t'-x\cdot\nu)/2} q(y)\partial_t^2 v_{n-1}(y,t-t'+y\cdot\nu,\nu)dzd\psi dt'.\end{aligned}$$

Calculate separately the first term in this equality, which we denote as $I_n(x,t,\nu)$. We have

$$\begin{aligned}I_n(x,t,\nu) &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_{-\infty}^{(t-x\cdot\nu)/2} q(y)v_{n-1}(y,y\cdot\nu,\nu)dzd\psi \\ &= -\frac{1}{4}q\left(x+\frac{t-x\cdot\nu}{2}\nu\right)v_{n-1}\left(x+\frac{t-x\cdot\nu}{2}\nu, x\cdot\nu+\frac{t-x\cdot\nu}{2}\right) \\ &\quad -\frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{(t-x\cdot\nu)/2} \nabla[q(y)v_{n-1}(y,y\cdot\nu,\nu)]\cdot(e_2\cos\psi+e_3\sin\psi) \\ &\quad \times\frac{t-x\cdot\nu-z}{\sqrt{(t-x\cdot\nu)(t-x\cdot\nu-2z)}}\,dzd\psi,\end{aligned}$$

where y is given by (37). Integrating by parts with respect to ψ, we obtain

$$\begin{aligned}I_n(x,t,\nu) &= -\frac{1}{4}q\left(x+\frac{t-x\cdot\nu}{2}\nu\right)v_{n-1}\left(x+\frac{t-x\cdot\nu}{2}\nu, x\cdot\nu+\frac{t-x\cdot\nu}{2}\right) \\ &\quad -\frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{(t-x\cdot\nu)/2} \nabla\{\nabla[q(y)v_{n-1}(y,y\cdot\nu,\nu)]\cdot(e_2\sin\psi-e_3\cos\psi)\} \\ &\quad \cdot(e_2\sin\psi-e_3\cos\psi)(t-x\cdot\nu-z)dzd\psi.\end{aligned}$$

Note that $v_n(x,x\cdot\nu,\nu)=0$ for $n\geq 1$ and

$$v_0(x,x\cdot\nu,\nu) = -\frac{1}{2}\int_{-\infty}^{0} q(x+z\nu)dz.$$

Therefore $I_n(x,t,\nu)=0$ for $n>1$ and $I_1(x,t,\nu)$ is a continuous function for $(x,t)\in\overline{G}(T,B,\nu)$ and for it the following estimate holds

$$|I_1(x,t,\nu)|\leq Cq_2^2, \quad (x,t)\in\overline{G}(T,B,\nu).$$

Using (51), we sequentially obtain that the functions $\partial_t^2 v_n(x,t,\nu)$, $n = 1,2,\dots$ are continuous for $(x,t) \in \overline{G}(T,B,\nu)$. Moreover, one can easily check that the following estimates hold

$$|\partial_t^2 v_n(x,t,\nu)| \le C q_4^{n+1}, \quad (x,t) \in \overline{G}(T,B,\nu),\ n = 1,2.$$

By (50), $\partial_t v_n(x, x\cdot\nu, \nu) = 0$ for $n \ge 2$. Hence, starting from $n = 3$ all functions $\partial_t^2 v_n(x,t,\nu)$ are determined by the relations

$$\partial_t^2 v_n(x,t,\nu) = -\frac{1}{4\pi} \int\limits_{t-x\cdot\nu}^{t} \int\limits_{0}^{2\pi} \int\limits_{-\infty}^{(t'-x\cdot\nu)/2} q(y)\partial_t^2 v_{n-1}(y, t-t'+y\cdot\nu, \nu) dz d\psi dt'.$$

From here, using (43), we obtain for $(x,t) \in \overline{G}(T,B,\nu)$ the estimates

$$|\partial_t^2 v_n(x,t,\nu)| \le C q_4^3 (q_0 B)^{n-2} \frac{(t - x\cdot\nu)^{n-2}}{(n-2)!} \le C q_4^3 (q_0 B)^{n-2} \frac{T^{n-2}}{(n-2)!}, n \ge 3.$$

Thus, the series

$$\sum_{n=0}^{\infty} \partial_t^2 v_n(x,t,\nu)$$

uniformly converges in the space $C\left(\overline{G}(T,B,\nu)\right)$ and its sum is the function $\partial_t^2 v(x,t,\nu) \in C\left(\overline{G}(T,B,\nu)\right)$. $\square$

4 Proof of Theorem 1

Consider again the solution $v(x,t,\nu)$ of the Cauchy problem (17), (18). By Theorem 2 and Corollary functions $\Delta_x v(x,t,\nu), \partial_t^k v(x,t,\nu) \in C\left(\overline{G}(T,B,\nu)\right), k = 0,1,2$. Let $\Phi \subset \mathbb{R}^3$ be an arbitrary bounded domain. We now refer to Lemma 6 of Chapter 10 of the book [27] as well as to Remark 3 after that lemma. It follows from these results that functions $\partial_t^k v(x,t,\nu), k = 0,1,2$ and $\Delta_x v(x,t,\nu)$ decay exponentially as $t \to \infty$ as longs as x reminds in the domain Φ. In other words, there exist constants $M = M(\Phi, q), c = c(\Phi, q) > 0$ such that

$$\left|\partial_t^k v(x,t,\nu)\right|, |\Delta v(x,t,\nu)| \le M e^{-ct} \text{ for all } t \ge |x\cdot\nu| \text{ and for all } x \in \Phi. \tag{52}$$

Hence, one can consider the Fourier transform $V(x,k,\nu)$ of the function v,

$$V(x,k,\nu) = \int\limits_{x\cdot\nu}^{\infty} v(x,t,\nu) \exp(-ikt)\, dt.$$

Now we again refer to Theorem 3.3 of the paper of [26] and Theorem 6 of Chapter 9 of [27]. These results guarantee that $V(x,k,\nu) = u_{sc}(x,k,\nu)$, where the function $u_{sc}(x,k,\nu)$ is the above solution of the problem (6)-(8).

Thus,

$$u_{sc}(x,k,\nu)=\int\limits_{x\cdot\nu}^{\infty}\exp(-ikt)v(x,t,\nu)dt. \tag{53}$$

$$v|_{t=x\cdot\nu+0}=a(x,\nu)=-\frac{1}{2}\int\limits_{L_-(x,\nu)}q(\xi)d\sigma=-\frac{1}{2}\int\limits_{0}^{\infty}q(x-s\nu)ds, \tag{54}$$

Applying the integration by parts Theorem 2 to (53), we obtain

$$u_{sc}(x,k,\nu)=-\frac{i\exp(-ikx\cdot\nu)}{k}v(x,(x\cdot\nu)^{+},\nu)+O\left(\frac{1}{k^2}\right),\ k\to\infty.$$

Hence, by (54)

$$u_{sc}(x,k,\nu)=\frac{i\exp(-ikx\cdot\nu)}{2k}\int\limits_{L_-(x,\nu)}q(\xi)d\sigma+O\left(\frac{1}{k^2}\right),k\to\infty.$$

Since by (2) $q\geq 0$, then

$$|u_{sc}(x,k,\nu)|=\frac{1}{2k}\int\limits_{L_-(x,\nu)}q(\xi)d\sigma+O\left(\frac{1}{k^2}\right),\ k\to\infty.$$

This formula coincides with the formula (14). □

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Michael V. Klibanov,
Department of Mathematics and Statistics University of North Carolina at Charlotte,
Charlotte, NC 28223, USA,
Email: mklibanv@uncc.edu,

Vladimir G. Romanov,
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,
Email: romanov@math.nsc.ru

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