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REGULARIZATION OF THE DECISION PROLONGATION PROBLEM FOR PARABOLIC AND ELLIPTIC ELLIPTIC EQUATIONS FROM BORDER PART

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Abstract. In this article continuation problems for parabolic and elliptic equations with data on the part of the boundary are considered. These are a Cauchy problem for heat transfer equation with data on part of boundary and a Cauchy problem for the Laplace equation. The continuation problem is formulated as the operator equation $Aq = f$. To solve the problem one has applied a gradient method for minimization of the objective functional $J(q) =< Aq - f, Aq - f >$.

Having estimated the conditional stability, we studied the convergence rate of Landweber iteration method. The study has shown that Landweber iteration method is a regularization method with an iteration number as the regularization parameter. As the result, the formulae to calculate the singular values of the continuation problem operator have been obtained.

Key words: continuation problem, inverse and ill-posed problem, singular values, regularization

AMS Mathematics Subject Classification: 34M50, 49N45, 65L09

1 Introduction

In this article we offer a unified approach to regularization of continuation problem for two types of equations of mathematical physics: parabolic and elliptic ones. For the first time a similar iteration approach was proposed by V.A. Kozlov, V.G. Maz'ya and A.V. Fomin in 1991 [20]. In our work we provide estimations of conversion rates based on the functional gradient methods and of strong convergence using estimations of conditional stability.

The continuation problems are related to inverse problems of mathematical physics. Its theoretical framework has been set in publications of A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, as well as of their students and followers. In many inverse problems the sought heterogeneities are located at a certain depth beneath a layer of the medium with known parameters (in geophysics these are, as a rule, either homogeneous or layered media). In this case the problem of continuation of geophysical fields from the land surface in the direction of the heterogeneity position becomes one of the important tools is the hands of a practitioner.

2 Estimations of conditional stability

2.1 Parabolic equations

Assume that you are observing a process of heat propagation in a medium (diffusion process). At the same time one is measuring the heat (matter) flux and temperature (matter concentration) in one part of the boundary of the domain under study, while in another part such measurements are either impossible or too difficult to be performed. Hence, one should determine the temperature (or matter concentration) inside the domain up to the inaccessibility boundary. Such problems are common for geophysics [4], nuclear reactor theory, aeronautics [5, 9], heat exchange problems [2, 3] and so on.

Let us consider a two-dimensional mathematical model of this physical process.

A process of heat propagation in the domain $\Omega = \{(x, y) : x \in (0, 1),\ y \in (0, 1)\}$ with the time $t \in (0, T)$, $T \in \mathbb{R}^+$ is described by an initial boundary value problem for parabolic equation:

$$u_t = u_{xx} + u_{yy}, \qquad (x, y) \in \Omega,\ t \in (0, T), \tag{1}$$

$$u(x, y, 0) = 0, \qquad (x, y) \in \Omega, \tag{2}$$

$$u_x(0, y, t) = 0, \qquad y \in (0, 1),\ t \in (0, T), \tag{3}$$

$$u(1, y, t) = q(y, t), \qquad y \in (0, 1),\quad t \in (0, T), \tag{4}$$

$$u(x, 0, t) = u(x, 1, t) = 0, \qquad x \in (0, 1),\ t \in (0, T). \tag{5}$$

Here the function $u(x, y, t)$ describes the medium temperature at the point $(x, y) \in \Omega$ at the moment of time $t \in [0, T]$.

The problem of determining the function $u(x, y, t)$ from the relations (1)–(5) is direct problem.

The inverse problem can be formulated as finding $q(y, t)$ in $\Omega \times (0, T)$ based on the temperature measurements $f(y, t)$ on the part of the boundary Ω

$$u(0, y, t) = f(y, t), \qquad y \in (0, 1), \quad t \in (0, T), \tag{6}$$

Let us introduce the operator

$$Aq : q(y, t) = u(1, y, t) \mapsto f(y, t) = u(0, y, t).$$

Here $u(x, y, t)$ is the solution of the direct problem (1)–(5). The inverse problem (1)–(6) to determine function $q(y, t)$ can be written as the operator equation $Aq = f$.

Definition 1. *Let the function $u \in L_2(\Omega \times (0, T))$ be called a generalized solution of the direct problem (1)–(5) if for any $w \in H^{2,1}(\Omega \times (0, T))$ that meet the conditions*

$$w_x(0, y, t) = 0, \qquad y \in (0, 1) \times (0, T),$$

$$w(1, y, t) = 0, \qquad y \in (0, 1) \times (0, T),$$

$$w(x, 0, t) = w(x, 1, t) = 0, \qquad x \in (0, 1) \times (0, T),$$

$$w(x, y, T) = 0, \qquad (x, y) \in \Omega,$$

the following equality is executed:

$$\int_0^T \int_\Omega u(w_t + \Delta w) d\Omega dt - \int_0^T \int_0^1 q(y,t) w_x(1,y,t) dy dt = 0.$$

In [16] it has been proved that the operator A acts from $L_2((0,1) \times (0,T))$ to $L_2((0,1) \times (0,T))$ and is limited.

Theorem 2.1. *(about well-posedness of the direct problem and existence of the trace [16]) If $q \in L_2((0,1) \times (0,T))$ the problem (1)–(5) has the only solution and the estimations*

$$\|u\|^2_{L_2(\Omega\times(0,T))} \leq M_1 \|q\|^2_{L_2((0,1)\times(0,T))},$$
$$\|u(0,y,t)\|^2_{L_2((0,1)\times(0,T))} \leq M_2 \|q\|^2_{L_2((0,1)\times(0,T))}$$

are correct.

Here and elsewhere M_j, $j = 1, 2, \ldots$ stands for positive constants.

We introduce a adjoint problem:

$$\begin{aligned}
&\psi_t + \psi_{xx} + \psi_{yy} = 0, && (x,y) \in \Omega,\ t \in (0,T), && (7)\\
&\psi(1,y,t) = 0, && y \in (0,1),\ t \in (0,T), && (8)\\
&\psi_x(0,y,t) = \mu(y,t), && y \in (0,1),\ t \in (0,T), && (9)\\
&\psi(x,0,t) = \psi(x,1,t) = 0, && x \in (0,1),\ t \in (0,T), && (10)\\
&\psi(x,y,T) = 0, && (x,y) \in \Omega, && (11)
\end{aligned}$$

where $\mu(y,t) \in L_2((0,1) \times (0,T))$ is given function.

So, analogous to the direct problem definition one can determine the solution of the adjoint problem and prove the theorem about well-posedness of the adjoint problem and existence of the derivative trace [16].

The operator equation $Aq = f$ in this case should be solved through minimization of the objective functional $J(q) =< Aq - f, Aq - f >$ using Landweber iteration method:

$$q_{n+1} = q_n - \alpha J' q_n, \qquad n = 0, 1, 2, \ldots,$$

where the descent parameter $\alpha \in (0, \|A\|^{-2})$ and $J' q_n$ turns out to be the gradient of functional $J(q_n)$

$$J' q = 2A^*(Aq - f).$$

For numerical realization of Landweber iteration method we use the expression of the gradient of functional [16]:

$$J'(q)(y,t) = \psi(0,y,t),$$

where $\psi(x,y,t)$ is the solution of the adjoint problem (7)–(11) in which

$$\mu(y,t) = 2\big[u(0,y,t) - f(y,t)\big].$$

Theorem 2.2. *(about functional conversion rate of the Landweber iteration method [16, 14]) Let it be the exact solution $q_T \in L_2((0,1)\times(0,T))$ of the problem $Aq = f$ for $f \in L_2((0,1)\times(0,T))$. Then the sequence $\{q_n\}$ has a functional convergence, so the estimation*

$$J(q_n) \le \frac{M_3\|q_0 - q_T\|^2}{n}, \quad n = 1, 2, \dots$$

holds true.

2.2 Elliptic equations

Let us consider the following continuation problem for an elliptic equation:

$$u_{xx} + L(y)u = 0, \qquad (x,y) \in \Omega, \tag{12}$$

$$u(0,y) = f(y), \qquad y \in \mathcal{D}, \tag{13}$$

$$u_x(0,y) = 0, \qquad y \in \mathcal{D}, \tag{14}$$

$$u(x,y) = 0, \qquad x \in (0,h), \quad y \in \partial\mathcal{D} \tag{15}$$

with the matching conditions

$$f(y) = 0, \qquad y \in \partial\mathcal{D}. \tag{16}$$

Here $\Omega = (0,h)\times\mathcal{D}$, $\mathcal{D} \in \mathbb{R}^n$ is the bounded domain with a Lipschitz boundary $\partial\mathcal{D}$,

$$L(y)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u}{\partial y_j}\right) - c(y)u,$$

$$M_4 \sum_{j=1}^{n} \nu_j^2 \le \sum_{i,j=1}^{n} a_{ij}(y)\nu_i\nu_j,$$

$$\forall \nu_i \in \mathbb{R}, \quad a_{ij} = a_{ji}, \quad i,j = 1,\dots,n,$$

$$0 \le c(y) \le M_5,$$

$$a_{ij} \in C^1(\overline{\mathcal{D}}), \qquad c \in C(\overline{\mathcal{D}}).$$

Let us consider the ill-posed continuation problem (12)–(16) as the inverse problem of the following direct problem:

$$u_{xx} + L(y)u = 0, \qquad (x,y) \in \Omega, \tag{17}$$

$$u_x(0,y) = 0, \qquad y \in \mathcal{D}, \tag{18}$$

$$u(h,y) = q(y), \qquad y \in \mathcal{D}, \tag{19}$$

$$u(x,y) = 0, \qquad x \in (0,h), \quad y \in \partial\mathcal{D} \tag{20}$$

with the matching conditions:

$$q(y) = 0, \qquad y \in \partial\mathcal{D}. \tag{21}$$

In the direct problem (17)–(21) one has to find $u(x,y)$ in the domain Ω for the function $q(y)$ set for a part of the boundary $x = h$ of the domain Ω.

The inverse problem is to determine $q(y)$ from conditions of (17)–(21) and known additional information

$$u(0,y) = f(y), \qquad y \in \partial\mathcal{D}. \tag{22}$$

To familiarize yourself with some the results based on the theory of direct and inverse problems address to [14, 23].

Definition 2. *[14] The function $u \in L_2(\Omega)$ is called a generalized solution of the direct problem (17)–(21) if for any $w \in H^2(\Omega)$ which satisfy to*

$$\begin{aligned} &w_x(0,y) = 0, \qquad y \in \mathcal{D}, \\ &w(h,y) = 0, \qquad y \in \mathcal{D}, \\ &w(x,y) = 0, \qquad x \in (0,h), \quad y \in \partial\mathcal{D} \end{aligned}$$

the following equality is executed:

$$\int\limits_{\Omega} u(w_{xx} + L(y)w)dxdy - \int\limits_{\mathcal{D}} q(y)w_x(h,y)\,\mathrm{d}y = 0.$$

Theorem 2.3. *(about well-posedness of the direct problem [14]) If $q \in L_2(\mathcal{D})$ then the direct problem (17)–(21) has unique generalized solution $u \in L_2(\Omega)$ and the estimations are true:*

$$\|u\|^2_{L_2(\Omega)} \le M_6\|q\|^2_{L_2(\mathcal{D})};$$
$$\|u(0,y)\|^2_{L_2(\mathcal{D})} \le M_7\|q\|^2_{L_2(\mathcal{D})}.$$

Theorem 2.4. *(estimation of the conditional stability, [14]). Let $u(h,y)$, $f \in L_2(\mathcal{D})$. If the continuation problem (12)–(16) has the solution $u \in C^2(\Omega)$ then following estimation is true [14, 21]*

$$\int\limits_{\mathcal{D}} u^2(x,y)\,\mathrm{d}y \le M_8\|q\|^{2x/h}_{L_2(\mathcal{D})}\|f\|^{2(h-x)/h}_{L_2(\mathcal{D})}, \quad x \in (0,h).$$

Now, we consider a gradient method to solve the continuation problem for elliptic equation. For that purpose consider the adjoint problem:

$$\psi_{xx} + L(y)\psi = 0, \qquad (x,y) \in \Omega, \tag{23}$$
$$\psi_x(0,y) = \mu(y), \qquad y \in \mathcal{D}, \tag{24}$$
$$\psi(h,y) = 0, \qquad y \in \mathcal{D}, \tag{25}$$
$$\psi|_{\partial\mathcal{D}} = 0, \qquad x \in (0,h). \tag{26}$$

The problem consists in finding the function $\psi(x,y)$ using given $\mu(y)$.

Theorem 2.5. *(about well-posedness of the adjoint problem [14]) If $\mu \in L_2(\mathcal{D})$ then the problem (23)–(26) has unique generalized solution $\psi \in L_2(\Omega)$ and the estimations are true:*

$$\|\psi\|^2_{L_2(\Omega)} \le M_9\|\mu\|^2_{L_2(\mathcal{D})};$$
$$\|\psi_x(h,y)\|^2_{L_2(\mathcal{D})} \le M_{10}\|\mu\|^2_{L_2(\mathcal{D})}.$$

We introduce an operator

$$A : q(y) \to u(0, y),$$

where $u(x, y)$ is the solution of the direct problem (17)–(21).

Therefore, the adjoint operator A^* is expressed as

$$A^* : \mu(y) \to \psi_x(h, y),$$

where $\psi(x, y)$ is a solution of the adjoint problem (23)–(26).

From theorem of 2.3 and 2.5 it follows that operators A and A^* map $L_2(\mathcal{D})$ to $L_2(\mathcal{D})$. Therefore, the inverse problem (17)–(22) can be written in the operator form

$$Aq = f. \tag{27}$$

To find the solution (27) we apply Landweber iteration method.

It should be noted that the gradient of functional $J'q$ is calculated based on the formula:

$$\big(J'q\big)(y) = \psi_x(h, y),$$

where $\psi(x, y)$ is a solution of the adjoint problem (23)–(26), in which

$$\mu(y) = 2\big[u(0, y) - f(y)\big].$$

Theorem 2.6. *(estimation of functional convergence rate). Let us the problem $Aq = f$ has the exact solution $q_T \in L_2(\mathcal{D})$. Then the following estimation holds true*

$$J(q_n) \le \frac{M_{11}}{n}, \quad n = 1, 2, \dots$$

Theorem 2.7. *Let the problem $Aq = f$ have the exact solution $q_T \in L_2(\mathcal{D})$. Then a sequence of solutions $\{u_n\}$ of the direct problems (17)–(21) for the correspondent iteration q_n converge to the exact solution $u_T \in L_2(\Omega)$ of the problem (12)–(16) and the following estimation is true [16]:*

$$\int\limits_{\mathcal{D}} (u_n(x, y) - u_T(x, y))^2 dy \le M_{12} n^{\frac{x-h}{h}}, \quad x \in (0, h). \tag{28}$$

The estimation (28) leads to uniqueness of the solution and conditional stability of the continuation problem (12)–(16).

Theorem 2.8. *Let the problem $Aq = f$ have the exact solution $q_T \in L_2(\mathcal{D})$. Let $\|f - f^\delta\| \le \delta$ and $\{q^n_\delta\}$ be an Landweber iteration sequence to solve the inverse problem (17)–(22) with the additional information $u^n_\delta(0, y) = f_\delta(y)$ Then, to solve the corresponding direct problem (17)–(21) the following estimation should be carried out [16]:*

$$\int\limits_{\mathcal{D}} (u^\delta_n(x, y) - u_T(x, y))^2 \, \mathrm{d}y \le M_{13}\Big(\beta(n)\delta + n^{\frac{x-h}{h}}\Big), \quad x \in (0, h). \tag{29}$$

Here $u_T \in L_2(\Omega)$ is the exact solution and

$$\beta(n) = \frac{(1 + 2\alpha\|A\|^2)^{n-1} - 1}{\|A\|}.$$

Analogous results have been obtained for steepest descent and conjugate gradients methods [16, 14].

The estimation (29) shows that the sequence $\{u_n^\delta\}$ is regularizing one where n is the regularization parameter. Actually, due to the fact that the fist member is going monotonously to infinity, while the second in the same way to zero, at $n \to \infty$, the stopping criterion for the corresponding number of iterations n_* can be selected based on the following rule. Having differentiated the right part (29) with respect to n, one finds the root n_r of the following equation:

$$\delta \frac{\ln(1+2\alpha\|A\|^2)}{\|A\|}(1+2\alpha\|A\|^2)^{n-1} + \frac{x-h}{h} n^{\frac{x-2h}{h}} = 0 \tag{30}$$

and can select the stopping number n_s to be a natural number closest to the equation root (30).

3 Analysis of singular values

When studying acoustic or electrodynamic problems, in many cases one shifts to harmonic motions and the Helmholtz equation. In this section we are going to analyze the singular values of a continuation problem operator for a complex-valued formulation of the Helmholtz equation in a case of simple geometry.

A Cauchy problem for the Helmholtz equation is a well-known example of an ill-posed problem. Its solution isn't stable relative to the small variations of the Cauchy data [13, 1, 14].

In [13] author has shown that estimation of conditional stability with respect to k turns out to the best logarithmic estimation.

In [11, 12] it has been demonstrated that the ill-posedness of the Cauchy problem for the Helmholtz equation depends on the wave number k and increases with its growth. The numerical calculations using different methods have been presented in the following publications, e.g.: a quasi-reversibility method [18], frequency space cut-off [27], iteration methods [20, 15, 24, 30], regularization methods [8, 25, 26, 28].

Let us consider a continuation problem for the Helmholtz equation in a homogeneous medium for simple geometry:

$$\Delta u + k^2 u = 0, \qquad x \in (0,h), \quad y \in (0,\pi), \tag{31}$$

$$u(0,y) = f(y), \qquad y \in (0,\pi), \tag{32}$$

$$u_x(0,y) = 0, \qquad y \in (0,\pi), \tag{33}$$

$$u(x,0) = u(x,\pi) = 0, \qquad x \in (0,h). \tag{34}$$

Here

$$k^2 = \varepsilon\omega^2 - i\sigma\omega,$$

ω is a frequency, ε and σ are positive constants.

The continuation problem (31)—(34)) includes determination of the function $u(x,y)$ in the domain $x \in (0,h)$, $y \in (0,\pi)$ based on the given boundary conditions (32)–(34).

Now we formulate the continuation problem as an inverse with respect to the direct problem:

$$\Delta u + k^2 u = 0, \qquad x \in (0,h), \quad y \in (0,\pi), \tag{35}$$

$$u_x(0,y) = 0, \quad u(h,y) = q(y), \qquad y \in (0,\pi), \tag{36}$$

$$u(x,0) = u(x,\pi) = 0, \qquad x \in (0,h). \tag{37}$$

The inverse problem in this case is finding the function $q(y)$ based on the additional information:

$$u(0,y) = f(y), \qquad y \in (0,\pi). \tag{38}$$

To find the solution of the direct problem (35)–(37) we assume that $q(y)$ is expansible:

$$q(y) = \sum_{m=1}^{\infty} q^{(m)} \sin(my)$$

The solution will be found, expressed as a Fourier series:

$$u(x,y) = \sum_{m=1}^{\infty} u^{(m)}(x) \sin(my).$$

Solving a sequence of corresponding direct problems

$$u^{(m)}_{xx} + k_m^2 u^{(m)} = 0, \qquad x \in (0,h), \tag{39}$$

$$u^{(m)}_x(0) = 0, \quad u^{(m)}(h) = q^{(m)}. \tag{40}$$

Here

$$k_m^2 = \varepsilon\omega^2 - m^2 - i\sigma\omega.$$

We obtain the general solution of the equation (39), expressed as:

$$u^{(m)}(x) = C_1 e^{\lambda_m x} + C_2 e^{-\lambda_m x}.$$

Here $\sqrt{-k_m^2} = \pm\lambda_m$, $\lambda_m = \alpha_m + i\beta_m$ and

$$\alpha_m = \sqrt{\frac{\sqrt{(m^2 - \varepsilon\omega^2)^2 + \sigma^2\omega^2} + m^2 - \varepsilon\omega^2}{2}},$$

$$\beta_m = \sqrt{\frac{\sqrt{(m^2 - \varepsilon\omega^2)^2 + \sigma^2\omega^2} - m^2 + \varepsilon\omega^2}{2}}.$$

Thus, the solution of the problem (39), (40) is expressed by the formula

$$u^{(m)}(x) = \frac{\cosh(\lambda_m x)}{\cosh(\lambda_m h)} q^{(m)}.$$

In this case, the solution of the initial direct problem (35)—(37) is presented as a Fourier series:

$$u(x,y) = \sum_{m=1}^{\infty} \frac{\cosh(\lambda_m x)}{\cosh(\lambda_m h)} q^{(m)} \sin(my).$$

as well as the solution of the inverse problem (35)—(38):

$$q(y) = \sum_{m=1}^{\infty} f^{(m)} \cosh(\lambda_m h) \sin(my). \tag{41}$$

Since the operator A is diagonal, the singular values are expressed as:

$$\sigma_m(A) = \frac{1}{|\cosh(\lambda_m h)|} = \frac{\sqrt{2}}{\sqrt{\cosh(2\alpha_m h) + \cos(2\beta_m h)}}. \tag{42}$$

Now, we consider a number of particular cases of singular values of operator A.

The singular values of electromagnetic field continuation problem operator ($\varepsilon \neq 0$ and $\sigma \neq 0$) are expressed as:

$$\sigma_m(A) = \frac{1}{|\cosh(\lambda_m h)|} = \frac{\sqrt{2}}{\sqrt{\cosh(2\alpha_m h) + \cos(2\beta_m h)}}.$$

In case of the acoustic equation ($\varepsilon \neq 0$ and $\sigma = 0$) the singular values formula is expressed as [12]:

$$\sigma_m(A) = \begin{cases} \frac{1}{|\cos(\sqrt{k_m} h)|}, & m^2 \leq \varepsilon\omega^2, \\ \frac{1}{\cosh(\sqrt{k_m} h)}, & \varepsilon\omega^2 < m^2. \end{cases}$$

We note that the singular values depend on the wave number $k_m^2 = \varepsilon\omega^2 - m^2$ and the ration of m, ε and ω [12]. In the case of $m^2 \leq \varepsilon\omega^2$ the singular values of the operator A are limited to 1 from below. At the same time in the case of $m^2 > \varepsilon\omega^2$ the singular values decay to zero exponentially.

Now, let's consider the parabolic equation ($\varepsilon = 0$ and $\sigma \neq 0$). In this case

$$\sigma_m(A) = \frac{\sqrt{2}}{\sqrt{\cosh(2\alpha_m h) + \cos(2\beta_m h)}},$$

$$\alpha_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} + m^2}{2}}, \quad \beta_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} - m^2}{2}}.$$

In case of the Laplace equation ($\varepsilon = 0$ and $\sigma = 0$) the singular values decay exponentially:

$$\sigma_m(A) = \frac{1}{\cosh(mh)}.$$

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