EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 2, Issue 2(2014) 102 –121

VARIATION FORMULATION OF THE INVERSE PROBLEM OF DETERMINING THE COMPLEX-COEFFICIENT OF EQUATION OF QUASIOPTICS

A.D. Iskenderov, G.Ya. Yagubov, N.S. Ibragimov, N.Y.Aksoy

Abstract We consider the inverse problem of determining the complex-coefficient of linear quasioptics equations. The variation formulation of the problem for which theorems of existence and uniqueness of solutions proves and a necessary condition of extremum establishes in the variation inequality case.

Key words: inverse problem, the quasioptics equation, complex-coefficient, nonlinear optics, the variation method, the necessary condition of extremum.

AMS Mathematics Subject Classification:35R30, 35Q60, 35Q93, 35Q04, 49J20, 49K20, 93C20

1 Introduction

It is known that one of the most famous scientific and technological achievements of the last century are the emergence of quantum theory and achievements of the field of nonlinear optics [1]. The development of these two areas are interrelated and have found numerous scientific and technical applications. Quasioptics is the section of nonlinear optics, where the mathematical model of the processes have found the greatest development.

Inverse problem of determining the unknown coefficients of quasioptics often arise in the studying of the propagation of a light beam in the inhomogeneous medium, where the unknown functions are the refractive index and absorption of the medium, and the initial phase of the emitted wave [1]. The indices of refraction and absorption of the medium are included in the equation of quasioptics as the complex-coefficient factor. Therefore, in practice, it is necessary to determine the complex-coefficients of the equation of quasioptics.

We consider the inverse problem of determining the complex-coefficient of linear dependent equation of quasioptics, where the real part of the complex coefficient is the refractive index and the imaginary part is a measure of the absorption of the inhomogeneous medium. In this paper we considered the variation formulation of the inverse problem of determining the unknown coefficients of the equation of quasioptics. The criteria of quality will be based by the Dirichlet-Neumann maps. This approach is widely used and validated in [2,3] and others to determining the coefficients of the main types of equations of mathematical physics. Earlier, the variation formulation of the inverse problem for the equation of quasioptics studied in [4-8] and others, when the quality criterion formulated by the final observation. Formulation of the inverse problem considered in this paper differs from the productions, previously studied similar inverse problems, as by the quality criterion, as by the functional spaces, where decided the solutions of these problems.

The traditional theory of inverse problems for equations of mathematical physics is built to determine the real-valued coefficients of these equations [9.10]. There are very few studies by the inverse problem of determining the coefficients of complex differential equations. Therefore, this work is also of interest to the development the variation methods for determining the coefficients of complex differential equations.

2 Statement of the Problem

Let D - be the bounded domain of n-dimensional Euclidean space E_n, Γ - the boundary of the domain D, which is assumed to be sufficiently smooth, $T > 0$, $L > 0$- are given numbers, $0 \le t \le T$, $0 \le z \le L$, $x = (x_1, x_2, ..., x_n) \in D$- the arbitrary point, $\Omega_t = D \times (0,t)$, $\Omega_z = D \times (0,z)$, $\Omega_{tz} = D \times (0,t) \times (0,z)$, $\Omega = \Omega_{TL}$,$S_{tz} = \Gamma \times (0,t) \times (0,z)$,$S = S_{TL}$; $C^k([0,T], B)$- Banach space with all defined and $k \ge 0$ times continuously differentiable functions on the interval $[0,T]$ with values in a Banach space B, $L_p(D)$ - Lebesgue space of functions integrable in D with the degree $p \ge 1$; $W_p^k(D)$, $W_p^{k,m}(Q)$, $p \ge 1$, $k \ge 0$, $m \ge 0$- Sobolev spaces, which are defined, for example, in [11]; $W_2^{0,1,1}(\Omega)$ Hilbert space with all elements $u = u(x,t,z)$ from $L_2(\Omega)$, with generalized derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial z}$ from the $L_2(\Omega)$ space, the scalar product and the norm are defined by the equations:

$$(u_1, u_2)_{W_2^{0,1,1}(\Omega)} = \int_\Omega \left(u_1 \bar{u}_2 + \frac{\partial u_1}{\partial t}\frac{\partial \bar{u}_2}{\partial t} + \frac{\partial u_1}{\partial z}\frac{\partial \bar{u}_2}{\partial z} \right) dxdtdz,$$

$$\|u\|_{W_2^{0,1,1}(\Omega)} = \sqrt{(u,u)_{W_2^{0,1,1}(\Omega)}} < +\infty;$$

$W_2^{2,0,0}(\Omega)$- Hilbert space with all elements $u = u(x,t,z)$ of the $L_2(\Omega)$ space with generalized derivatives $\frac{\partial u}{\partial x_j}$, $j = \overline{1,n}$, $\frac{\partial^2 u}{\partial x_j \partial x_p}$, $j,p = \overline{1,n}$ of the space $L_2(\Omega)$, the scalar product and the norm are defined by the equations:

$$(u_1, u_2)_{W_2^{2,0,0}(\Omega)} = \int_\Omega \left(u_1 \bar{u}_2 + \sum_{j=1}^n \frac{\partial u_1}{\partial x_j}\frac{\partial \bar{u}_2}{\partial x_j} + \sum_{j,p=1}^n \frac{\partial^2 u_1}{\partial x_j \partial x_p}\frac{\partial^2 \bar{u}_2}{\partial x_j \partial x_p} \right) dxdtdz,$$

$$\|u\|_{W_2^{2,0,0,}(\Omega)} = \sqrt{(u,u)_{W_2^{2,0,0}(\Omega)}} < +\infty;$$

$W_2^{2,1,1}(\Omega) \equiv W_2^{2,0,0}(\Omega) \bigcap W_2^{0,1,1}(\Omega)$; $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ - the subspace of the $W_2^{2,1,1}(\Omega)$ space, whose elements vanish on $S = \Gamma \times (0,T) \times (0,L)$; the symbol $\overset{0}{\forall}$ is mean that the this property has for almost all values of variable. Constants, independent of the estimated values, denote by $c_j, j = 0, 1, 2, ...$

Consider the initial-boundary value problem for a linear non-stationary equation of quasioptics, which often arises in nonlinear optics in the study of the propagation the light beam in an inhomogeneous medium, when the complex amplitude of the electric field of the light wave depends by the time [1]:

$$i\frac{\partial\psi}{\partial t} + ia_0\frac{\partial\psi}{\partial z} - \sum_{j,p=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jp}(x)\frac{\partial\psi}{\partial x_p}\right) + a(x)\psi + v(x,t,z)\psi =$$

$$= f(x,t,z), (x,t,z) \in \Omega, \tag{2.1}$$

$$\psi(x,0,z) = \phi_0(x,z), \quad (x,z) \in \Omega_L, \quad \psi(x,t,0) = \phi_1(x,t), \quad (x,t) \in \Omega_T, \tag{2.2}$$

$$\psi|_S = 0, \tag{2.3}$$

where $i = \sqrt{-1}$; $\psi = \psi(x,t,z)$ - is the wave function or the complex amplitude of the electric field of the light wave (beam), which extends along the axis z,$v(x,t,z) = v_0(x,t,z)+iv_1(x,t,z)$, and $v_0(x,t,z)$, $v_1(x,t,z)$ - the refractive index and absorption of the medium, $f(x,t,z)$ - given complex-valued function, $\phi_0(x,z)$ - given initial complex amplitude of the electric field, $\phi_1(x,t)$ - specifies the initial phase profile. $a_0 > 0$ - a real number, $a_{jp}(x)$, $j,p = \overline{1,n}$, $a(x)$ - given the real-valued bounded measurable functions with bounded measurable derivatives $\frac{\partial a_{jp}(x)}{\partial x_l}$, $j,p,l = \overline{1,n}$. Let the following conditions are satisfied:

$$a_{jp}(x) = a_{pj}(x), \quad \mu_0\sum_{j=1}^{n}|\xi_j|^2 \le \sum_{j,p=1}^{n}a_{jp}(x)\xi_j\bar{\xi}_p \le \mu_1\sum_{j=1}^{n}|\xi_j|^2, \quad \forall\xi_j \in C, \tag{2.4}$$

$$\left|\frac{\partial a_{jp}(x)}{\partial x_l}\right| \le \mu_2, \quad \overset{\circ}{\forall} x \in D, \ j,p,l = \overline{1,n}, \mu_0,\mu_1,\mu_2 = const > 0 \tag{2.5}$$

$$\mu_3 \le a(x) \le \mu_4, \quad \dot{\forall} x \in D, \ \mu_3,\mu_4 = const > 0; \tag{2.6}$$

the symbol $\overset{0}{\forall}$ means "for almost all".

We consider the inverse problem of determining the unknown coefficients $v_0(x,t,z)$, $v_1(x,t,z)$, and the function $\psi(x,t,z)$ of the conditions (2.1)-(2.3). Suppose further defined the second boundary condition for the equation (2.1)

$$\left.\frac{\partial\psi}{\partial N}\right|_S = \left.\sum_{j,p=1}^{n}a_{jp}(x)\frac{\partial\psi}{\partial x_p}\cos(\nu,x_j)\right|_S = 0, \tag{2.7}$$

where ν- is the external normal of boundary Γ of the domain D, and $N-$ is conormal. Unknown coefficients $v_0 = v_0(x,t,z)$,$v_1 = v_1(x,t,z)$ will be found on the set:

$$V \equiv \Big\{ v = (v_0, v_1)\,,\ v_m \in W_2^{0,1,1}(\Omega)\,,\ \ |v_m(x,t,z)| \le b_m\ ,$$

$$\left|\frac{\partial v_m(x,t,z)}{\partial t}\right| \le d_m, \left|\frac{\partial v_m(x,t,z)}{\partial z}\right| \le c_m, m = 0,1, \overset{0}{\forall}(x,t,z) \in \Omega \Big\},$$

where $b_m > 0$,$d_m > 0$, $c_m > 0$, $m = 0,1$ - are given numbers. The set V is called the set of admissible coefficients.

Following the methods of the paper [2,3] we will present a variation formulation of the inverse problem. Through $\psi_1 = \psi_1(x,t,z)$ will denote the solution of the first initial-boundary value problem (2.1) - (2.3) for the equation (2.1). Let $\psi_2 = \psi_2(x,t,z)$ is the solution of the second initial-boundary value problem (2.1), (2.2), (2.7)for the (2.1). Now, consider the problem of minimizing the functional:

$$J_\alpha(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} + \alpha\|v - \omega\|^2_H \tag{2.8}$$

on the set V with the following conditions:

$$i\frac{\partial \psi_k}{\partial t} + ia_0\frac{\partial \psi_k}{\partial z} - \sum_{j,p=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jp}(x)\frac{\partial \psi_k}{\partial x_p}\right) + a(x)\psi_k +$$

$$+v_0(x,t,z)\psi_k + iv_1(x,t,z)\psi_k = f_k(x,t,z)\,,\quad (x,t,z) \in \Omega, k = 1,2, \tag{2.9}$$

$$\psi_k(x,0,z) = \varphi_{0k}(x,z), \quad (x,z) \in \Omega_L,$$

$$\psi_k(x,t,0) = \varphi_{1k}(x,z), \quad (x,t) \in \Omega_T, \quad k = 1,2, \tag{2.10}$$

$$\psi_1|_S = 0, \quad \left.\frac{\partial \psi_2}{\partial N}\right|_S = \left.\sum_{j,p=1}^{n} a_{jp}(x)\frac{\partial \psi_2}{\partial x_p}\cos(\nu, x_j)\right|_S = 0. \tag{2.11}$$

where $\alpha \ge 0$ - is given number, $H \equiv W_2^{0,1,1}(\Omega) \times W_2^{0,1,1}(\Omega)$, $\omega = (\omega_0, \omega_1) \in H$ - is specified element, $f_k(x,t,z)\,, \varphi_{0k}(x,z)\,, \varphi_{1k}(x,t)\,,k = 1,2$ - is a given complex-valued function satisfying the conditions:

$$f_k \in W_2^{0,1,1}(\Omega)\,,\ k = 1,2, \tag{2.12}$$

$$\varphi_{01} \in \overset{0}{W}{}_2^{2,1}(\Omega_L)\,,\ \varphi_{02} \in W_2^{2,1}(\Omega_L)\,, \quad \frac{\partial \varphi_{02}}{\partial N}|_{S_L} = 0, \tag{2.13}$$

$$\varphi_{11} \in \overset{0}{W}{}_2^{2,1}(\Omega_T), \ \varphi_{12} \in W_2^{2,1}(\Omega_T), \quad \frac{\partial \varphi_{12}}{\partial N}|_{S_T} = 0, \tag{2.14}$$

where $S_L = \Gamma \times (0, L)$, $S_T = \Gamma \times (0, T)$. For each $v \in V$ the problem of the determining the functions $\psi_k = \psi_k(x, t, z) \equiv \psi_k(x, t, z; v)$, $k = 1, 2$ from the conditions (2.9) -(2.11) are the first and the second initial-boundary value problem for the equation (2.9). By the solution of the first initial-boundary value problem for each $v \in V$ we will mean a function $\psi_1(x, t, z)$ of the space $\overset{0}{W}{}_2^{2,1,1}(\Omega)$, with *k=1* satisfying (2.9)-(2.11) for almost all $(x, t, z) \in \Omega$. The same way, for every $v \in V$ the function $\psi_2(x, t, z)$ of the space $W_2^{2,1,1}(\Omega)$, is called a solution of the second initial-boundary value problem for the equation (2.9), if *k=2*, it satisfies the conditions(2.9) -(2.11) for almost all $(x, t, z) \in \Omega$.

Note that these initial-boundary value problems have been previously studied in [8]. In that paper, theorems of the existence and uniqueness of solutions of initial-boundary value problems have been proved by the Galerkin method. From the results of that paper, the following theorem holds.

Theorem 2.1. *Let the functions $a_{jp}(x)$, $j, p = \overline{1, n}$, $a(x)$, $f_k(x, t, z)$,$\varphi_{0k}(x, z)$,$\varphi_{1k}(x, t)$, $k = 1, 2$ satisfy respectively the conditions (2.4)-(2.6) and (2.12)-(2.14). Then the first and the second initial-boundary value problems for the equation(2.1), for each $v \in V$ has an unique solution respectively such that the $\psi_1 \in \overset{0}{W}{}_2^{2,1,1}(\Omega)$,$\psi_2 \in W_2^{2,1,1}(\Omega)$, and for these solutions have the following estimates:*

$$\|\psi_1\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)} \le c_2 \left(\|\varphi_{01}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_L)} + \|\varphi_{11}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_T)} + \|f_1\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{2.15}$$

$$\|\psi_2\|^2_{W_2^{2,1,1}(\Omega)} \le c_3 \left(\|\varphi_{02}\|^2_{W_2^{2,1}(\Omega_L)} + \|\varphi_{12}\|^2_{W_2^{2,1}(\Omega_T)} + \|f_2\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{2.16}$$

where $c_2 > 0, c_3 > 0-$is the constants independent of $\varphi_{0k}, \varphi_{1k}$ and f_k, $k = 1, 2$.

The following theorem proved in [12]:

Theorem 2.2. *Let X- is uniformly convex space, U- is a closed bounded set of X, the functional $I(v)$ on the U is semi-continuous and lower limited, $\alpha > 0$, $\beta \ge 1$ - is a given number. Then there exists a dense subset G of X such that for any $\omega \in G$ the functional*

$$J_\alpha(v) = I(v) + \alpha \|v - \omega\|_X^\beta \tag{2.17}$$

get the minimum value on U. If $\beta > 1$, then the minimum value of the functional $J_\alpha(v)$ on U will get on a single element.

3 Existence and uniqueness of solutions of the variation formulation of the inverse problem

Theorem 3.1. *Let functions $a_{jp}(x), j,p=\overline{1,n}$, $a(x), f(x,t,z), \varphi_0(x,z), \varphi_1(x,t)$ satisfy respectively the conditions (2.4) - (2.6)and (2.12) - (2.14). Then there exists the dense subset G of the space H such that for all $\omega \in G$ and $\alpha > 0$ the problem (2.8) - (2.11)has an unique solution.*

Proof. Firstly, we will prove the continuity of the functional

$$J_0(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} \tag{3.1}$$

on the set V. Let $\Delta v \in B \equiv W^{0,1,1}_{\infty}(\Omega) \times W^{0,1,1}_{\infty}(\Omega)$ - increment of any element $v \in V$ such that $v + \Delta v \in V$. Denote $\Delta\psi_k = \Delta\psi_k(x,t,z) \equiv \psi_k(x,t,z; v+\Delta v) - \psi_k(x,t,z;v)$, $k=1,2$. From the conditions (2.9)-(2.11) we have that $\Delta\psi_k = \Delta\psi_k(x,t,z)$, $k=1,2$ satisfies the conditions of the following system:

$$i\frac{\partial \Delta\psi_k}{\partial t} + ia_0\frac{\partial \Delta\psi_k}{\partial z} - \sum_{j,p=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jp}(x)\frac{\partial\Delta\psi_k}{\partial x_p}\right) + a(x)\Delta\psi_k +$$

$$+(v_0(x,t,z)+\Delta v_0(x,t,z))\Delta\psi_k + i(v_1(x,t,z)+\Delta v_1(x,t,z))\Delta\psi_k =$$

$$= -\Delta v_0(x,t,z)\psi_k - i\Delta v_1(x,t,z)\psi_k, \ (x,t,z)\in\Omega, k=1,2, \tag{3.2}$$

$$\Delta\psi_k(x,0,z) = 0, \ (x,z)\in\Omega_L, \ \Delta\psi_k(x,t,0)=0, \ (x,t)\in\Omega_T, k=1,2, \tag{3.3}$$

$$\Delta\psi_1|_S = 0, \quad \left.\frac{\partial\Delta\psi_2}{\partial N}\right|_S = 0. \tag{3.4}$$

We multiply both sides of (3.2) by the function $\Delta\bar{\psi}_k = \Delta\bar{\psi}_k(x,t,z), k=1,2$ and integrate by the domain Ω_{tz}. Further, from the above equalities subtract their complex conjugation, then we have:

$$\int_{\Omega_{tz}}\frac{\partial}{\partial t}|\Delta\psi_k|^2 dxd\tau d\theta + \int_{\Omega_{tz}}\frac{\partial}{\partial z}|\Delta\psi_k|^2 dxd\tau d\theta = -2\int_{\Omega_{tz}}(v_1+\Delta v_1)|\Delta\psi_k|^2 dxd\tau d\theta -$$
$$-2\int_{\Omega_{tz}} Im\left(\Delta v_0\psi_k\Delta\bar{\psi}_k\right)dxd\tau d\theta - 2\int_{\Omega_{tz}} Re\left(\Delta v_1\psi_k\Delta\bar{\psi}_k\right)dxd\tau d\theta,$$
$$\forall t\in[0,T], \forall z\in[0,L], \quad k=1,2.$$

From these equations by using the estimates (2.15) and (2.16) with these above conditions imply the inequalities:

$$\|\Delta\psi_k(\,\cdot,t,\cdot)\|^2_{L_2(\Omega_L)} + \|\Delta\psi_k(\,\cdot,\cdot,z)\|^2_{L_2(\Omega_T)} \le$$

$$\le c_4\left(\|\Delta v_0\|^2_{L_\infty(\Omega)} + \|\Delta v_1\|^2_{L_\infty(\Omega)}\right), \quad k=1,2\ , \tag{3.5}$$

for $\forall t \in [0,T]\,,\ \ \forall z \in [0,L]$. Integrating these inequalities we obtain:

$$\|\Delta\psi_k\|^2_{L_2(\Omega)} \le c_5\left(\|\Delta v_0\|^2_{L_\infty(\Omega)} + \|\Delta v_1\|^2_{L_\infty(\Omega)}\right), k = 1,2\ . \tag{3.6}$$

Now, consider the increment of the functional $J_0(v)$ on any element $v \in V$. Using formula (3.1), the increment of the functional $J_0(v)$ can be written as:

$$\Delta J_0(v) = \int\limits_\Omega Re\left[\left(\psi_1(x,t,z) - \psi_2(x,t,z)\right)\left(\Delta\bar\psi_1(x,t,z) - \Delta\bar\psi_2(x,t,z)\right)\right]dxdtdz +$$

$$\|\Delta\psi_1\|^2_{L_2(\Omega)} + \|\Delta\psi_2\|^2_{L_2(\Omega)}\,2\int\limits_\Omega Re\left(\Delta\psi_1(x,t,z)\,\Delta\bar\psi_2(x,t,z)\right)dxdtdz \tag{3.7}$$

Hence, using the estimates (2.15) and (2.16),(3.6) and the Cauchy-Schwarz inequality we obtain:

$$|\Delta J_0(v)| \le c_6\left(\|\Delta v_0\|_{L_\infty(\Omega)} + \|\Delta v_1\|_{L_\infty(\Omega)} + \|\Delta v_0\|^2_{L_\infty(\Omega)} + \|\Delta v_1\|^2_{L_\infty(\Omega)}\right).$$

From this inequality we obtain the following limit value:

$$|\Delta J_0(v)| \to 0 \qquad \|\Delta v\|_B \to 0 \tag{3.8}$$

for $\forall v \in V$. From this limit relation follows the continuity of the functional $J_0(v)$ on the set V. The bottom functional limitations $J_0(v)$ on the set V follows from the inequality $J_0(v) \ge 0,\ \forall v \in V$. It is easy to prove that the set V is closed and bounded convex set in a uniformly convex space $H \equiv W_2^{0,1,1}(\Omega) \times W_2^{0,1,1}(\Omega)$ [see. 13, 182 p.]. Then we can say that all the conditions of Theorem 2.2 satisfy. Therefore, by this theorem, we conclude that there exists a dense subset G of the space H such that for any $\omega \in G$ and for any $\alpha > 0$ the problem (2.8)-(2.11) has an unique solution. Theorem 3.1 is proved.

This theorem shows that the problem (2.8)-(2.11) has an unique solution if $\alpha > 0$ for any $\omega \in G$. The following result shows that this problem has at least one solution for any $\alpha \ge 0$ and for any $\omega \in H$.

Theorem 3.2. *Let the conditions of Theorem 2.1 satisfy and $\omega \in H-$is the given element. Then the problem (2.8)-(2.11)with $\alpha \ge 0$ has at least one solution.*

Proof. Take any minimizing sequence $\{v^m\} \subset V$:

$$\lim_{m\to\infty} J_\alpha(v^m) = J_{\alpha*} = \inf_{v\in V} J_\alpha(v).$$

Suppose that $\psi_{km} = \psi_{km}(x,t,z) \equiv \psi_k(x,t,z;v^m)$,$k = 1,2, m = 1,2,....$ By the Theorem 2.1, the initial-boundary value problems has an unique solutions for each

$v^m \subset V, m = 1,2,...$ such that $\psi_{1m} \in \overset{0}{W}{}_2^{2,1,1}(\Omega), \psi_{2m} \in W_2^{2,1,1}(\Omega), m = 1,2,...,$ respectively, and the following estimates:

$$\|\psi_{1m}\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)} \leq c_2\left(\|\varphi_{01}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_L)} + \|\varphi_{11}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_T)} + \|f_1\|^2_{W_2^{0,1,1}(\Omega)}\right), m = 1,2,..., \tag{3.9}$$

$$\|\psi_{2m}\|^2_{W_2^{2,1,1}(\Omega)} \leq c_3\left(\|\varphi_{02}\|^2_{W_2^{2,1}(\Omega_L)} + \|\varphi_{12}\|^2_{W_2^{2,1}(\Omega_T)} + \|f_2\|^2_{W_2^{0,1,1}(\Omega)}\right), m = 1,2,..., \tag{3.10}$$

V is a bounded set of a Banach space B, then from the sequence $\{v^m\} \subset V$ can be taken a subsequence $\{v^{m_l}\}$, which we again denote by $\{v^m\}$, the $v_p^m \to v_p$, $\frac{\partial v_p^m}{\partial t} \to \frac{\partial v_p}{\partial t}, \frac{\partial v_p^m}{\partial z} \to \frac{\partial v_p}{\partial z}$, $p = 0,1$, (*) is weak in $L_\infty(\Omega)$ with $k \to \infty$ (3.11).

We will prove that the limit function $v(x,t,z)$ belongs to the set V. Indeed, from the structure of this set is clear that we have the inequalities for the sequence $\{v^k\} \subset V$:

$$\|v_p^m\|_{L_\infty(\Omega)} \leq b_p, \quad \left\|\frac{\partial v_p^m}{\partial t}\right\|_{L_\infty(\Omega)} \leq$$

$$\leq d_p, \quad \left\|\frac{\partial v_p^m}{\partial z}\right\|_{L_\infty(\Omega)} \leq c_p, \quad p = 0,1, \quad m = 1,2,.... \tag{3.12}$$

By the weak lower semi-continuity of the norm of the space $L_\infty(\Omega)$ and the limit relations (3.11) for the limit function $v(x,t,z)$ with the transition to the lower limit in (3.12) we obtain the following inequality:

$$\|v_p\|_{L_\infty(\Omega)} \leq b_p, \quad \left\|\frac{\partial v_p}{\partial t}\right\|_{L_\infty(\Omega)} \leq d_p, \quad \left\|\frac{\partial v_p}{\partial z}\right\|_{L_\infty(\Omega)} \leq c_p, \quad p = 0,1. \tag{3.13}$$

It is clear that

$$|v_p(x,t,z)| \leq b_p, \left|\frac{\partial v_p(x,t,z)}{\partial t}\right| \leq d_p, \left|\frac{\partial v_p(x,t,z)}{\partial z}\right| \leq c_p, p = 0,1, \overset{\circ}{\forall}(x,t,z) \in \Omega.$$

We get that $v \in V$ from these inequalities and the structure of the set V. We can write the following limit relations by the (3.11):

$$\int_\Omega v_p^m(x,t,z)\, q(x,t,z)\, dxdtdz \to \int_\Omega v_p(x,t,z)\, q(x,t,z)\, dxdtdz, \quad p = 0,1, \tag{3.14}$$

if $m \to \infty$ for any function $q \in L_1(\Omega)$.

From estimates (3.9)and (3.10) follows that the sequence $\{\psi_{km}(x,t,z)\}, k = 1, 2$ is respectively uniformly bounded in the norm spaces $\overset{0}{W} 2^{2,1,1}(\Omega), W_2^{2,1,1}(\Omega)$. Then we can extract such subsequences $\{\psi_{km_l}(x,t,z)\}, k = 1, 2$ from these sequences, which we again denote by $\{\psi_{km}(x,t,z)\}, k = 1, 2$ for simplicity, that

$$\psi_{km} \to \psi_k, \quad k = 1, 2 \quad in \quad W_2^{2,1,1} \quad weakly \quad with \quad m \to \infty. \tag{3.15}$$

By the compactness of the embedding spaces $\overset{0}{W} 2^{2,1,1}(\Omega), W_2^{2,1,1}(\Omega)$ â$L_2(\Omega)$ we have:

$$\psi_{km} \to \psi_k, \quad k = 1, 2 \quad strongly \quad L_2(\Omega) \, \text{при} \quad m \to \infty. \tag{3.16}$$

Using the relations (3.14)-(3.16) and passing to the limit with $m \to \infty$ in the identities:

$$\int_\Omega \Big[i\tfrac{\partial\psi_{km}}{\partial t} + ia_0\tfrac{\partial\psi_{km}}{\partial z} - \textstyle\sum_{j,p=1}^{n} \tfrac{\partial}{\partial x_j}\left(a_{jp}(x) \tfrac{\partial\psi_{km}}{\partial x_p}\right) + a(x)\,\psi_{km} +$$
$$+ v_0^m(x,t,z)\,\psi_{km} + iv_1^m(x,t,z)\,\psi_{km} - f_k(x,t,z)\,]\,\bar{\eta}(x,t,z)]\,dxdtdz = 0, k = 1, 2$$

for any function $\eta \in L_2(\Omega)$, we obtain the validity of the fact that the limit functions $\psi_k(x,t,z), k = 1, 2$ of $W_2^{2,1,1}(\Omega)$, satisfy the equations of(2.9) for $\overset{0}{\forall}(x,t,z) \in \Omega$. Of the embedding space $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ in $L_2(S)$ we have:

$\|\psi_{1m} - \psi_1\|_{L_2(S)} \to 0 \; m \to \infty$.

Using this and the condition:

$$\psi_{1m}|_S = 0, m = 1, 2, ...$$

from inequality:

$$\|\psi_1\|_{L_2(S)} \le \|\psi_1 - \psi_{1m}\|_{L_2(S)} + \|\psi_{1m}\|_{L_2(S)}$$

we see that the limit function $\psi_1(x,t,z)$ satisfies the first boundary condition in (2.11) for almost all $(\xi,t,z) \in S$.

By the theorem of traces of functions of the space $W_2^{2,1,1}(\Omega)$ for a subsequence $\{\psi_{2m}(x,t,z)\}$ which converges weakly in the space $\upsilon(x,t,z)$, we have the relation:

$\frac{\partial\psi_{2m}}{\partial N}\Big|_S \to \frac{\partial\psi_2}{\partial N}\Big|_S$weakly in $L_2(S)$ ïğè $m \to \infty$.

Using this and the condition:

$$\left.\frac{\partial\psi_{2m}}{\partial N}\right|_S = 0, m = 1, 2, ...,$$

With the transition to the limit:

$$\int_S \frac{\partial\psi_2}{\partial N}\bar{\eta}ds = \int_S \left(\frac{\partial\psi_2}{\partial N} - \frac{\partial\psi_{2m}}{\partial N}\right)\bar{\eta}ds + \int_S \frac{\partial\psi_{2m}}{\partial N}\bar{\eta}ds$$

for any function $\eta \in L_2(S)$, we obtain the validity of

$$\int_S \frac{\partial \psi_2}{\partial N} \bar{\eta} ds = 0$$

for any function $\eta \in L_2(S)$. This gives that the limit function $\psi_2(x,t,z)$ of $W_2^{2,1,1}(\Omega)$ satisfies the second boundary condition of (2.11) for almost all $(\xi,t,z) \in S$. Now show that the limit functions $\psi_k(x,t,z), k = 1,2$ satisfies to the condition (2.10). It is clear that the elements $\{\psi_{1m}\} \in \overset{0}{W}{}_2^{2,1,1}(\Omega)$, $\{\psi_{2m}\} \in W_2^{2,1,1}(\Omega)$ satisfies the relations:

$$\psi_{1m} \in L_2\left(0,T;\ \overset{0}{W}{}_2^{2,1}(\Omega_L)\right), \quad \frac{\partial \psi_{1m}}{\partial t} \in L_2(0,T;\ L_2(\Omega_L)), \quad m = 1,2,..., \tag{3.17}$$

$$\psi_{2m} \in L_2\left(0,T;\ W_2^{2,1}(\Omega_L)\right), \quad \frac{\partial \psi_{2m}}{\partial t} \in L_2(0,T;\ L_2(\Omega_L)), \quad m = 1,2,... \tag{3.18}$$

and have the limit relations:

$$\psi_{km} \to \psi_k \ \ weakly\ in \quad L_2\left(0,T;\ W_2^{2,1}(\Omega_L)\right), \quad k = 1,2, \tag{3.19}$$

$$\frac{\partial \psi_{km}}{\partial t} \to \frac{\partial \psi_k}{\partial t} \ \ weakly\ in \quad L_2(0,T;\ L_2(\Omega_L)), \quad k = 1,2 \tag{3.20}$$

when $m \to \infty$. From these relations and the embedding theorem we establish that

$$\|\psi_{km}(\cdot,t,\cdot) - \psi_k(\cdot,t,\cdot)\|_{L_2(\Omega_L)} \to 0, \quad k = 1,2 \quad m \to \infty \quad , \quad \forall t \in [0,T]. \tag{3.21}$$

Similarly, it is established that

$$\|\psi_{km}(\cdot,\cdot,z) - \psi_k(\cdot,\cdot,z)\|_{L_2(\Omega_T)} \to 0, \quad k = 1,2\ with \quad m \to \infty, \quad \forall z \in [0,L]. \tag{3.22}$$

With the limit relations (3.21),(3.22) and conditions:

$$\psi_{km}(x,0,z) = \varphi_{0k}(x,z), \quad k = 1,2 \quad (x,z) \in \Omega_L, \quad m = 1,2,... \quad , \tag{3.23}$$

$$\psi_{km}(x,t,0) = \varphi_{1k}(x,t), \quad k = 1,2 \quad (x,t) \in \Omega_T, \quad m = 1,2,..., \tag{3.24}$$

with the transition to the limit by $m \to \infty$ in inequalities:

$$\|\psi_k(\cdot,0,\cdot) - \varphi_{0k}\|_{L_2(\Omega_L)} \leq \|\psi_k(\cdot,0,\cdot) - \psi_{km}(\cdot,0,\cdot)\|_{L_2(\Omega_L)} +$$

$$+ \|\psi_{km}(\cdot,0,\cdot) - \varphi_{0k}\|_{L_2(\Omega_L)}, k = 1,2, \tag{3.25}$$

$$\left\|\psi_k\left(\cdot,\cdot,0\right)-\varphi_{1k}\right\|_{L_2(\Omega_L)}\leq\left\|\psi_k\left(\cdot,\cdot,0\right)-\psi_{km}(\cdot,\cdot,0)\right\|_{L_2(\Omega_L)}+$$

$$+\left\|\psi_{km}\left(\cdot,\cdot,0\right)-\varphi_{1k}\right\|_{L_2(\Omega_L)},k=1,2, \tag{3.26}$$

obtain the validity of:

$$\left\|\psi_k\left(\cdot,0,\cdot\right)-\varphi_{0k}\right\|_{L_2(\Omega_L)}=0,\left\|\psi_k\left(\cdot,\cdot,0\right)-\varphi_{1k}\right\|_{L_2(\Omega_L)}=0,k=1,2, \tag{3.27}$$

It follows that the limit functions $\psi_k\left(x,t,z\right),k=1,2$ respectively satisfies the conditions (2.10) for almost all $(x,z)\in\Omega_L$ and $(x,t)\in\Omega_T$. Thus, we have proved that the limit function $\psi_1=\psi_1\left(x,t,z\right)$ of the space $\overset{0}{W}2^{2,1,1}\left(\Omega\right)$ and the limit function $\psi_2=\psi_2\left(x,t,z\right)$ of the space $W_2^{2,1,1}\left(\Omega\right)$ are the solutions of the corresponding initial-boundary value problems for the limit function $v=v\left(x,t,z\right)\in V$ of a subsequence $\{v^m\}\subset V$, that the $\psi_k=\psi_k\left(x,t,z\right)\equiv\psi_k\left(x,t,z;v\right),k=1,2$. These solutions have the estimates (2.15) and (2.16), which follow from (3.9), (3.10) with the transition to the lower limit of weakly convergent subsequences $\{\psi_{km}\left(x,t,z\right)\},k=1,2$ to the functions $\psi_k\left(x,t,z\right),k=1,2$. We obtain the relation by the limit relations (3.15) and the weak lower semi-continuity of norm spaces $L_2\left(\Omega\right)$ and H, and by $\alpha\geq0$ for $\forall\omega\in H$:

$$J_{\alpha*}\leq J_\alpha\left(v\right)\leq\underline{\lim}_{m\to\infty}J_\alpha\left(v^m\right)=\inf_{v\in V}J_\alpha\left(v\right)=J_{\alpha*}.$$

This means that the limit function $v=v\left(x,t,z\right)$ from of the subsequence $\{v^m\}$ of V gives the minimum to the functional $J_\alpha\left(v\right)$ on the set V, the $v\in V$ is the solution of the identification problem (2.8) - (2.11). Theorem 3.2 is proved.

4 Differentiability of a quality criteria and a necessary condition for the solution of the variation problem

Let $\varphi_k=\varphi_k\left(x,t,z\right),k=1,2$ are solutions of the following problems:

$$i\frac{\partial\varphi_k}{\partial t}+ia_0\frac{\partial\varphi_k}{\partial z}-\sum_{p,j=1}^{n}\frac{\partial}{\partial x_p}\left(a_{jp}\left(x\right)\frac{\partial\varphi_k}{\partial x_j}\right)+a\left(x\right)\varphi_k+v_0\left(x,t,z\right)\varphi_k-iv_1\left(x,t,z\right)\varphi_k=$$

$$=\left(\,-1\right)^k2\left(\psi_1-\psi_2\right),\ \left(x,t,z\right)\in\Omega,k=1,2, \tag{4.1}$$

$$\varphi_k\left(x,T,z\right)=0,\quad(x,z)\in\Omega_L,k=1,2, \tag{4.2}$$

$$\varphi_k\left(x,t,L\right)=0,\quad(x,t)\in\Omega_T,k=1,2, \tag{4.3}$$

$$\left.\varphi_1\right|_S=0,\left.\frac{\partial\varphi_2}{\partial N}\right|_S=0, \tag{4.4}$$

This problem is called the dual problem of the problem (2.8) - (2.11). The system (4.1) - (4.4) is similar to the system (2.8) - (2.11) . Therefore, the solution of the adjoint system means as analogous of the solution of the initial-boundary value problems. This function $\varphi_1 = \varphi_1(x,t,z)$ belongs to the space $\overset{0}{W}2^{2,1,1}(\Omega)$, and the function $\varphi_2 = \varphi_2(x,t,z)$ belongs to the space $W_2^{2,1,1}(\Omega)$ and the conditions (4.1) - (4.4) are satisfied for almost all $(x,t,z) \in \Omega$.

With substitutions $\tau = T - t$, $\theta = L - z$ the adjoint problem (4.1) - (4.4) can be reduced to an initial-boundary value problem, which is a problem of type complex conjugate of the problem (2.9) - (2.11). Therefore, from Theorem 2.1 we have:

Theorem 4.1. *Let we have the conditions of Theorem 2.1. The the adjoint problem (4.1) - (4.4) for each* $v \in V$ *has an unique solution* $\varphi_1 \in \overset{0}{W}2^{2,1,1}(\Omega)$,$\varphi_2 \in W_2^{2,1,1}(\Omega)$ *and for these functions the following estimates:*

$$\|\varphi_1\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)} \le c_7 \left(\|\psi_1 - \psi_2\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{4.5}$$

$$\|\varphi_2\|^2_{W_2^{2,1,1}(\Omega)} \le c_8 \left(\|\psi_1 - \psi_2\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{4.6}$$

where $c_7 > 0, c_8 > 0$*- is the constants, independent of* $\psi_k, k = 1,2$.

Firstly, we will prove the existence of the first variation of the functional $J_\alpha(v)$ *and will find its expression.*

Theorem 4.2. *Let we have the conditions of Theorem 2.1 and* $\omega \in H-$*is the given element. Then for any function* $w = w(x,t,z)$ *of* B *and for any* $v \in V$ *there exists the first variation of the functional* $J_\alpha(v)$ *and we have the following expression:*

$$\delta J_\alpha(v,w) = \int_\Omega Re\left(\psi_1(x,t,z)\bar{\varphi}_1(x,t,z) + \psi_2(x,t,z)\bar{\varphi}_2(x,t,z)\right) w_0(x,t,z)\,dxdtdz-$$

$$-\int_\Omega Im\left(\psi_1(x,t,z)\bar{\varphi}_1(x,t,z) + \psi_2(x,t,z)\bar{\varphi}_2(x,t,z)\right) w_1(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_\Omega \left(v_0(x,t,z) - \omega_0(x,t,z)\right) w_0(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_\Omega \left(v_1(x,t,z) - \omega_1(x,t,z)\right) w_1(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_\Omega \left(\frac{\partial v_0(x,t,z)}{\partial t} - \frac{\partial \omega_0(x,t,z)}{\partial t}\right)\frac{\partial w_0(x,t,z)}{\partial t}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0(x,t,z)}{\partial z}-\frac{\partial \omega_0(x,t,z)}{\partial z}\right)\frac{\partial w_0(x,t,z)}{\partial z}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial t}-\frac{\partial \omega_1(x,t,z)}{\partial t}\right)\frac{\partial w_1(x,t,z)}{\partial t}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial z}-\frac{\partial \omega_1(x,t,z)}{\partial z}\right)\frac{\partial w_1(x,t,z)}{\partial z}dxdtdz \qquad (4.7)$$

for $\forall w\in B$, *where* $\psi_k=\psi_k(x,t,z)\equiv\psi_k(x,t,z;v)$,$\varphi_k=\varphi_k(x,t,z)\equiv\varphi_k(x,t,z;v)$, $k=1,2,\ v\in V,\ w=(w_0,w_1)$.

Proof. Calculating the increment of the functional $J_\alpha(v)$ of any element $v\in V$ and using the conditions (3.2) - (3.4)and(4.1) - (4.4), we establish the validity of the formula:

$$\Delta J_\alpha(v)=J_\alpha(v+\Delta v)-J_\alpha(v)=$$

$$=\int_{\Omega}Re\left(\psi_1(x,t,z)\bar\varphi_1(x,t,z)+\psi_2(x,t,z)\bar\varphi_2(x,t,z)\right)\Delta v_0(x,t,z)\,dxdtdz-$$

$$-\int_{\Omega}Im\left(\psi_1(x,t,z)\bar\varphi_1(x,t,z)+\psi_2(x,t,z)\bar\varphi_2(x,t,z)\right)\Delta v_1(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_0(x,t,z)-\omega_0(x,t,z)\right)\Delta v_0(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_1(x,t,z)-\omega_1(x,t,z)\right)\Delta v_1(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0(x,t,z)}{\partial t}-\frac{\partial \omega_0(x,t,z)}{\partial t}\right)\frac{\partial \Delta v_0(x,t,z)}{\partial t}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0(x,t,z)}{\partial z}-\frac{\partial \omega_0(x,t,z)}{\partial z}\right)\frac{\partial \Delta v_0(x,t,z)}{\partial z}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial t}-\frac{\partial \omega_1(x,t,z)}{\partial t}\right)\frac{\partial \Delta v_1(x,t,z)}{\partial t}dxdtdz+$$

$$+2\alpha\int\limits_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial z}-\frac{\partial \omega_1(x,t,z)}{\partial z}\right)\frac{\partial \Delta v_1(x,t,z)}{\partial z}dxdtdz+\tilde{R}(\Delta v), \qquad (4.8)$$

where the residual term $\tilde{R}(\Delta v)$ calculate by the formula:

$$\tilde{R}(\Delta v)=\int_{\Omega}Re\left(\Delta\psi_1(x,t,z)\,\bar{\varphi}_1(x,t,z)+\Delta\psi_2(x,t,z)\,\bar{\varphi}_2(x,t,z)\right)\Delta v_0(x,t,z)\,dxdtdz-$$

$$-\int_{\Omega}Im\left(\Delta\psi_1(x,t,z)\,\bar{\varphi}_1(x,t,z)+\Delta\psi_2(x,t,z)\,\bar{\varphi}_2(x,t,z)\right)\Delta v_1(x,t,z)\,dxdtdz+\|\Delta\psi_1\|^2_{L_2(\Omega)}+$$

$$+\|\Delta\psi_2\|^2_{L_2(\Omega)}-2\int\limits_{\Omega}Re\left(\Delta\psi_1(x,t,z)\,\Delta\bar{\psi}_2(x,t,z)\right)dxdtdz+\alpha\,\|\Delta v\|^2_H, \qquad (4.9)$$

$\Delta v\in B$ - is increment of any element $v\in V$ such that $v+\Delta v\in V$, $\Delta\psi_k=\Delta\psi_k(x,t,z)\equiv\psi_k(x,t,z;v+\Delta v)-\psi_k(x,t,z;v)\,, k=1,2$. By the estimates (3.6), (4.5), (4.6)from (4.9) we obtain the inequality:

$$\left|\tilde{R}(\Delta v)\right|\le c_9\,\|\Delta v\|^2_B. \qquad (4.10)$$

It means that

$$\tilde{R}(\Delta v)=o\left(\|\Delta v\|_B\right). \qquad (4.11)$$

Using this relationship the formula (4.8) can be represented as:

$$\Delta J_\alpha(v)=\int_{\Omega}Re\left(\psi_1(x,t,z)\,\bar{\varphi}_1(x,t,z)+\psi_2(x,t,z)\,\bar{\varphi}_2(x,t,z)\right)\Delta v_0(x,t,z)\,dxdtdz-$$

$$-\int_{\Omega}Im\left(\psi_1(x,t,z)\,\bar{\varphi}_1(x,t,z)+\psi_2(x,t,z)\,\bar{\varphi}_2(x,t,z)\right)\Delta v_1(x,t,z)\,dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_0\left(x,t,z\right)-\omega_0\left(x,t,z\right)\right)\Delta v_0\left(x,t,z\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_1\left(x,t,z\right)-\omega_1\left(x,t,z\right)\right)\Delta v_1\left(x,t,z\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0\left(x,t,z\right)}{\partial t}-\frac{\partial \omega_0\left(x,t,z\right)}{\partial t}\right)\frac{\partial \Delta v_0\left(x,t,z\right)}{\partial t}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0\left(x,t,z\right)}{\partial z}-\frac{\partial \omega_0\left(x,t,z\right)}{\partial z}\right)\frac{\partial \Delta v_0\left(x,t,z\right)}{\partial z}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1\left(x,t,z\right)}{\partial t}-\frac{\partial \omega_1\left(x,t,z\right)}{\partial t}\right)\frac{\partial \Delta v_1\left(x,t,z\right)}{\partial t}dxdtdz+$$

$$+2\alpha\int\limits_{\Omega}\left(\frac{\partial v_1\left(x,t,z\right)}{\partial z}-\frac{\partial \omega_1\left(x,t,z\right)}{\partial z}\right)\frac{\partial \Delta v_1\left(x,t,z\right)}{\partial z}dxdtdz+o\left(\|\Delta v\|_B\right). \qquad (4.12)$$

We will take $\theta w\in B$ instead of $\Delta v\in B$, where $0<\theta<1$, $w\in B$ - is any element is satisfying to $v+\theta w\in V$. So, we have:

$$\Delta J_\alpha(v)=J_\alpha(v+\theta w)-J_\alpha(v)=$$

$$=\theta\left[\int_{\Omega}Re\left(\psi_1\left(x,t,z\right)\bar{\varphi}_1\left(x,t,z\right)+\psi_2\left(x,t,z\right)\bar{\varphi}_2\left(x,t,z\right)\right)w_0\left(x,t,z\right)dxdtdz-\right.$$

$$-\int_{\Omega}Im\left(\psi_1\left(x,t,z\right)\bar{\varphi}_1\left(x,t,z\right)+\psi_2\left(x,t,z\right)\bar{\varphi}_2\left(x,t,z\right)\right)w_1\left(x,t,z\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_0\left(x,t,z\right)-\omega_0\left(x,t,z\right)\right)w_0\left(x,t,z\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_1\left(x,t,z\right)-\omega_1\left(x,t,z\right)\right)w_1\left(x,t,z\right)dxdtdz+$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_0 (x,t,z)}{\partial t} - \frac{\partial \omega_0 (x,t,z)}{\partial t} \right) \frac{\partial w_0 (x,t,z)}{\partial t} dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_0 (x,t,z)}{\partial z} - \frac{\partial \omega_0 (x,t,z)}{\partial z} \right) \frac{\partial w_0 (x,t,z)}{\partial z} dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_1 (x,t,z)}{\partial t} - \frac{\partial \omega_1 (x,t,z)}{\partial t} \right) \frac{\partial w_1 (x,t,z)}{\partial t} dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_1 (x,t,z)}{\partial z} - \frac{\partial \omega_1 (x,t,z)}{\partial z} \right) \frac{\partial w_1 (x,t,z)}{\partial z} dxdtdz \Bigg] + o(\theta)$$

Using this relation we can compute the first variation of the functional in the form of:

$$\delta J_{\alpha} (v,w) = \lim_{\theta \to +0} \frac{J_{\alpha} (v + \theta \omega) - J_{\alpha} (v)}{\theta} =$$

$$= \int_{\Omega} Re \left(\psi_1 (x,t,z) \bar{\varphi}_1 (x,t,z) + \psi_2 (x,t,z) \bar{\varphi}_2 (x,t,z) \right) w_0 (x,t,z) \, dxdtdz -$$

$$- \int_{\Omega} Im \left(\psi_1 (x,t,z) \bar{\varphi}_1 (x,t,z) + \psi_2 (x,t,z) \bar{\varphi}_2 (x,t,z) \right) w_1 (x,t,z) \, dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(v_0 (x,t,z) - \omega_0 (x,t,z) \right) w_0 (x,t,z) \, dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(v_1 (x,t,z) - \omega_1 (x,t,z) \right) w_1 (x,t,z) \, dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_0 (x,t,z)}{\partial t} - \frac{\partial \omega_0 (x,t,z)}{\partial t} \right) \frac{\partial w_0 (x,t,z)}{\partial t} dxdtdz +$$

$$+2\alpha \int_{\Omega} \left(\frac{\partial v_0 (x,t,z)}{\partial z} - \frac{\partial \omega_0 (x,t,z)}{\partial z} \right) \frac{\partial w_0 (x,t,z)}{\partial z} dxdtdz +$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial t}-\frac{\partial \omega_1(x,t,z)}{\partial t}\right)\frac{\partial w_1(x,t,z)}{\partial t}dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1(x,t,z)}{\partial z}-\frac{\partial \omega_1(x,t,z)}{\partial z}\right)\frac{\partial w_1(x,t,z)}{\partial z}dxdtdz+\lim_{\theta\to+0}\frac{o(\theta)}{\theta},\forall w\in B.$$

We have that $\lim\limits_{\theta\to+0}\frac{o(\theta)}{\theta}=0$, we obtain the statement of the theorem from the last equality. Theorem 4.2 is proved.

Now, we will prove the necessary condition for the solution of the problem (2.8) - (2.11).

Theorem 4.3. *Let we have the conditions of Theorem 2.1. Suppose, that* $V_*\equiv\left\{v^*\in V:\ J(v^*)=J_*=\inf\limits_{v\in V}J(v)\right\}$ *- is the set of solutions of problem (2.8) - (2.11). Then for any element* $v^*\in V_*\subset V$ *necessary the following inequalities:*

$$\int_{\Omega}Re\left(\psi_1^*(x,t,z)\bar{\varphi}_1^*(x,t,z)+\psi_2^*(x,t,z)\bar{\varphi}_2^*(x,t,z)\right)\left(v_0(x,t,z)-v_0^*(x,t,z)\right)dxdtdz-$$

$$-\int_{\Omega}Im\left(\psi_1^*(x,t,z)\bar{\varphi}_1^*(x,t,z)+\psi_2^*(x,t,z)\bar{\varphi}_2^*(x,t,z)\right)\left(v_1(x,t,z)-v_1^*(x,t,z)\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_0^*(x,t,z)-\omega_0(x,t,z)\right)\left(v_0(x,t,z)-v_0^*(x,t,z)\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(v_1^*(x,t,z)-\omega_1(x,t,z)\right)\left(v_1(x,t,z)-v_1^*(x,t,z)\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0^*(x,t,z)}{\partial t}-\frac{\partial \omega_0(x,t,z)}{\partial t}\right)\left(\frac{\partial v_0(x,t,z)}{\partial t}-\frac{\partial v_0^*(x,t,z)}{\partial t}\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_0^*(x,t,z)}{\partial z}-\frac{\partial \omega_0(x,t,z)}{\partial z}\right)\left(\frac{\partial v_0(x,t,z)}{\partial z}-\frac{\partial v_0^*(x,t,z)}{\partial z}\right)dxdtdz+$$

$$+2\alpha\int_{\Omega}\left(\frac{\partial v_1^*(x,t,z)}{\partial t}-\frac{\partial \omega_1(x,t,z)}{\partial t}\right)\left(\frac{\partial v_1(x,t,z)}{\partial t}-\frac{\partial v_1^*(x,t,z)}{\partial t}\right)dxdtdz+$$

$$+2\alpha\int\limits_{\Omega}\left(\frac{\partial v_1^*(x,t,z)}{\partial z}-\frac{\partial\omega_1(x,t,z)}{\partial z}\right)\left(\frac{\partial v_1(x,t,z)}{\partial z}-\frac{\partial v_1^*(x,t,z)}{\partial z}\right)dxdtdz\geq 0,$$

$$\forall v\in V, \tag{4.13}$$

where $\psi_k^*(x,t,z)\equiv\psi_k(x,t,z;v^*)$;$\varphi_k^*(x,t,z)=\varphi_k(x,t,z;v^*)\,,k=1,2.$

Proof. Let $v=v(x,t,z)$ - is an arbitrary element of V, and $v^*=v^*(x,t,z)$ - is any element of the set V_*, i.e. the solution of the problem (2.8) - (2.11). It is clear that it is a convex set from the structure of the set V. For $\forall v^*\in V_*\subset V$,$\forall v\in V$, we have:

$$v^*+\theta(v-v^*)\in V,\quad\forall\theta\in(0,1) \tag{4.15}$$

Therefore, for $v^*\in V_*\subset V$ - is a point on the set of minimum functional $J(v)$ of the set Vwe need that for any inequality (see. [14], 408p):

$$\frac{d}{d\theta}J(v^*+\theta(v-v^*))|_{\theta=0}=\delta J(v^*,\ v-v^*)\geq 0. \tag{4.16}$$

Hence, by the formula (4.7) for $v=v^*$,$w=v-v^*$, we obtain the inequality (4.13). Theorem 4.4 is proved.

5 Conclusion

Quasioptics as a section of nonlinear optics develops intensively in the literature [1,8]. For solving inverse problems of quasioptics often used engineering heuristics like a trial. Grounded theory and computational methods underdeveloped for solving these problems.

It is known that variation methods are one of the universal and effective methods for solving both direct and inverse problems [9,10]. Variation methods are often used for the numerical solution of inverse problems. The theory of variation formulations of inverse problems proposed to develop in [2]. This approach developed and validated for the typical productions of inverse problems of mathematical physics and other types of equations, including the equations of quantum theory. In addition, the results found different applications. The variation formulation of the inverse problem of quasioptics considered in this paper using the advantages of independent significance of variation formulations of inverse problems. This raises a number of traditional questions about the relationship of variation and unvariation problems of the rule of selection the parameter $\alpha\geq 0$ and the element $\omega\in$H, about the recommendations of the numerical algorithm for solving the inverse problems, about attitude of variation formulations of inverse problems with the corresponding optimal control problems, and others. These issues interrelated and in the roots of most of them is incorrect setting this class of inverse problems.

We just note that the numerical parameter $\alpha \geq 0$ play a role of the regularization parameter in some cases. However, the theorems proved in section 2 and the results of [2,3], and others indicate that this parameter plays a deeper role in these tasks.

References

[1] M.A. Vorontsov, V.I. Shmalgauzen, *Principles of adaptive optics*, M .: Science, 1985.

[2] A.D. Iskenderov, *On the variation formulation of multidimensional inverse problems of mathematical physics*, Report of USSR Academy of Sciences, 1984, vol. 274, ³ 3, 531-533 p.

[3] A.D. Iskenderov, N.M. Mahmudov, *Optimal control of quantum-mechanics system with a quality criterion Lions*, Math. Azerb. Ser. Physics-tech.-Mat. Sciences, 1995, vol. XVI, ³5-6, 30-35 p.

[4] G.Ya. Yagubov, N.S. Ibragimov, *An optimal control problem for the non-stationary quasioptics equation*, "Problems of Mathematics and optimal control", Baku, 2001, 49-57 p.

[5] N.S. Ibragimov, *The problem of identification for non-stationary quasioptics equation* , Tauride Journal of Computer Science and Mathematics, 2010, ³ 2, 45-55 p.

[6] N.S. Ibragimov, *A problem of identification by the final observation for the linear non-stationary quasioptics equation*, The Journal of Computational and Applied Mathematics, T. Shevchenko Kiev University, 2010, ³ 4, 26-37 p.

[7] B. Yildiz, O. Kilicoglu, G. Yagubov, *TOptimal control problem for non-stationary Schrodinger equation*, Numerical methods for partial differential equations, 2009, 25, p. 1195-1203.

[8] A.D. Iskenderov, N.S. Ibragimov, *Ibragimov Initial-boundary value problems for the non-stationary quasioptics equatio*, he Vestnik of Lankaran State University, Natural Sciences, 2009, Lankaran, 47-66 p.

[9] M.M. Lavrentev , V.G Romanov , S.P. Shishatskii , *Ill-posed problems of mathematical physics and analysis*, AMS, 1986, 290 p.

[10] V.G. Romanov, *Inverse problems for differential equations*, VNU science press 1987, 239 p.

[11] O.A. Ladyzhenskaya, *Boundary-value problems of mathematical physics*, M.: Science, 1973. - 408 p.

[12] Goebel M., *On existence of optimal control*, Math Nachr.-1979, vol.93.-pp. 67-73.

[13] K. Iosida, *Functional analysis*, M.: World, 1967. -624 p.

[14] M. Minu, *Mathematical Programming*, M.: Science, 1990, 488 p.

A.D. Iskenderov,
Lankaran State University,
Azerbaijan, Email: office@lsu.edu.az,

G.Ya. Yagubov,
Caucasus University,
Turkey, Email: gabilya@mail.ru,

N.S. Ibragimov,
Baku State University,
Azerbaijan Email: natiqibrahimov@mail.ru,

N.Y.Aksoy,
Caucasus University,
TurkeyEmail: nyaksoy55@hotmail.com,

Received 21 July 2014, accepted 05 Sep 2014