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RECONSTRUCTION OF 2D SYMMETRIC 2-TENSOR FIELD BY MOMENTUM RAY TRANSFORMS

Svetov I.E. [1], **Polyakova A.P.**, **Maltseva S.V.**

Abstract Algorithms for reconstructing a two-dimensional symmetric 2-tensor field from the known longitudinal or transverse ray transforms of its moments are proposed and justified. The algorithms are based on the method of singular value decomposition. We use the known singular value decompositions of the Radon transform of functions and the ray transforms of 2-tensor fields. Numerical simulations demonstrate reliable results of reconstructing symmetric 2-tensor fields from values of the momentum ray transforms.

Key words: momentum ray transform, Radon transform, symmetric 2-tensor field, tensor field decomposition, reconstruction, singular value decomposition method.

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Introduction

In recent years, quite a lot of articles are devoted to the momentum ray transforms of tensor fields (see, for example, [1]–[4]). Traditional questions of reconstruction of m-tensor fields were studied in these articles. In particular, stability estimates were received, images of momentum ray transforms were described, uniqueness theorems were proven. Note that the longitudinal ray transforms of moments were used as data in almost all works. There are significantly less articles in which formulas and inversion procedures, constructive methods and algorithms of solving reconstruction problems of tensor fields by momentum ray transforms were suggested. The articles [1],[5] contain algorithms for reconstruction of a symmetric m-tensor filed in $\mathbb{R}^n$ by its known longitudinal ray transforms with the weight t^k, $k = 0, \dots, m$. Relatively simple reconstruction algorithms were earlier suggested in a special case at $n = 2$: recovery of vector fields by the longitudinal ray transforms of moments [3]; reconstructing vector and symmetric 2-tensor fields by the longitudinal and transverse ray transforms of moments [6]. We highlight the article [7], in which numerical simulations on reconstruction of a vector field by the momentum ray transforms were included.

Development of numerical methods and algorithms for reconstruction of a two-dimensional symmetric 2-tensor field by the momentum ray transforms is goal of the current article. Furthermore, we carry out comparative research of three algorithms using the computational experiment method.

The singular value decomposition (SV-decomposition) method is frequently used for inverting operators. In the SV-decomposition method an operator A is represented as

$$Af = \sum_{k=1}^{\infty} \sigma_k \langle f, f_k \rangle_H g_k,$$

where (f_k), (g_k) are orthonormal systems in the Hilbert spaces H and K, respectively, and the numbers $\sigma_k > 0$ are called singular values of the operator A (see, for example, [8]). If the

[1] Corresponding Author.

operator A has a singular value decomposition, its (pseudo)inverse operator is given by

$$A^\dagger g = \sum_{k=1}^{\infty} \sigma_k^{-1} \langle g, g_k \rangle_K f_k.$$

The operator $A^\dagger$ is unbounded if and only if $\sigma_{k_j} \to 0$ for some sequence $k_j \to \infty$. Then the operator $A^\dagger$ can be regularized using the truncated singular value decomposition

$$A_N^\dagger g = \sum_{k \leqslant N} \sigma_k^{-1} \langle g, g_k \rangle_K f_k.$$

Construction of SV-decompositions is one of classical questions in the frameworks of investigation of tomographic operators properties. We mention some articles. The SV-decompositions were constructed for the Radon transform operator [9]-[12] and for the ray transform operator [13] of functions. Later the SV-decompositions were constructed for the ray transforms operators (longitudinal, transverse and mixed) of two-dimensional vector [14], symmetric 2-tensor [15] and symmetric m-tensor fields [16]. In [17] an SV-decomposition for the operator of longitudinal ray transform of fan type acting on two-dimensional solenoidal tensor fields of arbitrary degree m was constructed. In [14],[18] algorithms based on the truncated SV-decomposition method were developed and numerically implemented for an approximate recovery of two-dimensional vector and symmetric 2-tensor fields.

The present article is organized as follows. We remind definitions of well known spaces and operators in Section 1 below. In Section 2, we establish connections between the operators of momentum ray transforms of 2-tensor fields and obtain some properties of these operators as a consequence. Further, in Section 3, we formulate the problem statements. Section 4 is devoted to justification and description of algorithms for solving formulated problems. In Sections 5 and 6, we give details of numerical realization of the algorithms and describe simulation results.

1 Mathematical basis

Let $x = (x_1, x_2)$, $B = \{x \in \mathbb{R}^2 \mid |x| = \sqrt{x_1^2 + x_2^2} < 1\}$ be a unit disk with the boundary $\partial B = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ and $Z = \{(s, \xi) \mid \xi \in \mathbb{R}^2, |\xi| = 1, s \in \mathbb{R}\}$ be a cylinder. The functional space $L_2(B)$ consists of functions, which are square integrable in B. A set of symmetric m-tensor fields $\mathbf{v}(x) = (v_{i_1 \dots i_m}(x))$, where $i_1, \dots, i_m = 1, 2$, defined in B is denoted by $S^m(B)$. In this article, we deal only with $m = 0$ (functions), $m = 1$ (vector fields) and $m = 2$ (symmetric 2-tensor fields). We need the space of square integrable symmetric m-tensor fields $L_2(S^m(B))$. The Sobolev spaces are denoted by $H^k(S^m(B))$ and $H_0^k(S^m(B))$. We also use functions in the weight space $L_2(Z, \rho)$, $\rho > 0$.

Let us fix restrictions and conventions that are accepted in the tensor tomography model used in this work. Sources of a physical field (the radiation) and their receivers are concentrated on the unit circle ∂B. Support of a symmetric m-tensor field $\mathbf{w}$ is separated from the observation system, $\operatorname{supp} \mathbf{w} \subset B$. Outside the support, including on the set $\mathbb{R}^2 \setminus B$, the field $\mathbf{w}$ vanishes. The listed conditions seem natural, when organizing a data collection within the framework of transmission tomography, and limitations arising from physical considerations and the nature of the objects being studied.

The operators of *inner derivation* d and *inner $\perp$-derivation* $\mathrm{d}^\perp$ are compositions of the operators of covariant derivation ∇, covariant $\perp$-derivation $\nabla^\perp$ and symmetrization σ, respectively,

$$\mathrm{d}, \mathrm{d}^\perp : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B))$$

and act on a function f and a vector field $\mathbf{v}$ according to the formulas

$$(\mathrm{d}f)_i = (\nabla f)_i = \frac{\partial f}{\partial x_i}, \qquad (\mathrm{d}^\perp f)_i = (\nabla^\perp f)_i = (-1)^i \frac{\partial f}{\partial x_{3-i}},$$
$$(\mathrm{d}\mathbf{v})_{ij} = (\sigma\nabla\mathbf{v})_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (\mathrm{d}^\perp\mathbf{v})_{ij} = (\sigma\nabla^\perp\mathbf{v})_{ij} = \frac{1}{2}\left((-1)^j\frac{\partial v_i}{\partial x_{3-j}} + (-1)^i\frac{\partial v_j}{\partial x_{3-i}}\right).$$

By direct verification it is established that

$$\nabla^2\varphi = \mathrm{d}^2\varphi, \quad (\nabla^\perp)^2\varphi = (\mathrm{d}^\perp)^2\varphi, \quad \mathrm{d}\mathrm{d}^\perp\varphi = \mathrm{d}^\perp\mathrm{d}\varphi, \quad \varphi \in H^2(B).$$

However, in the general case we have

$$\nabla\nabla^\perp\varphi \neq \nabla^\perp\nabla\varphi, \quad \nabla\nabla^\perp\varphi \neq \mathrm{d}\mathrm{d}^\perp\varphi, \quad \nabla^\perp\nabla\varphi \neq \mathrm{d}\mathrm{d}^\perp\varphi, \quad \varphi \in H^2(B). \tag{1}$$

The *divergence* operator $\delta : H^k(S^m(B)) \to H^{k-1}(S^{m-1}(B))$ acts on a vector field $\mathbf{v}$ and on a symmetric 2-tensor field $\mathbf{w}$ according to the formulas

$$\delta\,\mathbf{v} = \sum_{i=1}^{2}\frac{\partial v_i}{\partial x_i}, \quad (\delta\,\mathbf{w})_j = \sum_{i=1}^{2}\frac{\partial w_{ji}}{\partial x_i}.$$

Recall that a symmetric m-tensor field $\mathbf{w} \in H^k(S^m(B))$ is called *potential* if there is a tensor field $\mathbf{v} \in H^{k+1}(S^{m-1}(B))$ such that $\mathbf{w} = \mathrm{d}\mathbf{v}$. A tensor field $\mathbf{w} \in H^k(S^m(B))$ is called *solenoidal* if $\delta\,\mathbf{w} = 0 \in H^{k-1}(S^{m-1}(B))$. In particular, a symmetric 2-tensor field is solenoidal if and only if there exists a function ψ such that $\mathbf{w} = (\mathrm{d}^\perp)^2\psi$ (see, for example, [19]).

It is well known [20],[21] that there is a unique decomposition of an arbitrary symmetric m-tensor field $\mathbf{w}$ to the following sum of potential and solenoidal parts

$$\mathbf{w} = {}^s\mathbf{w} + \mathrm{d}\mathbf{v}, \quad \delta\,{}^s\mathbf{w} = 0, \quad \mathbf{v} \in H^1_0(S^{m-1}(B)).$$

There exists a more detailed decomposition [19],[22] of a symmetric 2-tensor field $\mathbf{w}$ to a sum of the three terms

$$\mathbf{w} = \mathrm{d}^2\varphi + \mathrm{d}\mathrm{d}^\perp\chi + (\mathrm{d}^\perp)^2\psi, \tag{2}$$

where

$$\varphi, \chi, \psi \in H^2(B), \quad \varphi\big|_{\partial B} = 0, \quad (\mathrm{d}\varphi + \mathrm{d}^\perp\chi)\big|_{\partial B} = 0.$$

Unit vectors $\xi = (\cos\theta, \sin\theta)$, $\eta = \xi^\perp = (-\sin\theta, \cos\theta)$, $\theta \in [0, 2\pi)$ and a number $s \in \mathbb{R}$ define a line $L_{\xi,s} = \{x \in \mathbb{R}^2 : x = s\xi + t\eta,\ t \in \mathbb{R}\}$.

The *Radon transform* $\mathcal{R} : H^k(B) \to H^k(Z, \rho)$ integrates a function $f \in H^k(B)$ along the lines $L_{\xi,s}$ for all s, ξ:

$$(\mathcal{R}f)(s, \xi) = (\mathcal{R}f)(s, \theta) = \int_{-\infty}^{\infty} f(s\xi + t\eta)dt.$$

Here $\rho(s) = (1 - s^2)^{-1/2}$, $|s| < 1$ is the weight function we use for realization of numerical algorithms using the SV-decomposition method.

In this work, we consider one of the variants of integral operators generated by the Radon transform. Namely, *momentum ray transforms* of tensor fields

$$(\mathcal{P}^{(j)}_{km}\mathbf{w})(s, \theta) = \int_{-\infty}^{\infty} t^k \sum_{i_1,..,i_m=1}^{2} w_{i_1\ldots i_m}(s\xi + t\eta)\xi^{i_1}\ldots\xi^{i_j}\eta^{i_{j+1}}\ldots\eta^{i_m}dt. \tag{3}$$

Here the index m determines the degree of the tensor field $\mathbf{w}$, $k \geqslant 0$ is the order of the moments, the index (j), $0 \leqslant j \leqslant m$ is responsible for components number of the normal vector ξ. If $j = m = 0$ we obtain the Radon transform $\mathcal{P}^{(0)}_{k0}$ with the weight t^k, which for $k = 0$ coincides with the Radon transform: $\mathcal{P}^{(0)}_{00} f = \mathcal{R} f$. Recall that for $j = 0$ the ray transforms are called *longitudinal*, for $j = m$, $m \geqslant 1$ are *transverse* (or *transversal*), and for $0 < j < m$, $m \geqslant 2$ are *mixed*.

2 Momentum ray transforms of symmetric 2-tensor fields

We are interested in the case $m = 2$ and connections with the case $m = 0$. From the operators definition (3) follows connections between the momentum ray transforms of symmetric 2-tensor fields and the momentum Radon transform of functions

$$\mathcal{P}^{(0)}_{k2}\mathbf{w} = \sum_{i,j=1}^{2} \eta^i\eta^j(\mathcal{P}^{(0)}_{k0} w_{ij}), \quad \mathcal{P}^{(1)}_{k2}\mathbf{w} = \sum_{i,j=1}^{2} \eta^i\xi^j(\mathcal{P}^{(0)}_{k0} w_{ij}), \quad \mathcal{P}^{(2)}_{k2}\mathbf{w} = \sum_{i,j=1}^{2} \xi^i\xi^j(\mathcal{P}^{(0)}_{k0} w_{ij}).$$

For any k, solving this system with respect to $\mathcal{P}^{(0)}_{k0} w_{ij}$, $i, j = 1, 2$, we obtain the equalities

$$\mathcal{P}^{(0)}_{k0} w_{ij} = \eta^i\eta^j(\mathcal{P}^{(0)}_{k2}\mathbf{w}) + (\eta^j\xi^i + \eta^i\xi^j)(\mathcal{P}^{(1)}_{k2}\mathbf{w}) + \xi^i\xi^j(\mathcal{P}^{(2)}_{k2}\mathbf{w}), \quad i, j = 1, 2. \tag{4}$$

Let us establish the connection between the longitudinal, mixed and transverse ray transforms of the moment k.

Theorem 2.1. *For an arbitrary $k \geqslant 0$ and a function $\varphi \in H^2(B)$ there are the following connections between values of the transverse, mixed and longitudinal ray transforms of the moment k:*

$$\begin{aligned}
\mathcal{P}^{(0)}_{k2}\mathrm{d}^2\varphi &= \mathcal{P}^{(2)}_{k2}(\mathrm{d}^\perp)^2\varphi, & \mathcal{P}^{(1)}_{k2}\mathrm{d}^2\varphi &= -\mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi,\\
\mathcal{P}^{(0)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi &= -\mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi, & \mathcal{P}^{(1)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi &= \frac{1}{2}\Big(\mathcal{P}^{(2)}_{k2}\mathrm{d}^2\varphi - \mathcal{P}^{(2)}_{k2}(\mathrm{d}^\perp)^2\varphi\Big),\\
\mathcal{P}^{(0)}_{k2}(\mathrm{d}^\perp)^2\varphi &= \mathcal{P}^{(2)}_{k2}\mathrm{d}^2\varphi, & \mathcal{P}^{(1)}_{k2}(\mathrm{d}^\perp)^2\varphi &= \mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi.
\end{aligned}$$

Proof. We need the following equalities, which can be established by a direct verification:

$$\langle \nabla^\perp\phi, \xi\rangle = -\langle \nabla\phi, \eta\rangle, \tag{5}$$

$$\langle \nabla\phi, \xi\rangle = \langle \nabla^\perp\phi, \eta\rangle. \tag{6}$$

Using the operators definition (3) and applying the formula (5) twice, we get

$$\mathcal{P}^{(0)}_{k2}\mathrm{d}^2\varphi = \int_{-\infty}^{\infty} t^k\langle \mathrm{d}^2\varphi, \eta^2\rangle\, dt = \int_{-\infty}^{\infty} t^k\langle \nabla\langle\nabla\varphi,\eta\rangle,\eta\rangle\, dt \overset{(5)}{=} -\int_{-\infty}^{\infty} t^k\langle \nabla\langle\nabla^\perp\varphi,\xi\rangle,\eta\rangle\, dt$$

$$\overset{(5)}{=} \int_{-\infty}^{\infty} t^k\langle \nabla^\perp\langle\nabla^\perp\varphi,\xi\rangle,\xi\rangle\, dt = \int_{-\infty}^{\infty} t^k\langle (\mathrm{d}^\perp)^2\varphi, \xi^2\rangle\, dt = \mathcal{P}^{(2)}_{k2}(\mathrm{d}^\perp)^2\varphi.$$

Note that after the first application of the formula (5) the momentum mixed ray transform $\mathcal{P}^{(1)}_{k2}$ arises. However, a direct verification establishes that $\mathcal{P}^{(1)}_{k2}$ is applied to the asymmetric field $\nabla\nabla^\perp\varphi$ (see (1)). The formula for calculating values of $\mathcal{P}^{(0)}_{k2}(\mathrm{d}^\perp)^2\varphi$ is obtained in a similar way by applying the formula (6) twice.

Applying the formulas (5), (6) and using the equality $\mathrm{d}\mathrm{d}^\perp\varphi = \mathrm{d}^\perp\mathrm{d}\varphi$, we obtain

$$\mathcal{P}^{(0)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi = \int_{-\infty}^{\infty} t^k\langle \mathrm{d}\mathrm{d}^\perp\varphi, \eta^2\rangle\,dt = \int_{-\infty}^{\infty} t^k\langle \nabla\langle\nabla^\perp\varphi,\eta\rangle,\eta\rangle\,dt \overset{(6)}{=} \int_{-\infty}^{\infty} t^k\langle \nabla\langle\nabla\varphi,\xi\rangle,\eta\rangle\,dt$$
$$\overset{(5)}{=} -\int_{-\infty}^{\infty} t^k\langle \nabla^\perp\langle\nabla\varphi,\xi\rangle,\xi\rangle\,dt = -\int_{-\infty}^{\infty} t^k\langle \mathrm{d}^\perp\mathrm{d}\varphi,\xi^2\rangle\,dt = -\mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi.$$

We note that after application of the formula (6) the momentum mixed ray transform $\mathcal{P}^{(1)}_{k2}\mathrm{d}^2\varphi$ arises. Thus, it is additionally established that $\mathcal{P}^{(1)}_{k2}\mathrm{d}^2\varphi = -\mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi$. In a similar way we obtain the equality $\mathcal{P}^{(1)}_{k2}(\mathrm{d}^\perp)^2\varphi = \mathcal{P}^{(2)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi$. To do this, it is enough in the reasoning to change the order of application of the formulas (5) and (6).

Finally, we obtain an expression for $\mathcal{P}^{(1)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi$. We have

$$\begin{aligned}
\mathcal{P}^{(1)}_{k2}\mathrm{d}\mathrm{d}^\perp\varphi &= \int_{-\infty}^{\infty} t^k\langle\langle \mathrm{d}\mathrm{d}^\perp\varphi,\xi\rangle,\eta\rangle\,dt \\
&= \int_{-\infty}^{\infty} t^k\Big(-\frac{\partial^2\varphi}{\partial x\partial y}\xi_1\eta_1 + \frac{1}{2}\Big(\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial y^2}\Big)(\xi_1\eta_2+\xi_2\eta_1) + \frac{\partial^2\varphi}{\partial x\partial y}\xi_2\eta_2\Big)\,dt \\
&= \int_{-\infty}^{\infty} t^k\Big(\frac{\partial^2\varphi}{\partial x\partial y}\xi_1\xi_2 + \frac{1}{2}\Big(\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial y^2}\Big)(\xi_1\xi_1-\xi_2\xi_2) + \frac{\partial^2\varphi}{\partial x\partial y}\xi_2\xi_1\Big)\,dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} t^k\Big(\Big(\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial^2 y}\Big)\xi_1\xi_1 + 4\frac{\partial^2\varphi}{\partial x\partial y}\xi_1\xi_2 + \Big(\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial x^2}\Big)\xi_2\xi_2\Big)\,dt \\
&= \frac{1}{2}\Big(\mathcal{P}^{(2)}_{k2}\mathrm{d}^2\varphi - \mathcal{P}^{(2)}_{k2}(\mathrm{d}^\perp)^2\varphi\Big).
\end{aligned}$$

□

From Th. 2.1 it follows that it is sufficient to study, for example, only the transverse ray transforms of moments. Values of the longitudinal and mixed ray transforms of moments of symmetric 2-tensor fields can be found using Th. 2.1.

Let us note one more consequence. For any k we have

$$\mathcal{P}^{(1)}_{k2}(\mathrm{d}^2\varphi + (\mathrm{d}^\perp)^2\varphi) = 0. \tag{7}$$

Relations for the transverse ray transforms of moments, which are necessary for constructing numerical algorithms, are collected in the following statement (see [22],[6] for details).

Proposition 2.1. *Let $\mathrm{d}^2\varphi$, $\mathrm{d}\mathrm{d}^\perp\chi$, $(\mathrm{d}^\perp)^2\psi$ be symmetric 2-tensor fields with potentials $\varphi,\chi,\psi \in H^2_0(\mathbb{R}^2)$, then for the momentum transverse ray transforms $\mathcal{P}^{(2)}_{k2}$, $k = 0,1,2$ the following equalities*

$$\begin{aligned}
&\mathcal{P}^{(2)}_{02}\mathrm{d}^2\varphi = (\mathcal{R}\varphi)''_{ss}, && \mathcal{P}^{(2)}_{02}\mathrm{d}\mathrm{d}^\perp\chi = 0, && \mathcal{P}^{(2)}_{02}(\mathrm{d}^\perp)^2\psi = 0,\\
&\mathcal{P}^{(2)}_{12}\mathrm{d}^2\varphi = -(\mathcal{R}\varphi)''_{s\theta}, && \mathcal{P}^{(2)}_{12}\mathrm{d}\mathrm{d}^\perp\chi = (\mathcal{R}\chi)'_{s}, && \mathcal{P}^{(2)}_{12}(\mathrm{d}^\perp)^2\psi = 0,\\
&\mathcal{P}^{(2)}_{22}\mathrm{d}^2\varphi = (\mathcal{R}\varphi)''_{\theta\theta} - (s\,\mathcal{R}\varphi)'_{s}, && \mathcal{P}^{(2)}_{22}\mathrm{d}\mathrm{d}^\perp\chi = -2(\mathcal{R}\chi)'_{\theta}, && \mathcal{P}^{(2)}_{22}(\mathrm{d}^\perp)^2\psi = 2\,\mathcal{R}\psi
\end{aligned}$$

hold.

From Th. 2.1 and Pr. 2.1 the results for the operators of longitudinal and transverse ray transforms of moments follow.

Corollary 2.1. *Let* $\mathrm{d}^2\varphi$, $\mathrm{d}\mathrm{d}^\perp\chi$, $(\mathrm{d}^\perp)^2\psi$ *be symmetric 2-tensor fields with potentials* $\varphi, \chi, \psi \in H_0^2(\mathbb{R}^2)$*, then there are equalities for the momentum longitudinal ray transform* $\mathcal{P}_{k2}^{(0)}$*,* $k = 0, 1, 2$

$$
\begin{aligned}
&\mathcal{P}_{02}^{(0)}\mathrm{d}^2\varphi = 0, && \mathcal{P}_{02}^{(0)}\mathrm{d}\mathrm{d}^\perp\chi = 0, && \mathcal{P}_{02}^{(0)}(\mathrm{d}^\perp)^2\psi = (\mathcal{R}\psi)''_{ss},\\
&\mathcal{P}_{12}^{(0)}\mathrm{d}^2\varphi = 0, && \mathcal{P}_{12}^{(0)}\mathrm{d}\mathrm{d}^\perp\chi = -(\mathcal{R}\chi)'_s, && \mathcal{P}_{12}^{(0)}(\mathrm{d}^\perp)^2\psi = -(\mathcal{R}\psi)''_{s\theta},\\
&\mathcal{P}_{22}^{(0)}\mathrm{d}^2\varphi = 2\,\mathcal{R}\varphi, && \mathcal{P}_{22}^{(0)}\mathrm{d}\mathrm{d}^\perp\chi = 2(\mathcal{R}\chi)'_\theta, && \mathcal{P}_{22}^{(0)}(\mathrm{d}^\perp)^2\psi = (\mathcal{R}\psi)''_{\theta\theta} - (s\,\mathcal{R}\psi)'_s,
\end{aligned}
$$

and for the mixed ray transform $\mathcal{P}_{k2}^{(1)}$ *of the moments* $k = 0, 1, 2$

$$
\begin{aligned}
&\mathcal{P}_{02}^{(1)}\mathrm{d}^2\varphi = 0, && \mathcal{P}_{02}^{(1)}\mathrm{d}\mathrm{d}^\perp\chi = \frac{1}{2}(\mathcal{R}\chi)''_{ss}, && \mathcal{P}_{02}^{(1)}(\mathrm{d}^\perp)^2\psi = 0,\\
&\mathcal{P}_{12}^{(1)}\mathrm{d}^2\varphi = -(\mathcal{R}\varphi)'_s, && \mathcal{P}_{12}^{(1)}\mathrm{d}\mathrm{d}^\perp\chi = -\frac{1}{2}(\mathcal{R}\chi)''_{s\theta}, && \mathcal{P}_{12}^{(1)}(\mathrm{d}^\perp)^2\psi = (\mathcal{R}\psi)'_s,\\
&\mathcal{P}_{22}^{(1)}\mathrm{d}^2\varphi = 2(\mathcal{R}\varphi)'_\theta, && \mathcal{P}_{22}^{(1)}\mathrm{d}\mathrm{d}^\perp\chi = \frac{1}{2}\left((\mathcal{R}\chi)''_{\theta\theta} - (s\,\mathcal{R}\chi)'_s - 2\,\mathcal{R}\chi\right), && \mathcal{P}_{22}^{(1)}(\mathrm{d}^\perp)^2\psi = -2(\mathcal{R}\psi)'_\theta.
\end{aligned}
$$

3 Statement of problems

We are now in a position to state the inverse problems we seek to solve in this work.

Problem 1. *Let for a symmetric 2-tensor field* $\mathbf{w}$ *values of the transverse ray transform* $\mathcal{P}_{k2}^{(2)}\mathbf{w}$ *of the moments* $k = 0, 1, 2$ *are given. We wish to recover* $\mathbf{w}$ *from this data.*

Problem 2. *Let values of the momentum longitudinal ray transforms* $\mathcal{P}_{k2}^{(0)}\mathbf{w}$*,* $k = 0, 1, 2$ *are given. It is necessary to reconstruct the symmetric 2-tensor field* $\mathbf{w}$*.*

Note that a problem of restoring a symmetric 2-tensor field $\mathbf{w}$ from values of the mixed ray transforms $\mathcal{P}_{k2}^{(1)}\mathbf{w}$ of moments $k = 0, 1, 2$ does not have a unique solution, since due to the equalities (7) these operators do not distinguish the solenoidal part $(\mathrm{d}^\perp)^2\varphi$ and the potential part $\mathrm{d}^2\varphi$ of the field $\mathbf{w}$. Therefore, this problem is not studied.

Additionally, the following problem is considered.

Problem 0. *Let values of the longitudinal* $\mathcal{P}_{02}^{(0)}\mathbf{w}$*, mixed* $\mathcal{P}_{02}^{(1)}\mathbf{w}$ *and transverse* $\mathcal{P}_{02}^{(2)}\mathbf{w}$ *ray transforms are given for a symmetric 2-tensor field* $\mathbf{w}$*. We need to restore the field* $\mathbf{w}$*.*

Problem 0 is well studied (see, for example, [19],[22]). In particular, it is known that from values of the longitudinal $\mathcal{P}_{02}^{(0)}\mathbf{w}$, mixed $\mathcal{P}_{02}^{(1)}\mathbf{w}$ and transverse $\mathcal{P}_{02}^{(2)}\mathbf{w}$ ray transforms the solenoidal part $(\mathrm{d}^\perp)^2\psi$, the potential parts $\mathrm{d}\mathrm{d}^\perp\chi$ and $\mathrm{d}^2\varphi$ of the field $\mathbf{w}$ can be restored, respectively. Algorithms for solving Problem 0 based on the SV-decomposition method were obtained in [14],[16].

4 Algorithms for solving the problems

The algorithms aimed at solving Pr. 1 and 2 are proposed in the current work. These algorithms use initial data in the form of values of the ray transforms of the same type (transverse or longitudinal) of the moments $k = 0, 1, 2$. This is a significant difference between Problem 0 and Problems 1 and 2. Consideration of Problem 0, while simultaneously solving Problem 1 or 2, aims to compare the results of restoration using the previously tested algorithms (for Problem 0) and the algorithms proposed in the present work for restoring 2-tensor fields from values of the momentum ray transforms (for Problems 1 and 2).

The following theorem establishes connections between the ray transforms $\mathcal{P}^{(0)}_{k2}, \mathcal{P}^{(2)}_{k2}$ of the moments $k = 0, 1, 2$ and the ray transforms $\mathcal{P}^{(j)}_{02}$, $j = 0, 1, 2$.

Theorem 4.1. *Let* $\mathbf{w} = \mathrm{d}^2\varphi + \mathrm{d}\mathrm{d}^\perp\chi + (\mathrm{d}^\perp)^2\psi$ *with potentials* $\varphi, \chi, \psi \in H^2_0(\mathbb{R}^2)$ *then there are connections between the ray transforms* $\mathcal{P}^{(j)}_{02}$, $j = 0, 1, 2$ *and the ray transforms* $\mathcal{P}^{(0)}_{k2}$ *of the moments* $k = 0, 1, 2$

$$\mathcal{P}^{(2)}_{02}\mathrm{d}^2\varphi = \frac{1}{2}\big((\mathcal{P}^{(0)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{P}^{(0)}_{12}\mathbf{w})''_{s\theta} + (\mathcal{P}^{(0)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(0)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(0)}_{02}\mathbf{w})'_s\big), \tag{8}$$

$$\mathcal{P}^{(1)}_{02}\mathrm{d}\mathrm{d}^\perp\chi = -\frac{1}{2}\big((\mathcal{P}^{(0)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(0)}_{02}\mathbf{w})'_\theta\big), \tag{9}$$

$$\mathcal{P}^{(0)}_{02}(\mathrm{d}^\perp)^2\psi = \mathcal{P}^{(0)}_{02}\mathbf{w} \tag{10}$$

and connections between the ray transforms $\mathcal{P}^{(j)}_{02}$, $j = 0, 1, 2$ *and* $\mathcal{P}^{(2)}_{k2}$, $k = 0, 1, 2$

$$\mathcal{P}^{(2)}_{02}\mathrm{d}^2\varphi = \mathcal{P}^{(2)}_{02}\mathbf{w}, \tag{11}$$

$$\mathcal{P}^{(1)}_{02}\mathrm{d}\mathrm{d}^\perp\chi = \frac{1}{2}\big((\mathcal{P}^{(2)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(2)}_{02}\mathbf{w})'_\theta\big), \tag{12}$$

$$\mathcal{P}^{(0)}_{02}(\mathrm{d}^\perp)^2\psi = \frac{1}{2}\big((\mathcal{P}^{(2)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{P}^{(2)}_{12}\mathbf{w})''_{s\theta} + (\mathcal{P}^{(2)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(2)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(2)}_{02}\mathbf{w})'_s\big). \tag{13}$$

Proof. Let us obtain formulas (11)–(13) for the momentum transverse ray transforms. Using Pr. 2.1 and the decomposition (2), we obtain:

$$\begin{aligned}
\mathcal{P}^{(2)}_{02}\mathrm{d}^2\varphi &= (\mathcal{R}\varphi)''_{ss} = \mathcal{P}^{(2)}_{02}\mathbf{w},\\
\mathcal{P}^{(1)}_{02}\mathrm{d}\mathrm{d}^\perp\chi &= \frac{1}{2}(\mathcal{R}\chi)''_{ss} = \frac{1}{2}(\mathcal{P}^{(2)}_{12}\mathrm{d}\mathrm{d}^\perp\chi)'_s = \frac{1}{2}\big(\mathcal{P}^{(2)}_{12}(\mathbf{w} - \mathrm{d}^2\varphi)\big)'_s = \frac{1}{2}\big(\mathcal{P}^{(2)}_{12}\mathbf{w} - \mathcal{P}^{(2)}_{12}\mathrm{d}^2\varphi\big)'_s\\
&= \frac{1}{2}\big((\mathcal{P}^{(2)}_{12}\mathbf{w})'_s + (\mathcal{R}\varphi)'''_{ss\theta}\big) = \frac{1}{2}\big((\mathcal{P}^{(2)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(2)}_{02}\mathbf{w})'_\theta\big),\\
\mathcal{P}^{(0)}_{02}(\mathrm{d}^\perp)^2\psi &= (\mathcal{R}\psi)''_{ss} = \frac{1}{2}(\mathcal{P}^{(2)}_{22}(\mathrm{d}^\perp)^2\psi)''_{ss} = \frac{1}{2}\big(\mathcal{P}^{(2)}_{22}(\mathbf{w} - \mathrm{d}\mathrm{d}^\perp\chi - \mathrm{d}^2\varphi)\big)''_{ss}\\
&= \frac{1}{2}\big(\mathcal{P}^{(2)}_{22}\mathbf{w} - \mathcal{P}^{(2)}_{22}\mathrm{d}\mathrm{d}^\perp\chi - \mathcal{P}^{(2)}_{22}\mathrm{d}^2\varphi\big)''_{ss}\\
&= \frac{1}{2}\big(\mathcal{P}^{(2)}_{22}\mathbf{w} + 2(\mathcal{R}\chi)'_\theta - (\mathcal{R}\varphi)''_{\theta\theta} + (s\,\mathcal{R}\varphi)'_s\big)''_{ss}\\
&= \frac{1}{2}\big((\mathcal{P}^{(2)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{R}\chi)'''_{ss\theta} - (\mathcal{R}\varphi)''''_{ss\theta\theta} + 3(\mathcal{R}\varphi)''_{ss} + s\,(\mathcal{R}\varphi)'''_{sss}\big)\\
&= \frac{1}{2}\big((\mathcal{P}^{(2)}_{22}\mathbf{w})''_{ss} + 2\big((\mathcal{P}^{(2)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(2)}_{02}\mathbf{w})'_\theta\big)'_\theta - (\mathcal{P}^{(2)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(2)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(2)}_{02}\mathbf{w})'_s\big)\\
&= \frac{1}{2}\big((\mathcal{P}^{(2)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{P}^{(2)}_{12}\mathbf{w})''_{s\theta} + (\mathcal{P}^{(2)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(2)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(2)}_{02}\mathbf{w})'_s\big).
\end{aligned}$$

The formulas (8)–(10) are obtained similarly. □

The algorithms schemes aimed at both solving Problems 1 and 2 are completely similar and differ only in the set of formulas used. Therefore, steps for solving Problems 0, 1 and 2 are given only for Algorithm 1. For Algorithms 2 and 3, the implementation stages of solution only for Problem 1 are presented.

4.1 Algorithm 1

The algorithm allows to recover the components w_{ij}, $i, j = 1, 2$, of the symmetric 2-tensor field $\mathbf{w}$, using the SV-decomposition of the Radon transform $\mathcal{R}$ of functions.

Problem 0. Let values of the ray transforms $\mathcal{P}^{(j)}_{02}\mathbf{w}$, $j = 0, 1, 2$ are given. To restore the field $\mathbf{w}$, one should perform the following steps:

1. using the formulas (4) with $k = 0$, calculate values of the expressions
$$\eta^i\eta^j(\mathcal{P}^{(0)}_{02}\mathbf{w}) + (\eta^j\xi^i + \eta^i\xi^j)(\mathcal{P}^{(1)}_{02}\mathbf{w}) + \xi^i\xi^j(\mathcal{P}^{(2)}_{02}\mathbf{w}), \quad i, j = 1, 2,$$
which are equal to $\mathcal{R}w_{ij}$, $i, j = 1, 2$;
2. reconstruct the components w_{ij}, $i, j = 1, 2$ of the symmetric 2-tensor field $\mathbf{w}$ from values of the expressions obtained in the first step.

Problem 1. Let values of the momentum transverse ray transforms $\mathcal{P}^{(2)}_{k2}\mathbf{w}$, $k = 0, 1, 2$ are known. To recovery the symmetric 2-tensor field $\mathbf{w}$ it is necessary to perform the following steps:

1. using the formulas (4) with $k = 0$ and the formulas (11)-(13), find values of the expressions
$$\frac{\eta^i\eta^j}{2}\big((\mathcal{P}^{(2)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{P}^{(2)}_{12}\mathbf{w})''_{s\theta} + (\mathcal{P}^{(2)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(2)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(2)}_{02}\mathbf{w})'_s\big)$$
$$+\ \frac{(\eta^j\xi^i + \eta^i\xi^j)}{2}\big((\mathcal{P}^{(2)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(2)}_{02}\mathbf{w})'_\theta\big) + \xi^i\xi^j(\mathcal{P}^{(2)}_{02}\mathbf{w}), \quad i, j = 1, 2,$$
which are equal to $\mathcal{R}w_{ij}$, $i, j = 1, 2$ (the derivatives are calculated using difference schemes);
2. reconstruct the components w_{ij}, $i, j = 1, 2$ of $\mathbf{w}$.

Problem 2. Let values of the momentum longitudinal ray transforms $\mathcal{P}^{(0)}_{k2}\mathbf{w}$, $k = 0, 1, 2$ are given. To restore the field $\mathbf{w}$, the following sequence of actions is implemented:

1. using the formulas (4) with $k = 0$ and the formulas (8)–(10), calculate the expressions
$$\eta^i\eta^j(\mathcal{P}^{(0)}_{02}\mathbf{w}) - \frac{(\eta^j\xi^i + \eta^i\xi^j)}{2}\big((\mathcal{P}^{(0)}_{12}\mathbf{w})'_s + (\mathcal{P}^{(0)}_{02}\mathbf{w})'_\theta\big)$$
$$+\ \frac{\xi^i\xi^j}{2}\big((\mathcal{P}^{(0)}_{22}\mathbf{w})''_{ss} + 2(\mathcal{P}^{(0)}_{12}\mathbf{w})''_{s\theta} + (\mathcal{P}^{(0)}_{02}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}^{(0)}_{02}\mathbf{w} + s\,(\mathcal{P}^{(0)}_{02}\mathbf{w})'_s\big), \quad i, j = 1, 2,$$
where the derivatives are calculated using difference schemes;
2. reconstruct the components w_{ij}, $i, j = 1, 2$ of the tensor field $\mathbf{w}$ from values of the expressions (as values of the Radon transform $\mathcal{R}w_{ij}$, $i, j = 1, 2$) obtained in the previous step.

Note that when implementing Algorithm 1, the components w_{ij}, $i, j = 1, 2$ of the field $\mathbf{w}$ are restored independently and therefore these calculations can be performed in parallel.

4.2 Algorithm 2

The algorithm allows to reconstruct the solenoidal part $(\mathrm{d}^\perp)^2\psi$ and the potential parts $\mathrm{d}\mathrm{d}^\perp\chi$ and $\mathrm{d}^2\varphi$ of the field $\mathbf{w}$ separately. We use the previously constructed SV-decompositions of the ray transforms $\mathcal{P}^{(j)}_{02}$, $j = 0, 1, 2$ of symmetric 2-tensor fields. This algorithm is implemented to solve Problems 0, 1 and 2, but we demonstrate the complete scheme for Problem 1 only.

Problem 1. Let values of the momentum transverse ray transforms $\mathcal{P}^{(2)}_{k2}\mathbf{w}$, $k = 0, 1, 2$ are known. To recovery the symmetric 2-tensor field $\mathbf{w}$ it is necessary to perform the following steps:

1. construct an approximation $\widetilde{\mathrm{d}^2\varphi}$ of the potential part $\mathrm{d}^2\varphi$ of the field $\mathbf{w}$ from $\mathcal{P}_{02}^{(2)}\mathbf{w}$;
2. construct an approximation $\widetilde{\mathrm{d}\mathrm{d}^\perp\chi}$ of the potential part $\mathrm{d}\mathrm{d}^\perp\chi$ of $\mathbf{w}$ from values of the expression
$$\frac{1}{2}\big((\mathcal{P}_{12}^{(2)}\mathbf{w})'_s + (\mathcal{P}_{02}^{(2)}\mathbf{w})'_\theta\big),$$
that, according to the formula (12), is equal to $\mathcal{P}_{02}^{(1)}\mathrm{d}\mathrm{d}^\perp\chi$ (the derivatives are computed numerically);
3. construct an approximation $\widetilde{(\mathrm{d}^\perp)^2\psi}$ of the solenoidal part $(\mathrm{d}^\perp)^2\psi$ of the tensor field $\mathbf{w}$ from values of the expression
$$\frac{1}{2}\big((\mathcal{P}_{22}^{(2)}\mathbf{w})''_{ss} + 2(\mathcal{P}_{12}^{(2)}\mathbf{w})''_{s\theta} + (\mathcal{P}_{02}^{(2)}\mathbf{w})''_{\theta\theta} + 3\mathcal{P}_{02}^{(2)}\mathbf{w} + s\,(\mathcal{P}_{02}^{(2)}\mathbf{w})'_s\big),$$
that, according to the formula (13), is equal to $\mathcal{P}_{02}^{(2)}\mathrm{d}^2\varphi$ (the derivatives are computed numerically);
4. finally, summing up the results obtained, we get a complete reconstruction $\widetilde{\mathbf{w}}$ of the field $\mathbf{w}$.

Note that since the solenoidal part $\widetilde{(\mathrm{d}^\perp)^2\psi}$ and the potential parts $\widetilde{\mathrm{d}^2\varphi}$, $\widetilde{\mathrm{d}\mathrm{d}^\perp\chi}$ of the field $\mathbf{w}$ are reconstructed separately, these calculations can be performed in parallel.

4.3 Algorithm 3

Algorithm 3 is a modification of Algorithm 2. The modification consists in partially avoiding numerical differentiation in the steps 2 and 3. In this case, the steps 1–3 must now be carried out sequentially. This algorithm is implemented to solve Problems 1 and 2, but we demonstrate the complete scheme for Prjblem 1 only.

Problem 1. Let values of the transverse ray transforms $\mathcal{P}_{k2}^{(2)}\mathbf{w}$ of the moments $k = 0, 1, 2$ are given. To reconstruct the field $\mathbf{w}$ it is needed to perform the steps:

1. construct an approximation $\widetilde{\mathrm{d}^2\varphi}$ of the potential part $\mathrm{d}^2\varphi$ of the tensor field $\mathbf{w}$ from $\mathcal{P}_{02}^{(2)}\mathbf{w}$;
2. construct an approximation $\widetilde{\mathrm{d}\mathrm{d}^\perp\chi}$ of the potential part $\mathrm{d}\mathrm{d}^\perp\chi$ of the field $\mathbf{w}$ from values of the expression
$$\frac{1}{2}\big((\mathcal{P}_{12}^{(2)}\mathbf{w})'_s + (\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})'_\theta\big),$$
that, according to the formula (12), is approximately equal to $\mathcal{P}_{02}^{(1)}\mathrm{d}\mathrm{d}^\perp\chi$ (here values of $(\mathcal{P}_{12}^{(2)}\mathbf{w})'_s$ are computed numerically, but values of $(\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})'_\theta$ are calculated analytically);
3. construct an approximation $\widetilde{(\mathrm{d}^\perp)^2\psi}$ of the solenoidal part $(\mathrm{d}^\perp)^2\psi$ of $\mathbf{w}$ from values of the expression
$$\frac{1}{2}\big((\mathcal{P}_{22}^{(2)}\mathbf{w})''_{ss} + 4(\mathcal{P}_{02}^{(1)}\widetilde{\mathrm{d}\mathrm{d}^\perp\chi})'_\theta - (\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})''_{\theta\theta} + 3\mathcal{P}_{02}^{(2)}\mathbf{w} + s\,(\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})'_s\big),$$
that, in accordance with the intermediate result when obtaining the formula (13), is approximately equal to $\mathcal{P}_{02}^{(2)}\mathrm{d}^2\varphi$ (values of $(\mathcal{P}_{22}^{(2)}\mathbf{w})''_{ss}$ are computed numerically, but values of $(\mathcal{P}_{02}^{(1)}\widetilde{\mathrm{d}\mathrm{d}^\perp\chi})'_\theta$, $(\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})''_{\theta\theta}$ and $(\mathcal{P}_{02}^{(2)}\widetilde{\mathrm{d}^2\varphi})'_s$ are calculated analytically);
4. summing up the results obtained in the previous steps, we get a full reconstruction of $\mathbf{w}$.

5 Details of numerical implementation

Using the SV-decompositions. The choice of the SV-decomposition method for solving the formulated problems is determined by good accuracy of approximately reconstruction of functions [14] and symmetric 2-tensor fields [16]. Moreover, the SV-decomposition method allows to calculate values of the ray transforms of approximations of tensor fields analytically.

We use the Jacobi polynomials $P_n^{(p,q)}(t)$, the Gegenbauer polynomials $C_n^{(\mu)}(t)$ and the trigonometric functions $Y_k^1(\theta) = \cos k\theta$, $Y_k^2(\theta) = \sin k\theta$ when constructing bases in the original space and in the space of images. Below, we get the results for arbitrary $m \geqslant 0$, but we use only $m = 0$ (Algorithm 1) and $m = 2$ (Algorithms 2 and 3) when carrying out simulations.

The SV-decompositions of the ray transforms operators $\mathcal{P}_{0m}^{(j)}$, $0 \leqslant j \leqslant m$ of symmetric m-tensor fields have the form

$$g(s,\theta) := (\mathcal{P}_{0m}^{(j)}\mathbf{u})(s,\theta) = \sum_{i=1}^{2}\sum_{k=i-1}^{\infty}\sum_{n=0}^{\infty} \sigma_{kn}^{jm}\langle \mathbf{u}, \mathbf{F}_{kn}^{ijm}\rangle_{L_2(S^m(B))} G_{kn}^{im}(s,\theta).$$

Values of (pseudo)inverse operators $(\mathcal{P}_{0m}^{(j)})^\dagger$, $0 \leqslant j \leqslant m$, acting on g may be calculated by the following formulas

$$[(\mathcal{P}_{0m}^{(j)})^\dagger g](x) = \sum_{i=1}^{2}\sum_{k=i-1}^{\infty}\sum_{n=0}^{\infty} \left(\sigma_{kn}^{jm}\right)^{-1} \langle g, G_{kn}^{im}\rangle_{L_2(Z,(1-s^2)^{-1/2})} \mathbf{F}_{kn}^{ijm}(x). \qquad (14)$$

The fields $\mathbf{F}_{kn}^{ijm}$ are defined

$$\mathbf{F}_{kn}^{ijm}(x) = \mathrm{d}^j\left(\mathrm{d}^\perp\right)^{m-j}\Phi_{kn}^{ijm}(x), \quad i = 1,2, \quad k \geqslant i-1, \quad n \geqslant 0, \quad 0 \leqslant j \leqslant m, \qquad (15)$$

where the potentials in polar coordinate system are

$$\Phi_{kn}^{ijm}(r\cos\varphi, r\sin\varphi) = \lambda_{kn}^{jm}\left(1-r^2\right)^m r^k Y_k^i(\varphi) P_n^{(k+1+m,k+1)}(r^2)$$

and the normalizing coefficients are defined by

$$\lambda_{kn}^{jm} = \binom{m}{j}^{1/2} \frac{b_k(n+k)!}{2^m k!(n+m)!}\left(\frac{2n+k+1+m}{\pi}\right)^{1/2}, \quad \text{here } b_k = \begin{cases}\sqrt{2}, & k \geqslant 1,\\ 1, & k = 0.\end{cases}$$

The functions G_{kn}^{im}, $i = 1,2$, $k \geqslant i-1$, $n \geqslant 0$, $0 \leqslant j \leqslant m$ are determined by the formulas

$$G_{kn}^{im}(s,\theta) = \frac{b_k(-1)^{n+m}}{\pi}(1-s^2)^{1/2} C_{2n+k+m}^{(1)}(s) Y_k^i(\theta).$$

The singular values σ_{kn}^{jm} are calculated by

$$\sigma_{kn}^{jm} = \left(\frac{4\pi}{2n+k+m+1}\right)^{1/2}\binom{m}{j}^{-1/2}.$$

The operators $(\mathcal{P}_m^{(j)})^\dagger$ are unbounded because $(\sigma_{kn}^{jm})^{-1} \to +\infty$ for $n, k \to \infty$. These operators can be regularized using the truncated SV-decomposition. Details can be found in [14], [16], [18]. We use the basis fields (15) whose components have degree no more than $N = 50$.

The following formulas are required for the realization of Algorithm 3:

$$(\mathcal{P}_{02}^{(j)}\mathbf{F}_{kn}^{ij2})'_\theta(s,\theta) = \frac{k\sigma_{kn}^{j2} b_k(-1)^{n+i}}{\pi}(1-s^2)^{1/2} C_{2n+k+2}^{(1)}(s) Y_k^{3-i}(\theta),$$

$$(\mathcal{P}_{02}^{(j)}\mathbf{F}_{kn}^{ij2})''_{\theta\theta}(s,\theta) = \frac{k^2\sigma_{kn}^{j2} b_k(-1)^{n+1}}{\pi}(1-s^2)^{1/2} C_{2n+k+2}^{(1)}(s) Y_k^{i}(\theta),$$

$$(\mathcal{P}_{02}^{(j)}\mathbf{F}_{kn}^{ij2})'_s(s,\theta) = \frac{\sigma_{kn}^{j2} b_k(-1)^{n}}{\pi(1-s^2)^{1/2}}\left(2(1-s^2)C_{2n+k+1}^{(2)}(s) - sC_{2n+k+2}^{(1)}(s)\right) Y_k^{i}(\theta).$$

Discretization of initial data. Values of the ray transforms $\mathcal{P}^{(2)}_{k2}(w)$, $k=0,1,2$ (Problem 1), $\mathcal{P}^{(0)}_{k2}(w)$, $k=0,1,2$ (Problem 2), or $\mathcal{P}^{(j)}_{02}(w)$, $j=0,1,2$ (Problem 0), known at points of an uniform grid are initial data in numerical simulations. Solving Problem 0, while simultaneously solving Problem 1 and 2, has a goal of comparison of calculation results using the previously tested algorithm and the algorithms proposed for reconstruction symmetric 2-tensor fields by the momentum ray transforms. Fixing a natural L, we get the following sequences

$$\begin{aligned} &s_p = p\cdot\Delta s, && p=-L+1,\dots,L-1, && \Delta s = 1/L,\\ &\theta_q = q\cdot\Delta\theta, && q=0,1,\dots,2L-1, && \Delta\theta=\pi/L, \end{aligned}$$

of discrete values of the variables s,θ. Choosing a pair s_p,θ_q means setting vectors $\xi_q = (\cos\theta_q,\sin\theta_q)$, $\eta_q=(-\sin\theta_q,\cos\theta_q)$ and a point

$$x_{pq} = \left(\cos\theta_q s_p + \sin\theta_q\sqrt{1-s_p^2}\,,\, \sin\theta_q s_p - \cos\theta_q\sqrt{1-s_p^2}\right)\in\partial B,$$

from which a ray propagate in the direction η_q. We use discretizations 200×200, 300×300 and 400×400 by (s,θ).

Numerical integration. We use the trapezoidal rule for calculation values of the ray transforms of test fields. This rule consists of integration along the straight line by the formulas (3) for $m=2$, $j,k=0,1,2$. The SV-decomposition method leads to a polynomial approximation of the (pseudo)inverse operators (14). We use the Simpson's rule, consisting of integration over the variables s,θ, when calculating the inner products $\langle g, G^{im}_{kn}\rangle_{L_2(Z,(1-s^2)^{-1/2})}$.

Numerical differentiation. It is necessary to use the first and the second derivatives of functions calculated in uniform grid when implementing Algorithms 1-3. In calculations we use the five-point difference schemes

$$\begin{aligned} f'(t_0) &\approx \frac{f(t_0-2h)-8f(t_0-h)+8f(t_0+h)-f(t_0+2h)}{12h},\\ f''(t_0) &\approx \frac{-f(t_0-2h)+16f(t_0-h)-30f(t_0)+16f(t_0+h)-f(t_0+2h)}{12h^2} \end{aligned}$$

with the high degree of accuracy $O(h^4)$.

6 Numerical simulations

We consider the symmetric 2-tensor fields

$$\mathbf{w}^{(l)} = (\mathrm{d}^\perp)^2\psi^{(l)} + \mathrm{d}\mathrm{d}^\perp\chi^{(l)} + \mathrm{d}^2\varphi^{(l)},\quad l=1,2,3,$$

generated by potentials of class C^{l+2}, which are given by the formulas

$$\psi^{(l)}(x) = \begin{cases} (0.36-x_1^2-(x_2+0.2)^2)^{l+2}, & \text{if } x_1^2+(x_2+0.2)^2<0.36,\\ 0, & \text{otherwise,}\end{cases}$$

$$\chi^{(l)}(x) = \begin{cases} (0.16-(x_1+0.2)^2-x_2^2)^{l+2}, & \text{if } (x_1+0.2)^2+x_2^2<0.16,\\ 0, & \text{otherwise,}\end{cases}$$

$$\varphi^{(l)}(x) = \begin{cases} (0.25-(x_1-0.2)^2-x_2^2)^{l+2}, & \text{if } (x_1-0.2)^2+x_2^2<0.25,\\ 0, & \text{otherwise.}\end{cases}$$

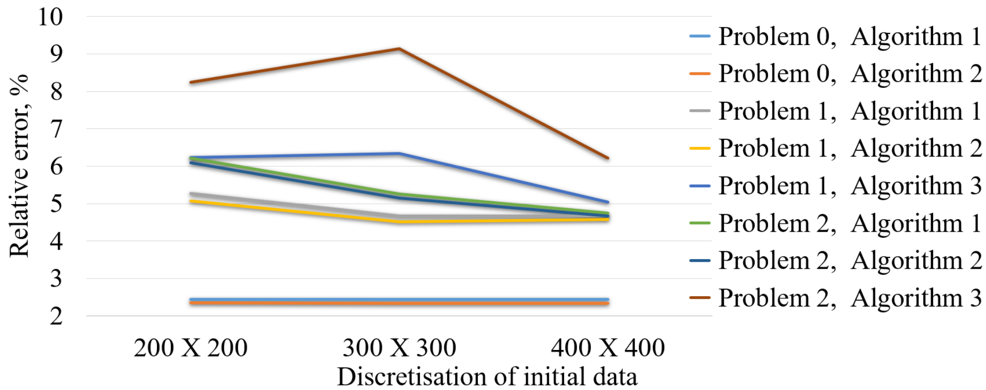

Figure 1: The dependence of the relative error of the field $\mathbf{w}^{(1)}$ reconstruction (vertical axis) on the discretization of initial data (horizontal axis).

We study the dependence of the relative error of reconstruction (in %) on the discretization of initial data — values of the operators $\mathcal{P}^{(2)}_{k2}$, $k = 0, 1, 2$ (Problem 1), $\mathcal{P}^{(0)}_{k2}$, $k = 0, 1, 2$ (Problem 2) and $\mathcal{P}^{(j)}_{02}$, $j = 0, 1, 2$ (Problem 0).

Test 1. The test results of the field $\mathbf{w}^{(1)}$ reconstruction are presented in Figure 1. Figure 1 demonstrates the performance of all proposed algorithms when solving Problems 1 and 2. However, the results of using Algorithms 1 and 2 are better than using Algorithm 3.

Figure 2 demonstrates the components of $\mathbf{w}^{(1)}$ (line a), their best approximations when solving Problem 0 using Algorithm 2 (line b) and when solving Problem 1 using Algorithms 2 (line c) and 3 (line d). The reconstruction results well present the behavior of $\mathbf{w}^{(1)}$ within its support. Outside the support of field, minor artifacts arise.

Test 2. The results of recovering the field $\mathbf{w}^{(2)}$ using Algorithms 1 and 2 are close, so in Table 1 we present only the results for Algorithm 2. If the optimal value of the degree N of the basis polynomials in the SV-decomposition is less than 50, its value is given in brackets. For example, the notation 1.61 (32) means that for a given discretization the smallest error is 1.61%, and it is achieved at $N = 32$.

Table 1: Dependence of field reconstruction accuracy $\mathbf{w}^{(2)}$ on discretization

Discretization by (s, θ)		200×200	300×300	400×400
Problem 0	Algorithm 2	0.36	0.36	0.36
Problem 1	Algorithm 2	0.39	0.36	0.36
	Algorithm 3	1.61 (32)	1.69 (32)	1.05 (37)
Problem 2	Algorithm 2	0.53 (46)	0.37	0.36
	Algorithm 3	2.41 (27)	2.72 (32)	1.57 (37)

Test 3. When solving all the problems of the field $\mathbf{w}^{(3)}$ restoration, Algorithms 1 and 2 demonstrated a restoration error in the range of 0.08–0.1%. When implementing Algorithm 3 to solve Problem 1, with increasing discretization, the recovery error changes from 0.23% to 0.19%. While when solving Problem 2 using Algorithm 3, the recovery error changes from 0.51% to 0.31%.

Conclusions. The results of numerical simulations demonstrate the performance of all proposed algorithms when solving Problems 1 and 2. However, the results of using Algorithms 1 and 2 are better than using Algorithm 3. For comparison, the results of reconstruction of

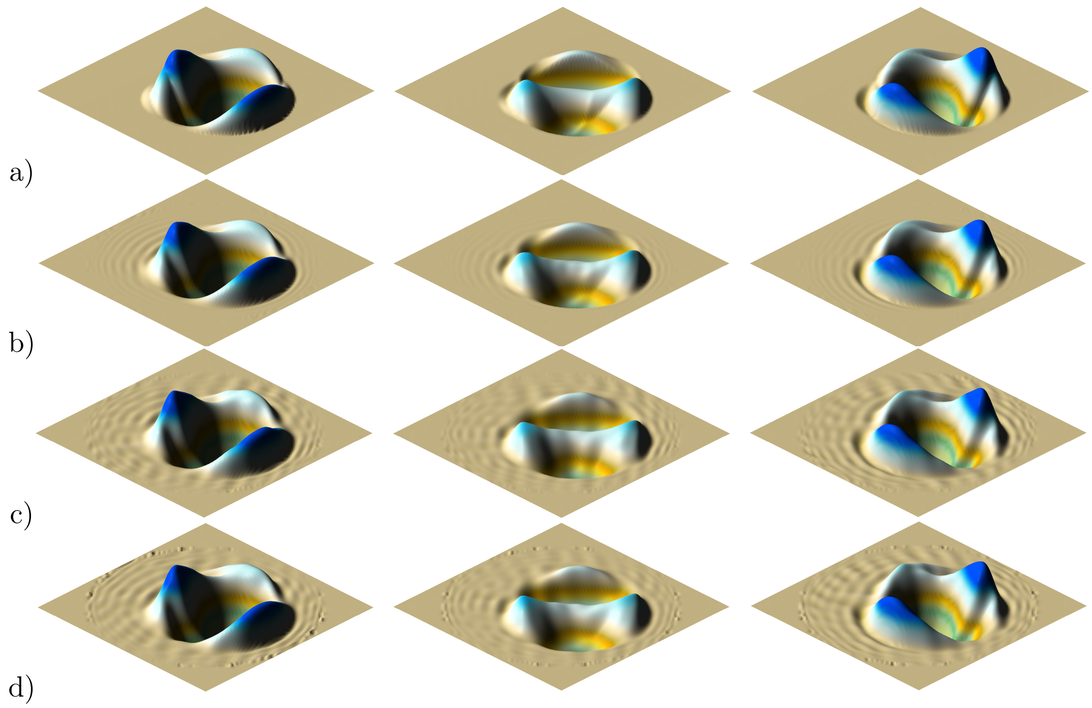

Figure 2: The components of field $\mathbf{w}^{(1)}$ (line a), their best approximations when solving Problem 0 using Algorithm 2 (line b) and when solving Problem 1 using Algorithms 2 (line c) and 3 (line d). The discretization of initial data is 400×400.

the fields $\mathbf{w}^{(l)}$, $l = 1, 2, 3$ from values of the ray transforms $\mathcal{P}_{02}^{(j)}$, $j = 0, 1, 2$ (Problem 0) are presented. It is evident that for the smoother fields $\mathbf{w}^{(l)}$, $l = 2, 3$ Algorithms 1 and 2 demonstrate reliable accuracy of solving Problems 1 and 2. With large data discretization, the accuracy coincides with the accuracy of solving Problem 0. This is probably due to a smaller error in calculating the numerical derivatives for these fields.

Conclusion

The article is devoted to the justification, development and implementation of the algorithms for restoring a symmetric 2-tensor field $\mathbf{w}$ from values of the ray transforms $\mathcal{P}_{k2}^{(0)}\mathbf{w}$ or $\mathcal{P}_{k2}^{(2)}\mathbf{w}$ of its moments $k = 0, 1, 2$. It was demonstrated that a symmetric 2-tensor field is uniquely reconstructed from the longitudinal or transverse ray transforms with the weight t^k, $k = 0, 1, 2$. The proposed algorithms for restoring a symmetric 2-tensor field are based on the properties of the ray transforms $\mathcal{P}_{k2}^{(0)}\mathbf{w}$, $\mathcal{P}_{k2}^{(2)}\mathbf{w}$, $k = 0, 1, 2$ established in the current article. The main numerical method is the SV-decompositions of the Radon transform $\mathcal{R}$ of functions and the ray transforms $\mathcal{P}_{02}^{(j)}$, $j = 0, 1, 2$ of 2-tensor fields. Numerical experiments were carried out aimed at studying the influence on the accuracy of the symmetric 2-tensor field reconstruction of such factors as discretization of the initial data and the smoothness of the field. Tests demonstrate the advantage of Algorithms 1 and 2 over Algorithm 3. When discretizing data is 300×300 or 400×400 for fields from C^l, $l \geqslant 2$, Algorithms 1 and 2 demonstrate the accuracy of solving Problems 1 and 2 coincides with the accuracy of solving Problem 0.

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Ivan Evgenyevich Svetov,
Sobolev Institute of Mathematics,
4, Koptyug Av., Novosibirsk 630090, Russia,
Novosibirsk State University,
1, Pirogova St., Novosibirsk 630090, Russia,
Email: `svetovie@math.nsc.ru`

Anna Petrovna Polyakova,
Sobolev Institute of Mathematics,
4, Koptyug Av., Novosibirsk 630090, Russia,
Novosibirsk State University,
1, Pirogova St., Novosibirsk 630090, Russia,
Email: `apolyakova@math.nsc.ru`

Svetlana Vasilievna Maltseva,
Sobolev Institute of Mathematics,
4, Koptyug Av., Novosibirsk 630090, Russia,
Novosibirsk State University,
1, Pirogova St., Novosibirsk 630090, Russia,
Email: `maltsevasv@math.nsc.ru`

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