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STRUCTURE AND DIFFERENTIAL PROPERTIES OF 2D TENSOR FIELDS OF SMALL RANKS

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Abstract The Radon transform generates many other integral transforms in integral geometry and tensor tomography. Together with complicated geometrical objects, the weighted ray and Radon transforms arise. Usually symmetric tensor fields are considered as the object should be reconstructed. We consider complete, symmetric and difference non-symmetric tensor fields of small ranks as the objects for applications in integral geometry and tensor tomography. The structure and differential properties of such fields are investigated. We establish a decomposition theorem for a complete tensor field, properties and attributes of solenoidal and potential fields.

Key words: tensor field, gradient, divergence, symmetric tensor field, potential field, solenoidal field.

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1 Introduction

The well known [1] Radon transform $\mathcal{R}$ over a function $\varphi(x)$, $x = (x^1, x^2) \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$, $s \in \mathbb{R}$,

$$(\mathcal{R}\varphi)(\theta, s) = \int\limits_{-\infty}^{\infty} \varphi(s\xi(\theta) + t\eta(\theta))dt, \tag{1}$$

generates numerous generalizations in tomography and integral geometry [2]–[6]. In (1) we use designations $\xi = (\xi^1, \xi^2) = (\cos\theta, \sin\theta)$, $\xi^\perp = \eta = (\eta^1, \eta^2) = (-\sin\theta, \cos\theta)$ for normal and direction vectors of a straight line $L_{\xi,s} \in \mathbb{R}^2$, along which the integration is carried out. The line is defined parametrically, $x = s\xi + t\eta$, or by the normal equations, $\langle \xi, x\rangle - s = 0$. Here $\langle \cdot\, , \cdot\rangle$ is the inner product in $\mathbb{R}^2$. By B, ∂B we designate the unit disk and the unit circle.

In mathematical models of 2D tomography of symmetric tensor fields [5] values of the ray transforms are used as initial data. The ray transforms $\mathcal{P}_m^{(j)}$, $m \geq 0$ integer, $j = 0, \ldots, m$, act on symmetric m-tensor fields $w(x)$, $x = s\xi + t\eta$, and convert them into functions $g_m^{(j)}(\xi(\theta), s)$, according to the rule

$$(\mathcal{P}_m^{(j)} w)(\xi, s) = \int\limits_{-\infty}^{\infty} w_{i_1\ldots i_m}(s\xi + t\eta)\, \xi^{i_1} \ldots \xi^{i_j} \eta^{i_{j+1}} \ldots \eta^{i_m} dt = g_m^{(j)}(\xi, s). \tag{2}$$

In (2) and formulas below we use the Einstein rule according to which by repeating above and below eponymous indices the summation from 1 to 2 is effected.

In recent years, the interest of researchers attracted to the ray transforms over moments of tensor fields

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$$(\mathcal{P}^{(j)}_{km}w)(\xi,s)=\int\limits_{-\infty}^{\infty}t^k w_{i_1\dots i_m}(s\xi+t\eta)\,\xi^{i_1}\dots\xi^{i_j}\eta^{i_{j+1}}\dots\eta^{i_m}dt=g^{(j)}_{km}(\xi,s), \tag{3}$$

As a rule, the researchers are talking about symmetric tensor fields, but we would like to use the fields belonging to three different classes. They are the complete tensor fields $w^{(c)}$, symmetric tensor fields $w^{(s)}$ and their difference non-symmetric fields (later on "difference fields") $w^{(cs)} := w^{(c)} - w^{(s)}$.

The ray transforms of the fields $w^{(c)}$, $w^{(cs)}$ in context of integral geometry or tensor tomography were rarely considered previously. With some exceptions (see [9]), the longitudinal ray transforms of symmetric tensor fields or their moments are usually considered. Note that the structure, geometric and differential properties of symmetric tensor fields, especially those defined on the plane, are well studied [3], [5], [7], [8], which cannot be said about the fields $w^{(c)}$, $w^{(cs)}$. We restrict the purpose of this work by a study of the structure, differential properties and relationships of the tensor fields $w^{(c)}$, $w^{(s)}$, $w^{(cs)}$ of small ranks. The research results are expected to be used in constructing inversion procedures and formulas, developing approaches, constructive methods and algorithms aimed at the problem of recovering tensor fields of types $w^{(c)}$, $w^{(cs)}$ from the ray transforms over moments of tensor fields (3).

Section 1 provides introductory notes and necessary definitions. In the next sections we consider tensor fields of small rank defined on the plane, belonging to three different classes. We distinguish between a complete, symmetric, and difference field. The structure and relationships of the fields of ranks 1–4 are studied in detail. We establish differential and geometric properties, signs of potentiality and solenoidality. The decomposition theorem for the complete tensor field $w^{(c)}$ is proved.

2 Definitions and preliminaries

Let us recall the notations and definitions of certain sets and functional spaces. $B=\{(x,y)\in\mathbb{R}^2\,\big|\,x^2+y^2<1\}$ is the unit disk with the boundary (unit circle) $\partial B=\{(x,y)\in\mathbb{R}^2\,\big|\,x^2+y^2=1\}$; $\mathbb{S}^1$ is a set of vectors of unit length, $\mathbb{S}^1=\{\xi\in\mathbb{R}^2\,|\,|\xi|=1\}$; $T^m(B)$ $(S^m(B))$ is a set of given in B (symmetric) tensor fields of rank m; $C^l(B)$ is a functional space of continuously differentiable including their derivatives up to the order l functions, defined in B; $L_2(B)$ is a space of integrable with square in B functions; $H^l(B)$ $(H^l_0(B))$ is Sobolev space of integrable with square, together with their derivatives up to the order l, l integer, $l\geq 0$, defined in B functions (vanishing on the boundary ∂B along with their derivatives up to the order $l-1$), $H^0(B)\equiv L_2(B)$.

The definitions of the functional spaces can be easily transferred to the spaces of tensor fields by applying them to each component. These are the spaces $C^l(T^m(B))$, $(C^l(S^m(B)))$ $H^l_0(T^m(B))$, $(H^l_0(S^m(B)))$ of the (symmetric) m-tensor fields, l,m integers, $l,m\geq 0$. Below the symbol B in the notation of the listed spaces will be omitted. Moreover, the most part of further considerations do not need in the values of a tensor field on the boundary, so the concrete domains of defining tensor field are not important. We consider m-tensor fields $R^{(c)},P^{(c)},Q^{(c)}\in H^l_0(T^m(B))$, $l,m\geq 0$, with components $R^{(c)}_{i_1\dots i_m}$, $P^{(c)}_{i_1\dots i_m}$, $Q^{(c)}_{i_1\dots i_m}$, $i_1,\dots i_m=1,2$. The operators $\nabla,\nabla^\perp : H^l_0(T^m)\to H^{l-1}(T^{m+1})$, $l\geq 1$, act on the complete fields, $R^{(c)}\in H^l_0(T^m(B))$, $P^{(c)},Q^{(c)}\in H^{l-1}(T^{m+1}(B))$, $l\geq 1$, $m\geq 0$, according to the rules

$$P^{(c)}_{i_1\dots i_m j}:=\big(\nabla R^{(c)}\big)_{i_1\dots i_m j}=\frac{\partial R^{(c)}_{i_1\dots i_m}}{\partial x^j},\quad Q^{(c)}_{i_1\dots i_m j}:=\big(\nabla^\perp R^{(c)}\big)_{i_1\dots i_m j}=(-1)^j\frac{\partial R^{(c)}_{i_1\dots i_m}}{\partial x^{3-j}}.$$

The operators ∇, $\nabla^\perp$ are non-commutative, $\nabla(\nabla^\perp)R^{(c)}\neq\nabla^\perp(\nabla)R^{(c)}$.

An m-tensor field $P^{(c)}$ is *potential* if there exists an $(m-1)$-tensor field $R^{(c)}$ such that $P^{(c)} = \nabla R^{(c)}$.

Designations d, $\mathrm{d}^\perp$ are used for the operators of inner differentiation and inner orthogonal differentiation, mapping a symmetric m-tensor field w into symmetric tensor fields $u^{(s)}$, $v^{(s)}$ of rank $m+1$. The image of the operator of inner differentiation $\mathrm{d} : H^l_0(S^m) \to H^{l-1}(S^{m+1})$, $l \geq 1$, is the field $u^{(s)}$ with the following components:

$$u^{(s)}_{i_1\ldots i_m j} := (\mathrm{d}w^{(s)})_{i_1\ldots i_m j} = \frac{1}{m+1}\Big(\frac{\partial w^{(s)}_{i_1\ldots i_m}}{\partial x^j} + \sum_{k=1}^{m} \frac{\partial w^{(s)}_{i_1\ldots i_{k-1} j i_{k+1}\ldots i_m}}{\partial x^{i_k}}\Big). \tag{4}$$

The operator of inner orthogonal differentiation $\mathrm{d}^\perp$ provides the field $v^{(s)}$ according to the rule

$$v^{(s)}_{i_1\ldots i_m j} := (\mathrm{d}^\perp w^{(s)})_{i_1\ldots i_m j} = \frac{1}{m+1}\Big((-1)^j \frac{\partial w^{(s)}_{i_1\ldots i_m}}{\partial x^{3-j}} + \sum_{k=1}^{m} (-1)^{i_k} \frac{\partial w^{(s)}_{i_1\ldots i_{k-1} j i_{k+1}\ldots i_m}}{\partial x^{3-i_k}}\Big), \tag{5}$$

$\mathrm{d}^\perp : H^l_0(S^m) \to H^{l-1}(S^{m+1})$, $l \geq 1$. The operators d and $\mathrm{d}^\perp$ are commutative,

$$(\mathrm{d}\,\mathrm{d}^\perp) w^{(s)} = (\mathrm{d}^\perp\,\mathrm{d}) w^{(s)}.$$

The divergence and orthogonal divergence δ, $\delta^\perp : H^l_0(T^m) \to H^{l-1}(T^{m-1})$, $l \geq 1$, act on the complete m-tensor field $w^{(c)}$,

$$u^{(c)}_{i_1\ldots i_{m-1}} := (\delta w^{(c)})_{i_1\ldots i_{m-1}} = \frac{\partial w^{(c)}_{i_1\ldots i_{m-1} j}}{\partial x^j} \equiv \frac{\partial w^{(c)}_{i_1\ldots i_{m-1} 1}}{\partial x^1} + \frac{\partial w^{(c)}_{i_1\ldots i_{m-1} 2}}{\partial x^2},$$

$$v^{(c)}_{i_1\ldots i_{m-1}} := (\delta^\perp w^{(c)})_{i_1\ldots i_{m-1}} = (-1)^j \frac{\partial w^{(c)}_{i_1\ldots i_{m-1} j}}{\partial x^{3-j}} \equiv -\frac{\partial w^{(c)}_{i_1\ldots i_{m-1} 1}}{\partial x^2} + \frac{\partial w^{(c)}_{i_1\ldots i_{m-1} 2}}{\partial x^1},$$

and produce the tensor fields $u^{(c)}$, $v^{(c)}$ of rank $m-1$.

A tensor field $R^{(c)}$ of rank $m \geq 1$ is *solenoidal* if its divergence is equal to zero, $\delta R^{(c)} = \mathbb{O}$.

It is known [5] that a symmetric tensor field of rank m can be decompose to the sum of $(m+1)$ symmetric tensor fields generated by potentials $\psi^{(j)}$, $j = 0, 1, \ldots, m$,

$$w^{(s)} = \sum_{j=0}^{m} u^{(s)j} \equiv \sum_{j=0}^{m} (\mathrm{d}^\perp)^{m-j} \mathrm{d}^j \psi^{(j)},$$

where the operators d, $\mathrm{d}^\perp$ are defined by the formulas (4), (5). The potentials $\psi^{(j)}$ generate m-tensor fields $u^{(s)j}$, $j = 0, \ldots, m$.

The following assertions related to the main properties of the complete tensor fields $w^{(c)}$. Proofs of them are based on the results obtained in the next sections. Thus the theorems are direct sequences of established in Sections 2 and 3 statements.

Theorem 2.1. *Let in the unit disk B a complete C^l-smooth m-tensor field $w^{(c)}$, $l \geq 1$, $m \leq 4$, be given.*
1. The field $w^{(c)}$ is solenoidal, $\delta w^{(c)} = \mathbb{O}$, if and only if it has a form $\nabla^\perp u^{(c)}$, where $u^{(c)}$ is certain field of rank $(m-1)$.
2. The field $w^{(c)}$ is potential, $w^{(c)} = \nabla u^{(c)}$ where $u^{(c)}$ is a field of rank $(m-1)$, if and only if the relation $\delta^\perp w^{(c)} = \mathbb{O}$ is fairly.

Theorem 2.2. *Let in the unit disk B a complete C^l-smooth m-tensor field $w^{(c)}$, $l \geq 1$, $m \leq 4$, and the operator $\mathcal{G}$ consisting of an arbitrary composition of the operators ∇, $\nabla^\perp$ acting on a potential φ, be given.*
1. For a solenoidal field $w^{(c)} = \nabla^\perp \mathcal{G}\varphi$ the relation $\delta^\perp w^{(c)} = \mathcal{G}(\Delta\varphi)$ is valid.
2. For a potential field $w^{(c)} = \nabla \mathcal{G}\varphi$ the relation $\delta w^{(c)} = \mathcal{G}(\Delta\varphi)$ is valid.

The result below is analogous to the decomposition theorem for a symmetric tensor field [5].

Theorem 2.3. *Let in the unit disk B a complete C^l-smooth m-tensor field $w^{(c)}$, $1 \le m \le 4$, be given. Then there exist $(m-1)$-tensor fields $w_p^{(c)}, w_s^{(c)}$ and potentials $\varphi^{(j)}$, $j = 1, \dots, N$, $N = 2^m$, such that the field $w^{(c)}(m)$ is decomposed to the sum of the potential and solenoidal parts,*

$$w^{(c)} = \nabla w_p^{(c)} + \nabla^{\perp} w_s^{(c)} = \sum_{k=1}^{N} \mathcal{G}_k \varphi^{(k)}.$$

The operators $\mathcal{G}_k$, $k = 1, \dots, N$, are the compositions of m operators $\nabla, \nabla^{\perp}$ in certain order, acting on the potentials $\varphi^{(k)}$, $k = 1, \dots, N$ ($N = 2^m$ is a number of allocations from 2 elements by m with recurrences).

3 Structure of tensor fields of ranks 1–3

3.1 Vector fields

Let in B the potentials $\varphi, \psi \in H_0^l$, $l \ge 1$, be given. We construct the potential vector field $u = \nabla\varphi \equiv \mathrm{d}\varphi \in H^{l-1}(T) \equiv H^{l-1}(S)$,

$$u = \nabla\varphi \equiv \mathrm{d}\varphi, \quad \nabla\varphi = (v_1, v_2) = \Big(\frac{\partial\varphi}{\partial x^1}, \frac{\partial\varphi}{\partial x^2}\Big),$$

and the solenoidal vector field $v = \nabla^{\perp}\psi \equiv \mathrm{d}^{\perp}\psi \in H^{l-1}(T) \equiv H^{l-1}(S)$,

$$v = \nabla^{\perp}\psi \equiv \mathrm{d}^{\perp}\psi, \quad \nabla^{\perp}\psi = \Big(-\frac{\partial\psi}{\partial x^2}, \frac{\partial\psi}{\partial x^1}\Big).$$

For vector fields the complete and symmetric fields are identical, and the differences between them are identically equal to zero. We use the designations: $w^{(c)} = \nabla\varphi + \nabla^{\perp}\psi$, $w^{(s)} = \mathrm{d}\varphi + \mathrm{d}^{\perp}\psi$.

3.2 Tensor fields of rank 2

Let $w = (w_i) \in H_0^l(T)$, $l \ge 1$, be a complete vector field. The gradient $\nabla : H_0^l(T) \to H^{l-1}(T^2)$ of the vector field w is defined in the following way: $(\nabla w)_{ij}^{(c)} = \Big(\dfrac{\partial w_i^{(c)}}{\partial x^j}\Big)$. The opthogonal gradient $\nabla^{\perp} : H_0^l(T) \to H^{l-1}(T^2)$ of the vector field w is defined as $(\nabla^{\perp} w)_{ij}^{(c)} = (-1)^j \dfrac{\partial w_i^{(c)}}{\partial x^{3-j}}$. The operator of inner differentiation $\mathrm{d} : H_0^l(S) \to H^{l-1}(S^2)$ acts on the field w in the following way: $(\mathrm{d}w)_{ij} = \dfrac{1}{2}\Big(\dfrac{\partial w_i}{\partial x^j} + \dfrac{\partial w_j}{\partial x^i}\Big)$. The operator of inner orthogonal differentiation $\mathrm{d}^{\perp} : H_0^l(S) \to H^{l-1}(S^2)$ acts on the field w as follows: $(\mathrm{d}^{\perp} w)_{ij} = \dfrac{1}{2}\Big((-1)^j \dfrac{\partial w_i}{\partial x^{3-j}} + (-1)^i \dfrac{\partial w_j}{\partial x^{3-i}}\Big)$.

Let's represent 2-tensor fields through potentials $\varphi, \phi, \chi, \psi \in H_0^l$, $l \ge 2$,

$$\nabla^2\varphi = \big(\varphi_{;1;1}, \varphi_{;1;2}, \varphi_{;2;1}, \varphi_{;2;2}\big) = \Big(\frac{\partial^2\varphi}{\partial x^2}, \frac{\partial^2\varphi}{\partial x\partial y}, \frac{\partial^2\varphi}{\partial x\partial y}, \frac{\partial^2\varphi}{\partial y^2}\Big) = \mathrm{d}^2\varphi,$$

$$\nabla(\nabla^\perp\phi) = \Big(-\frac{\partial^2\chi}{\partial x\partial y}, -\frac{\partial^2\chi}{\partial y^2}, \frac{\partial^2\chi}{\partial x^2}, \frac{\partial^2\chi}{\partial x\partial y}\Big),$$

$$\nabla^\perp(\nabla\chi) = \Big(-\frac{\partial^2\phi}{\partial x\partial y}, \frac{\partial^2\phi}{\partial x^2}, -\frac{\partial^2\phi}{\partial y^2}, \frac{\partial^2\phi}{\partial x\partial y}\Big),$$

$$(\nabla^\perp)^2\psi = \Big(\frac{\partial^2\psi}{\partial y^2}, -\frac{\partial^2\psi}{\partial x\partial y}, -\frac{\partial^2\psi}{\partial x\partial y}, \frac{\partial^2\psi}{\partial x^2}\Big) = (\mathrm{d}^\perp)^2\psi.$$

It is well known the decomposition of a vector field [5] on potential and solenoidal parts, generated by the operators d, $\mathrm{d}^\perp$. On its base we obtain the representation of a symmetric 2-tensor field w^s_{ij} through the potentials $\varphi, \phi \equiv \chi, \psi \in H^l_0$, $l \geq 2$,

$$u_{ij} = (\mathrm{d}^2\varphi)_{ij} = \frac{\partial^2\varphi}{\partial x^i\partial x^j}, \quad v_{ij} = \big((\mathrm{d}^\perp)^2\psi\big)_{ij} = (-1)^{i+j}\frac{\partial^2\psi}{\partial x^{3-i}\partial x^{3-j}},$$

$$\widetilde{u}_{ij} = \big(\mathrm{d}^\perp(\mathrm{d}\chi)\big)_{ij} = \frac{1}{2}\Big((-1)^j\frac{\partial^2\chi}{\partial x^{3-j}\partial x^i} + (-1)^i\frac{\partial^2\chi}{\partial x^{3-i}\partial x^j}\Big) = \big(\mathrm{d}(\mathrm{d}^\perp\phi)\big)_{ij}.$$

The components of this field in more details are:

$$\widetilde{u}_{11} = -\frac{\partial\chi}{\partial x^1\partial x^2}, \quad \widetilde{u}_{22} = \frac{\partial\chi}{\partial x^1\partial x^2}, \quad \widetilde{u}_{12} = \frac{1}{2}\Big(\frac{\partial^2\chi}{\partial(x^1)^2} - \frac{\partial^2\chi}{\partial(x^2)^2}\Big).$$

Direct computations show that the operators d and $\mathrm{d}^\perp$ are commutative, $\mathrm{d}(\mathrm{d}^\perp\chi) = \mathrm{d}^\perp(\mathrm{d}\chi)$.

Considering the 2-tensor field $w^{(c)} = \nabla^2\varphi + \nabla(\nabla^\perp\phi) + \nabla^\perp(\nabla\chi) + (\nabla^\perp)^2\psi$ and the symmetric field $w^{(s)} = \mathrm{d}^2\varphi + \mathrm{d}(\mathrm{d}^\perp\phi) + \mathrm{d}^\perp(\mathrm{d}\chi) + (\mathrm{d}^\perp)^2\psi$, we find their difference field $w^{(cs)}$,

$$w^{(cs)} = w^{(c)} - w^{(s)} = \big(0, \frac{1}{2}\Delta\phi, -\frac{1}{2}\Delta\phi, 0\big) + \big(0, -\frac{1}{2}\Delta\chi, \frac{1}{2}\Delta\chi, 0\big).$$

It is easy to see that both the first $\nabla(\nabla^\perp\phi) - \mathrm{d}(\mathrm{d}^\perp\phi)$ and the second $\nabla^\perp(\nabla\chi) - (\mathrm{d}^\perp\mathrm{d}\chi)$ fields are skew-symmetric and differ from each other in sign. Note that if the potentials ϕ, χ are harmonic functions, then the indicated 2-tensor fields vanish.

The results of action of the divergence and orthogonal divergence operators on the tensor fields $w^{(c)}$, $w^{(s)}$, $w^{(cs)}$ of rank 2 are established in the following statement. All listed relationships are verified directly.

Lemma 3.1. *Let the $C^2(B)$-smooth 2-tensor fields $\nabla^2\varphi$, $\nabla(\nabla^\perp\phi)$, $\nabla^\perp(\nabla\chi)$, $(\nabla^\perp)^2\psi$, $\mathrm{d}^2\varphi$, $\mathrm{d}(\mathrm{d}^\perp)^2\phi$, $(\mathrm{d}^\perp)^2\psi$, $w^{(cs1)} := \nabla(\nabla^\perp\phi) - \mathrm{d}(\mathrm{d}^\perp\phi)$, $w^{(cs2)} = -w^{(cs1)} := \nabla^\perp(\nabla\phi) - \mathrm{d}(\mathrm{d}^\perp\phi)$ be given. The following relations are valid:*

$$\delta(\nabla^\perp(\nabla\chi)) = \mathbb{O}, \qquad \delta((\nabla^\perp)^2\phi) = \mathbb{O},$$

$$\delta^\perp(\nabla^2\varphi) = \mathbb{O}, \qquad \delta^\perp(\nabla(\nabla^\perp\phi)) = \mathbb{O},$$

$$(\delta^\perp\delta)(\nabla^2\varphi) = 0 = (\delta^\perp\delta)(\mathrm{d}^2\varphi), \qquad (\delta\delta^\perp)((\nabla^\perp)^2\psi) = 0 = (\delta\delta^\perp)((\mathrm{d}^\perp)^2\psi),$$

$$(\delta^2)(\nabla(\nabla^\perp\phi)) = 0, \qquad \delta^2(\mathrm{d}\mathrm{d}^\perp\phi) = 0,$$

$$(\delta^\perp)^2(\nabla^\perp(\nabla\chi)) = 0, \qquad (\delta^\perp)^2(\mathrm{d}\mathrm{d}^\perp\phi) = 0,$$

$$\delta^2 w^{(cs1)} = -\delta^2 w^{(cs2)} = 0, \qquad (\delta^\perp)^2 w^{(cs1)} = -(\delta^\perp)^2 w^{(cs2)} = 0,$$

$$\delta^2(\nabla^2\varphi) = \Delta^2\varphi = \delta^2(\mathrm{d}^2\varphi), \qquad (\delta^\perp\delta)(\nabla(\nabla^\perp\phi)) = \Delta^2\phi,$$

$$(\delta\delta^\perp)((\nabla^\perp)\nabla\chi) = \Delta^2\chi, \qquad (\delta^\perp)^2((\nabla^\perp)^2\psi) = \Delta^2\psi = (\delta^\perp)^2((\mathrm{d}^\perp)^2\psi),$$

$$(\delta^\perp\delta)((\mathrm{d}\mathrm{d}^\perp)\phi) = \frac{1}{2}\Delta^2\phi, \qquad (\delta\delta^\perp)((\mathrm{d}\mathrm{d}^\perp)\chi) = \frac{1}{2}\Delta^2\chi,$$

$$(\delta^\perp\delta)w^{(cs1)} = -\frac{1}{2}\Delta^2\phi, \qquad (\delta\delta^\perp)w^{(cs2)} = -\frac{1}{2}\Delta^2\chi.$$

The fields $\nabla^\perp(\nabla\phi)$, $(\nabla^\perp)^2\psi$ are solenoidal, the fields $\nabla^2\varphi$, $\nabla(\nabla^\perp\phi)$ are potential.

3.3 Tensor fields of rank 3

Let $w^{(c)} = (w^{(c)}_{ij}) \in H^l_0(T^2)$, $l \geq 1$ be a complete 2-tensor field. The operator of gradient $\nabla : H^l_0(T^2) \to H^{l-1}(T^3)$ transforms the field $w^{(c)}$ according to the rule:

$$(\nabla w)^{(c)}_{ijk} = w^{(c)}_{ij;k} = \Big(\frac{\partial w^{(c)}_{ij}}{\partial x^k}\Big).$$

The operator of orthogonal gradient $\nabla^\perp : H^l_0(T^2) \to H^{l-1}(T^3)$ acts on the field $w^{(c)}$ as follows:

$$(\nabla^\perp w)^{(c)}_{ijk} = w^{(c)}_{ij;\bar{k}} = (-1)^k \frac{\partial w^{(c)}_{ij}}{\partial x^{3-k}}.$$

The operator of inner differentiation $\mathrm{d} : H^l_0(S^2) \to H^{l-1}(S^3)$ is defined as follows:

$$(\mathrm{d}w)^{(s)}_{ijk} = \frac{1}{3}\Big(\frac{\partial w^{(s)}_{ij}}{\partial x^k} + \frac{\partial w^{(s)}_{jk}}{\partial x^i} + \frac{\partial w^{(s)}_{ki}}{\partial x^j}\Big).$$

The operator of inner orthogonal differentiation $\mathrm{d}^\perp : H^l_0(S^2) \to H^{l-1}(S^3)$ acts on the field w in the following way:

$$(\mathrm{d}^\perp w)^{(s)}_{ijk} = \frac{1}{3}\Big((-1)^k \frac{\partial w_{ij}}{\partial x^{3-k}} + (-1)^i \frac{\partial w_{jk}}{\partial x^{3-i}} + (-1)^j \frac{\partial w_{ki}}{\partial x^{3-j}}\Big).$$

Here and below we introduce the notation $\partial^m_{kl} := \frac{\partial^m}{\partial x^k \partial y^l}$, $0 \leq k, l \leq m$, $k + l = m$, where m is the rank of the field, for the operator of partial differentiation over potential. Let's fix an order of location for the components of 3-tensor field $u^{(c)}_{ijk}$, generated by the potential φ.

$$\begin{aligned}
w^{(c)} &= \big(w^{(c)}_{111}, w^{(c)}_{112}, w^{(c)}_{121}, w^{(c)}_{122}, w^{(c)}_{211}, w^{(c)}_{212}, w^{(c)}_{221}, w^{(c)}_{222}\big),\\
u^{(c)} &= \big(\partial^3_{30}\varphi, \partial^3_{21}\varphi, \partial^3_{21}\varphi, \partial^3_{12}\varphi, \partial^3_{21}\varphi, \partial^3_{12}\varphi, \partial^3_{12}\varphi, \partial^3_{03}\varphi\big).
\end{aligned}$$

We represent eight 3-tensor fields according to this order in operator form, omitting the signs of potentials generating the corresponding tensor fields.

$$\begin{aligned}
&\nabla(\nabla^2) = \big(\partial^3_{30}, \partial^3_{21}, \partial^3_{21}, \partial^3_{12}, \partial^3_{21}, \partial^3_{12}, \partial^3_{12}, \partial^3_{03}\big) = \mathrm{d}^3,\\
&\nabla(\nabla\nabla^\perp) = \big(-\partial^3_{21}, -\partial^3_{12}, -\partial^3_{12}, -\partial^3_{03}, \partial^3_{30}, \partial^3_{21}, \partial^3_{21}, \partial^3_{12}\big),\\
&\nabla(\nabla^\perp\nabla) = \big(-\partial^3_{21}, -\partial^3_{12}, \partial^3_{30}, \partial^3_{21}, -\partial^3_{12}, -\partial^3_{03}, \partial^3_{21}, \partial^3_{12}\big),\\
&\nabla(\nabla^\perp)^2 = \big(\partial^3_{12}, \partial^3_{03}, -\partial^3_{21}, -\partial^3_{12}, -\partial^3_{21}, -\partial^3_{12}, \partial^3_{30}, \partial^3_{21}\big),\\
&\nabla^\perp(\nabla^2) = \big(-\partial^3_{21}, \partial^3_{30}, -\partial^3_{12}, \partial^3_{21}, -\partial^3_{12}, \partial^3_{21}, -\partial^3_{03}, \partial^3_{12}\big),\\
&\nabla^\perp(\nabla\nabla^\perp) = \big(\partial^3_{12}, -\partial^3_{12}, \partial^3_{03}, -\partial^3_{12}, -\partial^3_{21}, \partial^3_{30}, -\partial^3_{12}, \partial^3_{21}\big),\\
&\nabla^\perp(\nabla^\perp\nabla) = \big(\partial^3_{12}, -\partial^3_{21}, -\partial^3_{21}, \partial^3_{30}, \partial^3_{03}, -\partial^3_{12}, -\partial^3_{12}, \partial^3_{21}\big),\\
&\nabla^\perp(\nabla^\perp)^2 = \big(-\partial^3_{03}, \partial^3_{12}, \partial^3_{12}, -\partial^3_{21}, \partial^3_{12}, -\partial^3_{21}, -\partial^3_{21}, \partial^3_{30}\big) = (\mathrm{d}^\perp)^3.
\end{aligned} \tag{6}$$

Let's define the operator $\mathcal{H}$, consisting of m operators $\nabla, \nabla^\perp$ (or $\mathrm{d}, \mathrm{d}^\perp$) in arbitrary order. We fix the order of the operators in $\mathcal{H}$ and set the correspondence between the indicated m operators and the permutation $(1, 2, \ldots, m)$. Changing the operators $\nabla, \nabla^\perp$ (or $\mathrm{d}, \mathrm{d}^\perp$) on the operators $\delta, \delta^\perp$, we obtain corresponding sequence $(1, 2, \ldots, m)$, containing these operators in the same order as the operators $\nabla, \nabla^\perp$ (or $\mathrm{d}, \mathrm{d}^\perp$). The last step consists in changing the order of the sequence of the operators $\delta, \delta^\perp$ from $(1, 2, \ldots, m)$ into $(m, m-1, \ldots, 1)$. The operator $\mathcal{H}$ containing the fixed sequence of the operators $\nabla, \nabla^\perp$ is designated as $\mathcal{H}(nab)_{(1,2,\ldots,m)}$. By $\mathcal{H}(nab+sym)_{(1,2,\ldots,m)}$ we designate the operator $\mathcal{H}$ containing the fixed sequence of the operators $\mathrm{d}, \mathrm{d}^\perp$. In this paper we apply the operators $\mathcal{H}(nab)_{(1,2,\ldots,m)}$ and $\mathcal{H}(nab+sym)_{(1,2,\ldots,m)}$

only to functions (potentials). The corresponding "inverse" sequence of the operators $\delta, \delta^\perp$ is designated as $\mathcal{H}(del)_{(m,m-1,\dots,1)}$. We define an operator $\mathcal{G}$ as the sequence of $(m-1)$ operators $\nabla, \nabla^\perp$ in an arbitrary order.

The results of action of the divergence and orthogonal divergence operators on the tensor fields $w^{(c)}$ are given in the following assertion, which is checked directly.

Proposition 3.1. *Let in the unit disk B the C^l-smooth, $l \geq 3$, 3-tensor fields (6). The operators $\mathcal{G}$ and $\mathcal{H}(nab)_{(1,2,3)}$ consist of two and three operators $\nabla, \nabla^\perp$ in an arbitrary order, respectively. Then the following relations are valid:*

1. For the fields of a form $\nabla\mathcal{G}\varphi$ one have $\delta^\perp\nabla\mathcal{G}\varphi = \mathbb{O}$ and $\delta\nabla\mathcal{G}\varphi = \mathcal{G}\Delta\varphi$.

2. For the fields of a form $\nabla^\perp\mathcal{G}\varphi$ the equalities $\delta\nabla^\perp\mathcal{G}\varphi = \mathbb{O}$ and $\delta^\perp\nabla^\perp\mathcal{G}\varphi = \mathcal{G}\Delta\varphi$ hold.

3. The fields of the form $\nabla\mathcal{G}\varphi$ are potential, the fields of the form $\nabla^\perp\mathcal{G}\varphi$ are solenoidal.

4. For the fixed operator $\mathcal{H}(nab)_{(1,2,3)}$ there exists the unique operator consisting of 3 operators $\delta, \delta^\perp$, namely $\mathcal{H}(del)_{(3,2,1)}$, the action of which on the operator $\mathcal{H}(nab)_{(1,2,3)}$ gives the operator Δ^3. The action of any other operator consisting of three operators $\delta, \delta^\perp$ in other order gives zero.

Symmetric 3-tensor fields $w^{(s)}$ have four various components w_{111}, $w_{112} = w_{121} = w_{211}$, $w_{122} = w_{212} = w_{221}$, w_{222}. They can be written in the form

$$\begin{aligned}
&w = \big(w_{111}, w_{112}, w_{122}, w_{222}\big),\\
&\mathrm{d}^3\varphi = \big(\partial^3_{30}\varphi, \partial^3_{21}\varphi, \partial^3_{12}\varphi, \partial^3_{03}\varphi\big),\\
&3\,\mathrm{d}^2(\mathrm{d}^\perp\phi) = \big(-3\partial^3_{21}\phi, (\partial^3_{30}\phi - 2\partial^3_{12}\phi), (-\partial^3_{03}\phi + 2\partial^3_{21}\phi), 3\partial^3_{12}\phi\big),\\
&3\,\mathrm{d}(\mathrm{d}^\perp)^2\chi = \big(3\partial^3_{12}\chi, (\partial^3_{03}\chi - 2\partial^3_{21}\chi), (\partial^3_{30}\chi - 2\partial^3_{12}\chi), 3\partial^3_{21}\chi\big),\\
&(\mathrm{d}^\perp)^3\psi = \big(-\partial^3_{03}\psi, \partial^3_{12}\psi, -\partial^3_{21}\psi, \partial^3_{30}\big).
\end{aligned}$$

Notice, that $\mathrm{d}^3 = \nabla^3$, and $(\mathrm{d}^\perp)^3 = (\nabla^\perp)^3$ (see proposition 2.1), so we restrict ourselves to considerations of action of the divergence and orthogonal divergence on the fields $\mathrm{d}^2\mathrm{d}^\perp$ and $\mathrm{d}(\mathrm{d}^\perp)^2$. Direct calculations lead to the following result.

Lemma 3.2. *Let in the unit disk B the C^l-smooth, $l \geq 3$, 3-tensor fields $(\mathrm{d}^2\mathrm{d}^\perp)\varphi$, $(\mathrm{d}(\mathrm{d}^\perp)^2)\varphi$, and an operator $\mathcal{H}(nab+sym)_{(1,2,3)}$ consisting of three operators $\mathrm{d}, \mathrm{d}^\perp$ in an arbitrary up to the permutations order, be given. Then the following relations are valid:*

1. An application of the operators δ, $\delta^\perp$ to the fields $\mathrm{d}^2\mathrm{d}^\perp\varphi$, $\mathrm{d}(\mathrm{d}^\perp)^2\varphi$ gives the non-zero symmetric 2-tensor fields according to the rules

$$\delta(\mathrm{d}^2\mathrm{d}^\perp\varphi) = \delta^\perp(\mathrm{d}(\mathrm{d}^\perp)^2\varphi) = \frac{2}{3}\mathrm{d}\mathrm{d}^\perp\Delta\varphi,\ \ \delta^\perp(\mathrm{d}^2\mathrm{d}^\perp\varphi) = \frac{1}{3}\mathrm{d}^2\Delta\varphi,\ \ \delta(\mathrm{d}(\mathrm{d}^\perp)^2\varphi) = \frac{1}{3}(\mathrm{d}^\perp)^2\Delta\varphi.$$

2. The tensor fields $(\mathrm{d}^2\mathrm{d}^\perp)\varphi$, $(\mathrm{d}(\mathrm{d}^\perp)^2)\varphi$ are neither potential nor solenoidal.

3. For the fixed up to the permutations operator $\mathcal{H}(nab+sym)_{(1,2,3)}$ there exists an operator consisting of 3 operators $\delta, \delta^\perp$, namely $\mathcal{H}(del)_{(3,2,1)}$, the action of which on the operator $\mathcal{H}(nab+sym)_{(1,2,3)}$ gives the operator Δ^3 up to the constant. The action of any other operator consisting of other up to the permutations operators $\delta, \delta^\perp$ gives zero.

There are 6 nonzero 3-tensor fields $w^{(cs)}$, which are the differences of the fields $w^{(c)}$ and $w^{(s)}$. The first group of three fields, differing from each other by the permutations of their components, consists of the fields formed by action of the operators $u^{(cs1)} := \nabla^2\nabla^\perp - \mathrm{d}^2\mathrm{d}^\perp$, $u^{(cs2)} := \nabla\nabla^\perp\nabla - \mathrm{d}^2\mathrm{d}^\perp$, $u^{(cs3)} := \nabla^\perp\nabla^2 - \mathrm{d}^2\mathrm{d}^\perp$ on a potential,

$$\begin{aligned}
&3u^{(cs1)}\varphi = \big(0, -\partial^1_{10}\Delta\varphi, -\partial^1_{10}\Delta\varphi, -2\partial^1_{01}\Delta\varphi, 2\partial^1_{10}\Delta\varphi, \partial^1_{01}\Delta\varphi, \partial^1_{01}\Delta\varphi, 0\big),\\
&3u^{(cs2)}\varphi = \big(0, -\partial^1_{10}\Delta\varphi, 2\partial^1_{10}\Delta\varphi, \partial^1_{01}\Delta\varphi, -\partial^1_{10}\Delta\varphi, -2\partial^1_{01}\Delta\varphi, \partial^1_{01}\Delta\varphi, 0\big),\\
&3u^{(cs3)}\varphi = \big(0, 2\partial^1_{10}\Delta\varphi, -\partial^1_{10}\Delta\varphi, \partial^1_{01}\Delta\varphi, -\partial^1_{10}\Delta\varphi, \partial^1_{01}\Delta\varphi, -2\partial^1_{01}\Delta\varphi, 0\big).
\end{aligned} \tag{7}$$

Let the order of the components of the first field from (7) corresponds to the permutation (1 2 3 4 5 6 7 8). Then the components of the second of fields (7) are obtained by the permutation (1 2 5 6 3 4 7 8), and the components of the third field are the result of the permutation (1 5 3 7 2 6 4 8).

The second group forms three fields w_{cs} arising as a result of the action of the operators $\nabla(\nabla^{\perp})^2 - \mathrm{d}(\mathrm{d}^{\perp})^2$, $\nabla^{\perp}\nabla\nabla^{\perp} - \mathrm{d}(\mathrm{d}^{\perp})^2$, $(\nabla^{\perp})^2\nabla - \mathrm{d}(\mathrm{d}^{\perp})^2$ on a potential. These fields also differ from each other only by the permutations of the components,

$$\begin{aligned}
3v^{(cs1)}\chi &= \big(0, 2\partial^1_{01}\Delta\chi, -\partial^1_{01}\Delta\chi, -\partial^1_{10}\Delta\chi, -\partial^1_{01}\Delta\chi, -\partial^1_{10}\Delta\chi, 2\partial^1_{10}\Delta\chi, 0\big),\\
3v^{(cs2)}\chi &= \big(0, -\partial^1_{01}\Delta\chi, 2\partial^1_{01}\Delta\chi, -\partial^1_{10}\Delta\chi, -\partial^1_{01}\Delta\chi, 2\partial^1_{10}\Delta\chi, -\partial^1_{10}\Delta\chi, 0\big),\\
3v^{(cs3)}\chi &= \big(0, -\partial^1_{01}\Delta\chi, -\partial^1_{01}\Delta\chi, 2\partial^1_{10}\Delta\chi, 2\partial^1_{01}\Delta\chi, -\partial^1_{10}\Delta\chi, -\partial^1_{10}\Delta\chi, 0\big),
\end{aligned} \tag{8}$$

Let the order of the components of the first field from (8) corresponds to the permutation (1 2 3 4 5 6 7 8). Then the components of the second of the fields (8) are obtained by the permutation (1 3 2 4 5 7 6 8), and the components of the third field are the result of the permutation (1 5 2 6 3 7 4 8).

The first field from (8) can be obtained from the first field from (7) by changing the sign before the components 2, 3, 5 in the field from (7), and then by the permutation (1 4 6 2 7 3 5 8).

Corollary 3.1. *Let in the unit disk B the C^l-smooth, $l \geq 3$, defined by* (7), (8) 3*-tensor fields* $u^{(csi)}$, $v^{(csj)}$, $i, j = 1, 2, 3$. *The operators $\mathcal{H}(nab)_{(1,2,3)}$ and $\mathcal{H}(nab+sym)_{(1,2,3)}$ consist of three operators $\nabla, \nabla^{\perp}$ and $\mathrm{d}, \mathrm{d}^{\perp}$, respectively, in an arbitrary order. Then*
1. The fields (7), (8) *are neither potential nor solenoidal.*
2. The fields $u^{(csi)} := w^{(ci)} - w^{(si)}$, $i = 1, 2, 3$ (7), *$\mathcal{H}v^{(csj)} := w^{(c,j+3)} - w^{(s,j+3)}$, $j = 1, 2, 3$* (8), *are equal to zero if and only if the fields $w^{(ci)}$, $w^{(si)}$, $i = 1, \ldots, 6$ are equal to zero.*
3. For the fixed 3*-tensor field from* (7), (8) *with the corresponding operator $(\mathcal{H}(nab)_{(1,2,3)} - \mathcal{H}(nab+sym)_{(1,2,3)})$ there exists the operator consisting of* 3 *operators $\delta, \delta^{\perp}$, namely $\mathcal{H}(del)_{(3,2,1)}$, the action of which on the operator $(\mathcal{H}(nab)_{(1,2,3)} - \mathcal{H}(nab+sym)_{(1,2,3)})$ gives the operator Δ^3 up to the constant.*

4 Structure of tensor fields of rank 4

A set of the 4-tensor fields $w^{(c)}$ in a sense of different order of the operators ∇, $\nabla^{\perp}$ contains 16 (i.e. 2^4) elements. We remind that $\nabla^4 = d^4$, $(\nabla^{\perp})^4 = (\mathrm{d}^{\perp})^4$. Besides we remind the notations $\partial^4_{kl} := \frac{\partial^4}{\partial x^k \partial y^l}$, $0 \leq k, l \leq 4$, $k + l = 4$. In these notations 16 different types of 4-tensor fields can be written as follows (we restricting ourselves by writing only operators without potentials):

$$
\begin{aligned}
&\nabla^4 = \left(\partial^4_{40}, \partial^4_{31}, \partial^4_{31}, \partial^4_{22}, \partial^4_{31}, \partial^4_{22}, \partial^4_{22}, \partial^4_{13}, \partial^4_{31}, \partial^4_{22}, \partial^4_{22}, \partial^4_{13}, \partial^4_{22}, \partial^4_{13}, \partial^4_{13}, \partial^4_{04}\right), \\
&\nabla^3\nabla^\perp = \left(-\partial^4_{31}, -\partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{13}, -\partial^4_{04}, \partial^4_{40}, \partial^4_{31}, \partial^4_{31}, \partial^4_{22}, \partial^4_{31}, \partial^4_{22}, \partial^4_{22}, \partial^4_{13}\right), \\
&\nabla^2\nabla^\perp\nabla = \left(-\partial^4_{31}, -\partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{40}, \partial^4_{31}, \partial^4_{31}, \partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{13}, -\partial^4_{04}, \partial^4_{31}, \partial^4_{22}, \partial^4_{22}, \partial^4_{13}\right), \\
&\nabla^2(\nabla^\perp)^2 = \left(\partial^4_{22}, \partial^4_{13}, \partial^4_{13}, \partial^4_{04}, -\partial^4_{31}, -\partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{40}, \partial^4_{31}, \partial^4_{31}, \partial^4_{22}\right), \\
&\nabla(\nabla^\perp)\nabla^2 = \left(-\partial^4_{31}, -\partial^4_{22}, \partial^4_{40}, \partial^4_{31}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{31}, \partial^4_{22}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{31}, \partial^4_{22}, -\partial^4_{13}, -\partial^4_{04}, \partial^4_{22}, \partial^4_{13}\right), \\
&\nabla\nabla^\perp\nabla\nabla^\perp = \left(\partial^4_{22}, \partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, \partial^4_{13}, \partial^4_{04}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, \partial^4_{40}, \partial^4_{31}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{31}, \partial^4_{22}\right), \\
&\nabla(\nabla^\perp)^2\nabla = \left(\partial^4_{22}, \partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, -\partial^4_{31}, -\partial^4_{22}, \partial^4_{40}, \partial^4_{31}, \partial^4_{13}, \partial^4_{04}, -\partial^4_{22}, -\partial^4_{13}, -\partial^4_{22}, -\partial^4_{13}, \partial^4_{31}, \partial^4_{22}\right), \\
&\nabla(\nabla^\perp)^3 = \left(-\partial^4_{13}, -\partial^4_{04}, \partial^4_{22}, \partial^4_{13}, \partial^4_{22}, \partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, \partial^4_{22}, \partial^4_{13}, -\partial^4_{31}, -\partial^4_{22}, -\partial^4_{31}, -\partial^4_{22}, \partial^4_{40}, \partial^4_{31}\right), \\
&\nabla^\perp\nabla^3 = \left(-\partial^4_{31}, \partial^4_{40}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{13}, \partial^4_{22}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{13}, \partial^4_{22}, -\partial^4_{13}, \partial^4_{22}, -\partial^4_{04}, \partial^4_{13}\right), \qquad (9) \\
&\nabla^\perp\nabla^2\nabla^\perp = \left(\partial^4_{22}, -\partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{04}, -\partial^4_{13}, -\partial^4_{31}, \partial^4_{40}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{13}, \partial^4_{22}\right), \\
&\nabla^\perp\nabla\nabla^\perp\nabla = \left(\partial^4_{22}, -\partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{31}, \partial^4_{40}, -\partial^4_{22}, \partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{04}, -\partial^4_{13}, -\partial^4_{22}, \partial^4_{31}, -\partial^4_{13}, \partial^4_{22}\right), \\
&\nabla^\perp\nabla(\nabla^\perp)^2 = \left(-\partial^4_{13}, \partial^4_{22}, -\partial^4_{04}, \partial^4_{13}, \partial^4_{22}, -\partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{31}, \partial^4_{40}, -\partial^4_{22}, \partial^4_{31}\right), \\
&(\nabla^\perp)^2\nabla^2 = \left(\partial^4_{22}, -\partial^4_{31}, -\partial^4_{31}, \partial^4_{40}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{22}, \partial^4_{31}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{22}, \partial^4_{31}, \partial^4_{04}, -\partial^4_{13}, -\partial^4_{13}, \partial^4_{22}\right), \\
&(\nabla^\perp)^2\nabla\nabla^\perp = \left(-\partial^4_{13}, \partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, -\partial^4_{04}, \partial^4_{13}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, -\partial^4_{31}, \partial^4_{40}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{22}, \partial^4_{31}\right), \\
&(\nabla^\perp)^3\nabla = \left(-\partial^4_{13}, \partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, \partial^4_{22}, -\partial^4_{31}, -\partial^4_{31}, \partial^4_{40}, -\partial^4_{04}, \partial^4_{13}, \partial^4_{13}, -\partial^4_{22}, \partial^4_{13}, -\partial^4_{22}, -\partial^4_{22}, \partial^4_{31}\right), \\
&(\nabla^\perp)^4 = \left(\partial^4_{40}, -\partial^4_{13}, -\partial^4_{13}, \partial^4_{22}, -\partial^4_{13}, \partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, -\partial^4_{13}, \partial^4_{22}, \partial^4_{22}, -\partial^4_{31}, \partial^4_{22}, -\partial^4_{31}, -\partial^4_{31}, \partial^4_{40}\right).
\end{aligned}
$$

We repeat the definition of the operator $\mathcal{H}$ for the special case when it consists of 4 operators $\nabla, \nabla^\perp$ in an arbitrary order. We fix the order of the operators in $\mathcal{H}$ and set the correspondence between the indicated 4 operators and the permutation $(1,2,3,4)$. Changing the operators $\nabla, \nabla^\perp$ on the operators $\delta, \delta^\perp$, we obtain the corresponding sequence $(1,2,3,4)$, containing these operators in the same order as the operators $\nabla, \nabla^\perp$. The last step consists in changing the order of the sequence of the operators $\delta, \delta^\perp$ from $(1,2,3,4)$ into $(4,3,2,1)$. The operator $\mathcal{H}$, containing the fixed sequence of the operators $\nabla, \nabla^\perp$, we designate as $\mathcal{H}(nab)_{(1,2,3,4)}$. The operator $\mathcal{H}(nab + sym)_{(1,2,3,4)}$ is defined similarly. The corresponding "inverse" sequence of the operators $\delta, \delta^\perp$ is designated as $\mathcal{H}(del)_{(4,3,2,1)}$. The operator $\mathcal{G}$ differs from $\mathcal{H}$ only by a number of the the operators $\nabla, \nabla^\perp$ which is equal to 3.

The results of actions of the divergence and orthogonal divergence operators on tensor fields $w^{(c)}$ are given in the following assertion, which is checked directly.

Proposition 4.1. *Let in the unit disk B the C^l-smooth, $l \geq 4$, 4-tensor fields* (9). *The operators $\mathcal{G}$ and $\mathcal{H}(nab)_{(1,2,3,4)}$ consist of three and four operators $\nabla, \nabla^\perp$ in an arbitrary order, respectively. Then the following relations are valid:*

1. For fields of the form $\nabla\mathcal{G}\varphi$ the following equalities hold: $\delta^\perp\nabla\mathcal{G}\varphi = \mathbb{O}$ and $\delta\nabla\mathcal{G}\varphi = \mathcal{G}\Delta\varphi$.

2. For fields of the form $\nabla^\perp\mathcal{G}\varphi$ the following equalities hold: $\delta\nabla^\perp\mathcal{G}\varphi = \mathbb{O}$ and $\delta^\perp\nabla^\perp\mathcal{G}\varphi = \mathcal{G}\Delta\varphi$.

3. The fields of the form $\nabla\mathcal{G}\varphi$ are potential, the fields of the form $\nabla^\perp\mathcal{G}\varphi$ are solenoidal.

4. For the fixed operator $\mathcal{H}(nab)_{(1,2,3,4)}$ there exists the unique operator consisting of 4 operators $\delta, \delta^\perp$, namely $\mathcal{H}(del)_{(4,3,2,1)}$, the action of which on the operator $\mathcal{H}(nab)_{(1,2,3,4)}$ gives the operator Δ^4. The action of any other operator consisting of four operators $\delta, \delta^\perp$ in the other order gives zero.

Let's consider the symmetric 4-tensor fields $w_{ijkl}(x,y)$. Due to symmetry the following equalities hold

$$
\begin{aligned}
&w_{1112} = w_{1121} = w_{1211} = w_{2111}, \quad w_{1222} = w_{2122} = w_{2212} = w_{2221}, \\
&w_{1122} = w_{1212} = w_{1221} = w_{2112} = w_{2121} = w_{2211}.
\end{aligned}
$$

That's why we give these fields in the form $(w^{(s)}_{1111}, w^{(s)}_{1112}, w^{(s)}_{1122}, w^{(s)}_{1222}, w^{(s)}_{2222})$.

$$
\begin{aligned}
&\mathrm{d}^4 = \left(\partial^4_{40}, \partial^4_{31}, \partial^4_{22}, \partial^4_{13}, \partial^4_{04}\right),\\
&4\mathrm{d}^3\mathrm{d}^\perp = \left(-4\partial^4_{31}, \left(\partial^4_{40} - 3\partial^4_{22}\right), 2\left(\partial^4_{31} - \partial^4_{13}\right), \left(3\partial^4_{22} - \partial^4_{04}\right), 4\partial^4_{13}\right),\\
&6\mathrm{d}^2(\mathrm{d}^\perp)^2 = \left(6\partial^4_{22}, 3\left(\partial^4_{13} - \partial^4_{31}\right), 6\left(\partial^4_{04} - 4\partial^4_{22} + \partial^4_{40}\right), 3\left(\partial^4_{31} - \partial^4_{13}\right), 6\partial^4_{22}\right),\\
&4\mathrm{d}(\mathrm{d}^\perp)^3 = \left(-4\partial^4_{13}, \left(3\partial^4_{22} - \partial^4_{04}\right), 2\left(\partial^4_{13} - \partial^4_{31}\right), \left(\partial^4_{40} - 3\partial^4_{22}\right), 4\partial^4_{31}\right),\\
&(\mathrm{d}^\perp)^4 = \left(\partial^4_{04}, -\partial^4_{13}, \partial^4_{22}, -\partial^4_{31}, \partial^4_{40}\right).
\end{aligned}
$$

Notice, that $\mathrm{d}^4 = \nabla^4$, and $(\mathrm{d}^\perp)^4 = (\nabla^\perp)^4$ (see proposition 3.1), so we restrict ourselves to considerations of action of the divergence and orthogonal divergence on the fields $\mathrm{d}^3\mathrm{d}^\perp$, $\mathrm{d}^2(\mathrm{d}^\perp)^2$ and $\mathrm{d}(\mathrm{d}^\perp)^3$. Direct calculations lead to the following result.

Lemma 4.1. *Let in the unit disk B the C^l-smooth, $l \geq 4$, 4-tensor fields $\mathrm{d}^3\mathrm{d}^\perp\varphi$, $\mathrm{d}^2(\mathrm{d}^\perp)^2\varphi$, $\mathrm{d}(\mathrm{d}^\perp)^3\varphi$, and the operator $\mathcal{H}(nab+sym)_{(1,2,3,4)}$ consisting of four operators $\mathrm{d}, \mathrm{d}^\perp$ in arbitrary up to the permutations order, be given. Then the following relations are valid:*

1. An application of the operators δ, $\delta^\perp$ to the fields $\mathrm{d}^3\mathrm{d}^\perp\varphi$, $\mathrm{d}^2(\mathrm{d}^\perp)^2\varphi$, $\mathrm{d}(\mathrm{d}^\perp)^3\varphi$ give the non-zero symmetric 3-tensor fields according to the rules

$$
\delta(\mathrm{d}^3\mathrm{d}^\perp\varphi) = \frac{3}{4}\mathrm{d}^2\mathrm{d}^\perp\Delta\varphi, \quad \delta(\mathrm{d}^2(\mathrm{d}^\perp)^2\varphi) = \frac{1}{2}\mathrm{d}(\mathrm{d}^\perp)^2\Delta\varphi, \quad \delta(\mathrm{d}(\mathrm{d}^\perp)^3\varphi) = \frac{1}{4}(\mathrm{d}^\perp)^3\Delta\varphi,
$$
$$
\delta^\perp(\mathrm{d}^3\mathrm{d}^\perp\varphi) = \frac{1}{4}\mathrm{d}^3\Delta\varphi, \quad \delta^\perp(\mathrm{d}^2(\mathrm{d}^\perp)^2\varphi) = \frac{1}{2}\mathrm{d}^2\mathrm{d}^\perp\Delta\varphi, \quad \delta^\perp(\mathrm{d}(\mathrm{d}^\perp)^3\varphi) = \frac{3}{4}\mathrm{d}(\mathrm{d}^\perp)^2\Delta\varphi.
$$

2. The tensor fields $\mathrm{d}^3\mathrm{d}^\perp\varphi$, $\mathrm{d}^2(\mathrm{d}^\perp)^2\varphi$, $\mathrm{d}(\mathrm{d}^\perp)^3\varphi$ are neither potential nor solenoidal.

3. For the fixed up to the permutations operator $\mathcal{H}(nab)_{(1,2,3,4)}$ there exists the operator consisting of 4 operators $\delta, \delta^\perp$, namely $\mathcal{H}(del)_{(4,3,2,1)}$, the action of which on the operator $\mathcal{H}(nab)_{(1,2,3,4)}$ gives the operator Δ^4 multiplied by some constant. The action of any other operator consisting of other up to the permutations operators $\delta, \delta^\perp$ gives equal to $\mathbb{O}$ field.

We point out the difference fields $w^{(cs)}$ between the fields $w^{(c)}$ and $w^{(s)}$, and introduce the designations for these difference fields,

$$
\begin{aligned}
u^{(cs1)} &= \nabla^3\nabla^\perp - \mathrm{d}^3\mathrm{d}^\perp, & u^{(cs2)} &= \nabla^2\nabla^\perp\nabla - \mathrm{d}^3\mathrm{d}^\perp,\\
u^{(cs3)} &= \nabla\nabla^\perp\nabla^2 - \mathrm{d}^3\mathrm{d}^\perp, & u^{(cs4)} &= \nabla^\perp\nabla^3 - \mathrm{d}^3\mathrm{d}^\perp,\\
w^{(cs1)} &= \nabla^2(\nabla^\perp)^2 - \mathrm{d}^2(\mathrm{d}^\perp)^2, & w^{(cs2)} &= (\nabla\nabla^\perp)^2 - \mathrm{d}^2(\mathrm{d}^\perp)^2,\\
w^{(cs3)} &= \nabla(\nabla^\perp)^2\nabla - \mathrm{d}^2(\mathrm{d}^\perp)^2, & w^{(cs4)} &= \nabla^\perp(\nabla)^2\nabla\perp - \mathrm{d}^2(\mathrm{d}^\perp)^2,\\
w^{(cs5)} &= (\nabla^\perp\nabla)^2 - \mathrm{d}^2(\mathrm{d}^\perp)^2, & w^{(cs6)} &= (\nabla^\perp)^2\nabla^2 - \mathrm{d}^2(\mathrm{d}^\perp)^2,\\
v^{(cs1)} &= \nabla(\nabla^\perp)^3 - \mathrm{d}(\mathrm{d}^\perp)^3, & v^{(cs2)} &= \nabla^\perp\nabla(\nabla^\perp)^2 - \mathrm{d}(\mathrm{d}^\perp)^3,\\
v^{(cs3)} &= (\nabla^\perp)^2\nabla\nabla^\perp - \mathrm{d}(\mathrm{d}^\perp)^3, & v^{(cs4)} &= (\nabla^\perp)^3\nabla - \mathrm{d}(\mathrm{d}^\perp)^3.
\end{aligned}
$$

Besides we use the following designations:

$$
\begin{aligned}
A_0 &= \frac{1}{4}\partial^2_{20}\Delta, & A_1 &= \frac{1}{4}\partial^2_{11}\Delta, & A_2 &= \frac{1}{4}\partial^2_{02}\Delta,\\
B_1 &= \frac{1}{2}\partial^2_{11}\Delta = 2A_1, & B_2 &= -\frac{1}{6}\Delta^2, & B_3 &= \frac{5}{6}\Delta^2 - \partial^2_{20}\Delta, \quad B_4 = \frac{5}{6}\Delta^2 - \partial^2_{02}\Delta
\end{aligned}
$$

With usage of these designations we write three groups of the difference 4-tensor fields. The first group of four fields, differing from each other by the permutations of components, consists of the fields formed by the action of the operators $u^{(csj)}$, $j = 1, 2, 3, 4$ on a potential,

$$u^{(cs1)}=(0,-A_0,-A_0,-2A_1,-A_0,-2A_1,-2A_1,-3A_2,3A_0,2A_1,2A_1,A_2,2A_1,A_2,A_2,0)\,, \tag{10}$$
$$u^{(cs2)}=(0,-A_0,-A_0,-2A_1,3A_0,2A_1,2A_1,A_2,-A_0,-2A_1,-2A_1,-3A_2,2A_1,A_2,A_2,0)\,, \tag{11}$$
$$u^{(cs3)}=(0,-A_0,3A_0,2A_1,-A_0,-2A_1,2A_1,A_2,-A_0,-2A_1,2A_1,A_2,-2A_1,-3A_2,A_2,0)\,, \tag{12}$$
$$u^{(cs4)}=(0,3A_0,-A_0,2A_1,-A_0,2A_1,-2A_1,A_2,-A_0,2A_1,-2A_1,A_2,-2A_1,A_2,-3A_2,0)\,. \tag{13}$$

Let the order of components of the field (10) corresponds to the permutation

$$(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16).$$

Then the components of the fields (11)-(13) are obtained by the permutations

$$(1\,2\,3\,4\,9\,10\,11\,12\,5\,6\,7\,8\,13\,14\,15\,16),$$
$$(1\,2\,9\,13\,5\,6\,10\,14\,3\,7\,11\,12\,4\,8\,15\,16),$$
$$(1\,9\,2\,13\,5\,10\,6\,14\,3\,11\,7\,12\,4\,15\,8\,16),$$

respectively.

The second group is formed by six fields arising as a result of action of the operators $w^{(csk)}$, $k = 1, 2, 3, 4, 5, 6$ on a potential,

$$w^{(cs1)} = (0, B_1, B_1, B_3, -B_1, B_2, B_2, -B_1, -B_1, B_2, B_2, -B_1, B_4, B_1, B_1, 0)\,, \tag{14}$$
$$w^{(cs2)} = (0, B_1, -B_1, B_2, B_1, B_3, B_2, -B_1, -B_1, B_2, B_4, B_1, B_2, -B_1, B_1, 0)\,, \tag{15}$$
$$w^{(cs3)} = (0, B_1, -B_1, B_2, -B_1, B_2, B_4, B_1, B_1, B_3, B_2, -B_1, B_2, -B_1, B_1, 0)\,, \tag{16}$$
$$w^{(cs4)} = (0, -B_1, B_1, B_2, B_1, B_2, B_3, -B_1, -B_1, B_4, B_2, B_1, B_2, B_1, -B_1, 0)\,, \tag{17}$$
$$w^{(cs5)} = (0, -B_1, B_1, B_2, -B_1, B_4, B_2, B_1, B_1, B_2, B_3, -B_1, B_2, B_1, -B_1, 0)\,, \tag{18}$$
$$w^{(cs6)} = (0, -B_1, -B_1, B_4, B_1, B_2, B_2, B_1, B_1, B_2, B_2, B_1, B_3, -B_1, -B_1, 0)\,. \tag{19}$$

These fields also differ from each other only by the permutations of components. Let the order of the components of the field (14) corresponds to the permutation

$$(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16).$$

Then the components of the fields (15)–(19) are obtained by the permutations

$$(1\,2\,5\,6\,3\,4\,7\,8\,9\,10\,13\,14\,11\,12\,15\,16),$$
$$(1\,2\,8\,7\,5\,6\,13\,3\,14\,4\,11\,12\,10\,9\,15\,16),$$
$$(1\,5\,3\,7\,2\,6\,4\,8\,9\,13\,11\,15\,10\,12\,12\,16),$$
$$(1\,8\,3\,6\,5\,13\,7\,2\,15\,10\,4\,12\,11\,14\,9\,16),$$
$$(1\,5\,8\,13\,2\,6\,7\,3\,14\,10\,11\,15\,4\,9\,12\,16),$$

respectively.

The third group of four fields, differing from each other by the permutations of components, consists of the fields formed by the action of the operators $v^{(csl)}$, $l = 1, 2, 3, 4$ on a potential,

$$v^{(cs1)}=(0,-3A_2,A_2,2A_1,A_2,2A_1,-2A_1,-A_0,A_2,2A_1,-2A_1,-A_0,-2A_1,-A_0,3A_0,0)\,, \tag{20}$$
$$v^{(cs2)}=(0,A_2,-3A_2,2A_1,A_2,-2A_1,2A_1,-A_0,A_2,-2A_1,2A_1,-A_0,-2A_1,3A_0,-A_0,0)\,, \tag{21}$$

$$v^{(cs3)}=(0,A_2,A_2,-2A_1,-3A_2,2A_1,2A_1,-A_0,A_2,-2A_1,-2A_1,3A_0,2A_1,-A_0,-A_0,0)\,, \tag{22}$$
$$v^{(cs4)}=(0,A_2,A_2,-2A_1,A_2,-2A_1,-2A_1,3A_0,-3A_2,2A_1,2A_1,-A_0,2A_1,-A_0,-A_0,0)\,. \tag{23}$$

Let the order of components of the field (20) corresponds to the permutation

$$(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16).$$

Then the components of the fields (21)-(23) are obtained by the permutations

$$(1\,3\,2\,4\,5\,7\,6\,8\,9\,11\,10\,12\,13\,15\,14\,16),$$
$$(1\,5\,3\,7\,2\,6\,4\,8\,9\,13\,11\,15\,10\,14\,12\,16),$$
$$(1\,9\,3\,13\,5\,11\,7\,15\,2\,10\,4\,12\,6\,14\,8\,16),$$

respectively.

Lemma 4.2. *For the 4-tensor fields with the operators $u^{(csj)}$ (10)-(13), $w^{(csk)}$ (14)-(19), $v^{(csl)}$ (20)-(23) the following relations are valid: $\sum_{j=1}^{4} u^{(csj)}=0$, $\sum_{k=1}^{6} w^{(csk)}=0$, $\sum_{l=1}^{4} v^{(csl)}=0$.*

Proof. For the first and the third sets of the difference fields it is easy to prove lemma by direct calculations. For the second set we need to use the equality $4B_2+B_3+B_4=0$, which is proven as follows:

$$4B_2+B_3+B_4=-\frac{4}{6}\Delta^2+\frac{5}{6}\Delta^2-\partial^2_{20}\Delta+\frac{5}{6}\Delta^2-\partial^2_{02}\Delta=\Delta^2-\partial^2_{20}\Delta-\partial^2_{02}\Delta$$
$$=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}-\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=0.$$

□

Proposition 4.2. *The quadruple actions of the operators δ and $\delta^\perp$ on the difference fields between the complete 4-tensor fields $w^{(c)}$ and the symmetric 4-tensor fields $w^{(s)}$ give the following equalities:*

$$\delta^\perp\delta^3(u^{(cs1)})=\delta\delta^\perp\delta^2(u^{(cs2)})=\delta^2\delta^\perp\delta(u^{(cs3)})=\delta^3\delta^\perp(u^{(cs4)})=\frac{3}{4}\Delta^4,$$
$$(\delta^\perp)^2\delta^2(w^{(cs1)})=\delta^\perp\delta\delta^\perp\delta(w^{(cs2)})=\delta(\delta^\perp)^2\delta(w^{(cs3)})$$
$$=\delta^\perp\delta^2\delta^\perp(w^{(cs4)})=\delta\delta^\perp\delta\delta^\perp(w^{(cs5)})=\delta^2(\delta^\perp)^2(w^{(cs6)})=\frac{5}{6}\Delta^4,$$
$$(\delta^\perp)^3\delta(v^{(cs1)})=(\delta^\perp)^2\delta\delta^\perp(v^{(cs2)})=\delta^\perp\delta(\delta^\perp)^2(v^{(cs3)})=\delta(\delta^\perp)^3(v^{(cs4)})=\frac{3}{4}\Delta^4.$$

According to the general definition of difference non-symmetric tensor fields, let's introduce the unified notations $\tilde{w}^{(csi)}=\tilde{w}^{(ci)}-\tilde{w}^{(si)}$, $i=1,\dots 14$, for the defined by (10)-(13), (14)-(19), (20)-(23) operators.

Corollary 4.1. *Let in the unit disk B the C^l-smooth, $l\ge 4$, 4-tensor fields constructed by the operators $u^{(csj)}$, $j=1,\dots,4$, (10)-(13), $w^{(csk)}$ (14)-(19), $k=1,\dots,6$, $v^{(csl)}$ (20)-(23), $l=1,\dots,4$. The operators $\mathcal{H}(nab)_{(1,2,3,4)}$ and $\mathcal{H}(nab+sym)_{(1,2,3,4)}$ consist of four operators $\nabla,\nabla^\perp$ and $\mathrm{d},\mathrm{d}^\perp$, respectively, in an arbitrary order. Then*

1. The fields (10)-(13), (14)-(19), (20)-(23) are neither potential nor solenoidal.

2. Any field $\tilde{w}^{(csi)}$, $i=1,\dots,14$ is equal to zero if and only if the corresponding fields $\tilde{w}^{(ci)}$, $\tilde{w}^{(si)}$, $i=1,\dots,14$ are equal to zero.

3. For the fixed 4-tensor field constructed by the operator

$$(\mathcal{H}(nab)_{(1,2,3,4)}-\mathcal{H}(nab+sym)_{(1,2,3,4)})$$

there exists the operator consisting of 4 operators $\delta,\delta^\perp$, namely $\mathcal{H}(del)_{(4,3,2,1)}$, the action of which on the operator $(\mathcal{H}(nab)_{(1,2,3,4)}-\mathcal{H}(nab+sym)_{(1,2,3,4)})$ gives the operator Δ^4 up to the constant.

Conclusion

On the way of generalization of the Radon transform there arise many other transforms applied in integral geometry and tensor tomography. Together with complicated geometrical objects, the weighted ray and Radon transforms appear. Usually the symmetric tensor fields are considered as the object should be reconstructed. In the paper along with well known symmetric fields we consider poorly studied the complete and difference fields of small ranks. The structure and differential properties of the fields of ranks 1–4 were studied in detail. We established the decomposition theorem for complete tensor fields, the differential and geometric properties and the attributes of solenoidal and potential fields.

The work is the first step in investigations of the momentum ray transforms of previously unused in integral geometry and tensor tomography tensor fields. The goal of such studies along with theoretical and applied interest should be directed to the problem of recovering previously unused tensor fields by their momentum ray transforms.

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