EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 12, Issue 2 (2024) 134–154

AN INVERSE PROBLEM FOR A NONLINEAR HYPERBOLIC EQUATION

Romanov V.G. , Bugueva T.V.

Abstract For a second-order hyperbolic equation with inhomogeneity $|u|^{m-1}u$, $m > 1$, a forward and an one-dimensional inverse problems are studied. The inverse problem is devoted to determining the coefficient under heterogeneity. As an additional information, the trace of the derivative with respect to x of the solution to the forward initial-boundary value problem is given at $x = 0$ on a finite interval. Conditions for the unique solvability of the forward problem are found. For the inverse problem a local existence and uniqueness theorems are established and a stability estimate of its solutions is found.

Key words: nonlinear equation, inverse problem, uniqueness, stability estimate.

AMS Mathematics Subject Classification: 35R30.

DOI: 10.32523/2306-6172-2024-12-2-134-154

1 Introduction

In recent years, there has been an increasing number of scientific papers devoted to solving both forward and inverse problems for nonlinear wave equations. Equations containing nonlinearities of the form $|u|^{p-1}u$ are called defocusing. For example, various formulations of forward problems and methods for solving them are considered in articles [1–8].

Thus, in the paper [1] the asymptotic behavior of finite energy solutions of one-dimensional defocusing nonlinear wave equation $\Box u + |u|^{p-1}u = 0$, $p > 1$, is studied. In [2], the internal stabilization and control of the critical nonlinear Klein–Gordon equation $\Box u + u + |u|^4 u = g$ on 3-D compact manifolds are studied. In work [3], the authors prove the exponential stabilization of the semilinear wave equation $\Box u = \gamma(x)\partial_t u + \beta u + f(u)$, with an effective damping in a zone satisfying a geometric control condition only. The nonlinearity is assumed to be subcritical, defocusing and analytic. In [4], the global behaviors of solutions to defocusing semilinear wave equation $\Box\phi = |\phi|^{p-1}\phi$ in $\mathbb{R}^{1+d}$, $d \geqslant 3$, is investigated. For the case $p > 1 + 2/(d-1)$, an uniform weighted energy estimate for the solution is obtained, as well as an inverse polynomial attenuation of the energy flow through hypersurfaces away from the light cone is found.

In [5], a wave equation $\Box u + |u|^{p-1}u = 0$ with a power nonlinearity is considered , defined outside the unit ball in $\mathbb{R}^n$, $n \geqslant 3$, with Dirichlet boundary conditions. It is proved that if $p > n + 3$ and the initial data are nonradial perturbations of large radial data, then there exists a global smooth solution. The solution is unique in the energy class solutions satisfying an energy inequality. Work [6] is devoted to the study of the asymptotic behavior of solutions to the one-dimensional wave equation $\Box u + |u|^{p-1}u = 0$. It is proved that the solution with finite energy tends to zero in the pointwise sense, moreover, for sufficiently localized data belonging to some weighted energy space, the solution decays in time with inverse polynomial velocity. In [7], the equation $\Box\phi + |\phi|^{p-1}\phi = 0$ is studied on $\mathbb{R} \times \mathbb{R}^2 \setminus \mathcal{K}$ with the Dirichlet boundary condition. Here, $\mathcal{K}$ is is a star-shaped obstacle with smooth boundary. It is proven that the solution scatters both in energy space and the critical Sobolev space. In paper [8], a mixed boundary value problem for the equation $u_{tt} = \big(k(x)u_x\big)_x + c|u|^{p-1}u$, is considered. Here $p > 1$ and $c > 0$ are constants. Using the method of energy inequalities, estimates for the solution of the differential and difference problems are obtained.

Inverse problems for nonlinear wave equations have been studied relatively recently, but many results have already been obtained in solving these problems. Thus, in [9–11] various formulations of inverse problems related to the determination of the Lorentz metric or the coefficients included in these equations are considered.

In [9], nonlinear inverse problems for the wave equation $\Box_g u(x) + H(x, u(x))$ are considered on a Lorentzian manifold M with Laplacian—Beltrami operator. It is shown that, on a given space-time (M, g), the source-to-solution map determines some coefficients of the Taylor expansion of H in u. In [10], for the semilinear wave equation $\Box_g u + w(x, u, \nabla_g u) = 0$ on Lorentzian manifolds, the inverse problem of determining the background Lorentzian metric is studied. In [11], the inverse boundary value problem is considered for a semilinear wave equation $\Box u + H(x, u(x)) = 0$ on a time-dependent Lorentzian manifold $\mathcal{M}$, with a time-like boundary. It is assumed that $H(x, z) \sim \sum\limits_{k=2}^{\infty} h_k(x)z^k$, where $h_k \in C^\infty(\mathcal{M})$. The time-dependent coefficients in the nonlinear terms of the equation can be reconstructed using knowledge of the Neumann-Dirichlet mapping, which allows for the reconstruction of the time-dependent terms. It was shown that either distorted plane waves or Gaussian beams can be used to derive uniqueness.

In [12], the inverse problem of recovering the nonlinearity $f(x, u)$ in the differential equation $\Box u + f(x, u) = 0$ is considered. It is demonstrated that it is possible to recover the function $f(x, u)$ when it is odd in u, and it is also possible to recover the function $\alpha(x)$ when $f(x, u) = \alpha(x)u^{2m}$. In [13], the geometric non-linear inverse problem of recovering a Hermitian connection A from the source-to-solution map of the cubic wave equation $\Box_A u + \kappa|u|^2 u = f$, is considered. Here $\kappa \neq 0$, $\Box_A$ is the connection wave operator in Minkowski space $\mathbb{R}^{1+3}$. The microlocal analysis is used for this nonlinear wave interactions.

In [14], it is shown that the scattering operator for defocusing energy critical semilinear wave equations $\Box u + f(u) = 0$, $f \in C^\infty(\mathbb{R})$, $f \sim u^5$, defines the function f. In [15], the recovery of a potential associated with a semi-linear wave equation $\Box u + au^m = 0$ in $\mathbb{R}^{n+1}$, $n \geqslant 1$, is investigated, where m is integer number, $m \geqslant 2$. The Hölder stability estimate for the recovery of an unknown potential $a(x, t)$ from its Dirichlet-to-Neumann map is proved. In [16] the equation $\Box u + \alpha(x)|u|^2 u = 0$ is considered in two-dimensional and three-dimensional spaces. The inverse problems of restoring the function $\alpha(x)$, $0 \leq \alpha(x) \in C_0^\infty$ are investigated, and it is shown that using the

Radon transform, an unknown coefficient can be restored.

In [17] the inverse problem of determining the coefficient of a nonlinear term in the equation $\Box u = q(x)u^2 + \theta_0(t)\delta(x-y)$, where $\theta_0(t)$ is the Heavisaid step-function, is considered. The properties of the solutions to the forward problem are studied. In particular, the existence and uniqueness of a bounded solution in a neighborhood of a characteristic plain is established, and the structure of the solution is described. In [18], the equation $\Box u = f(x,u)$, $(x,t) \in \mathbb{R}^4$, is considered, where $f(x,u)$ is a smooth function by x and u and finite in x. The forward Cauchy problem is studied, and the existence of a unique bounded solution in a neighborhood of a characteristic plane is stated. An amplitude formula for the derivative of the solution with respect to t on the front of the wave is derived. It is demonstrated that the solution of the inverse problem reduces to a series of X-ray tomography problems.

In [19], for the nonlinear partial differential equation $\Box u = q(x)u^{\gamma+1}$, where $\gamma > 0$, the inverse problem of determining the function $q(x)$ from boundary data is considered. Here, it is assumed that the desired function q is a continuous and finite function for $x \in \mathbb{R}^3$. It is shown that solutions to the corresponding forward problem for the given differential equation are bounded in some neighborhood of the characteristic curve, and an asymptotic expansion for the solution in this neighborhood is obtained. A theorem on the uniqueness of solutions to the inverse problem is proved.

In [20] the equation $\Box u = q(t)(u_x)^m$, where $m > 1$ is a number, is considered. Theorems of the existence and uniqueness of the solution of the forward problem and a local existence and stability of the solution of the inverse problem are proved. In [21], an one-dimensional inverse problem of determining the nonlinear coefficient for a second-order hyperbolic equation with nonlinear absorption: $\Box u + \sigma(x)|u_t|^m u_t = 0$, is studied, here $m > 0$ is a real number. For the inverse problem, a local existence and uniqueness theorem and a global stability estimate of its solutions are stated.

In the present paper we consider an one-dimensional inverse problem for equation $u_{tt} - \big(k^2(x)u_x\big)_x - q(x)|u|^{m-1}u = 0$ on semi-axis $x > 0$ with zero initial data and the boundary condition $u(0,t) = f(t)$. The main goal is to recover coefficient $q(x)$ from given $k(x)$ and the derivative $u_x(0,t)$ given for $t \in [0,T]$. We prove an uniqueness and existence theorem for the forward problem when coefficients $k(x)$ and $q(a)$ as well as function $f(t)$ are given. Then we study the inverse problem and state a local uniqueness and existence theorem for this problem. Moreover, a stability estimate of solutions to the inverse problem is also found. Both theorems for forward and inverse problems are new in the theory of inverse problems.

2 Posing of problems

Let T be a real positive number.

A forward problem. Determine the function $u(x,t)$ satisfying the relations

$$\begin{gathered} u_{tt} - \big(k^2(x)u_x\big)_x - q(x)|u|^{m-1}u = 0,\ x > 0,\ 0 < t \le T, \\ u|_{t=0} = u_t|_{t=0} = 0, \quad u|_{x=0} = f(t), \quad t \in (0,T], \end{gathered} \tag{1}$$

where $m > 1$, $k(x)$ is the continuously differentiable function, $0 < k_1 \leqslant k(x) \leqslant k_2$;

$q(x)$ is the continuous function; $f(t)$ is the twice continuously differentiable function and $f(+0) = 1$.

An inverse problem. Let $f(t)$, $k(x)$ be the given functions. Determine the function $q(x)$ in equation (1), given the following information about the solution to the forward problem:

$$u_x|_{x=0} = h(t), \quad t \in [0, T]. \tag{2}$$

Let's write equations (1) in form

$$u_{tt} - k^2(x)u_{xx} = 2k'(x)k(x)u_x + q(x)|u|^{m-1}u, \; x > 0, \; 0 < t \le T, \tag{3}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{4}$$

$$u|_{x=0} = f(t), \quad t \in (0, T]. \tag{5}$$

We make the change of variables

$$z = z(x) = \int_0^x \frac{d\xi}{k(\xi)}, \quad z \in [0, \zeta(T/2)], \quad z(0) = 0$$

in equalities (3)–(5). Here $x = x(z)$ is the inverse function to $z = z(x)$ and define functions

$$\hat{k}(z) := k(x(z)), \quad \hat{q}(z) := q(x(z)), \quad U(z,t) := u(x(z), t).$$

Rewrite equations (3)–(5) in terms of these functions

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial z^2} = -K(z)\frac{\partial U}{\partial z} + \hat{q}(z)|U|^{m-1}U, \quad z > 0, \; t \in [0, T], \tag{6}$$

$$U|_{t=0} = U_t|_{t=0} = 0, \tag{7}$$

$$U|_{z=0} = f(t), \quad t \in (0, T], \tag{8}$$

where $K(z) = -\hat{k}'(z)/\hat{k}(z)$.

Let us introduce the function $V(z,t) = \dfrac{U(z,t)}{S(z)}$, where function $S(z)$ is determined from equations $\dfrac{2S'(z)}{S(z)} = K(z)$, $S(0) = 1$, therefore

$$S(z) = \exp\left\{\frac{1}{2}\int_0^z K(\xi)\, d\xi\right\} = \exp\left\{-\frac{1}{2}\int_0^z \frac{\hat{k}'(\xi)}{\hat{k}(\xi)}\, d\xi\right\} = \sqrt{\frac{\hat{k}(0)}{\hat{k}(z)}}.$$

Rewrite equations (6)–(8) in term of function $V(z,t)$:

$$V_{tt} - V_{zz} = p(z)V + \bar{q}(z)|V|^{m-1}V := \Phi(z,t), \quad z > 0, \; t \in [0, T], \tag{9}$$

$$V|_{t=0} = V_t|_{t=0} = 0, \tag{10}$$

$$V|_{z=0} = f(t), \quad t \in (0, T], \tag{11}$$

where

$$p(z) = \frac{K'(z)}{2} - \frac{K^2(z)}{4} = -\frac{\hat{k}''(z)}{2\hat{k}(z)} + \frac{1}{4}\Big(\frac{\hat{k}'(z)}{\hat{k}(z)}\Big)^2, \quad \bar{q}(z) = \hat{q}(z)\bigg(\sqrt{\frac{\hat{k}(0)}{\hat{k}(z)}}\bigg)^{m-1}.$$

Equation (2) in term of function $V(z,t)$ will be as follows

$$V_z|_{z=0} = h(t) - \frac{K(0)}{2} f(t) \equiv \mu(t), \quad t \in (0, T]. \tag{12}$$

3 An analysis of the forward problem

The solution of the problem (9)–(11) can be rewritten in the form

$$V(z,t) = f(t-z) + \frac{1}{2}\int\limits_0^t d\tau \int\limits_{|z-t+\tau|}^{z+t-\tau} \big[p(\xi)V(\xi,\tau) + \bar{q}(\xi)|V(\xi,\tau)|^{m-1}V(\xi,\tau)\big]\, d\xi, \tag{13}$$

or, since $V = 0$ for $t < z$, the equality (13) can be written in the form

$$V(z,t) = f(t-z) + \frac{1}{2}\iint\limits_{D(z,t)} \big[p(\xi)V(\xi,\tau) + \bar{q}(\xi)|V(\xi,\tau)|^{m-1}V(\xi,\tau)\big]\, d\xi d\tau, \tag{14}$$

where $D(z,t)$ is a rectangle bounded by the characteristics

$$\xi + \tau = t + z, \quad \xi + \tau = t - z, \quad \xi - \tau = z - t, \quad \xi - \tau = 0.$$

We'll assume, that $\max\limits_{0\leqslant z\leqslant T/2} |p(z)| \leqslant p_0$, $\max\limits_{0\leqslant z\leqslant T/2} |\bar{q}(z)| \leqslant q_0$. Consider the domain $G(T) = \{(z,t) \mid z < t \leqslant T - z\}$.

Lemma 3.1. *Let $p(z)$, $q(z) \in C[0, T/2]$, $f(t) \in C^2[0,T]$, $1 \leqslant f(t) \leqslant F$, $f(+0) = 1$. Then there exists a number T_1, $0 < T_1 \leqslant T$, such that function $V(z,t)$ is the unique solution of the forward problem* (9)–(11), *and it is non-negative and continuous in the domain $G(T_1)$.*

Proof. Rewrite equation (14) in operator form:

$$V = A_1 V, \tag{15}$$

where

$$A_1 V(z,t) = f(t-z) + \frac{1}{2}\iint\limits_{D_1(z,t)} \big[p(\xi)V(\xi,\tau) + \bar{q}(\xi)|V(\xi,\tau)|^{m-1}V(\xi,\tau)\big]\, d\xi d\tau. \tag{16}$$

For $(z,t) \in G(T)$ assume $\hat{f}(z,t) := f(t-z)$, then $\hat{f} \in C(G(T))$. The norm in space $C(G(T))$ is defined, as usual, by the formula

$$\|\varphi\|_{C(G(T))} = \sup_{(z,t)\in G(T)} |\varphi(z,t)|.$$

Consider the closed ball

$$\mathbf{B}_T^1 := \{\varphi \in C(G(T)) \mid \|\varphi - \hat{f}\|_{C(G(T))} \leqslant 1\}. \tag{17}$$

Since the closed subset of the complete metric space is complete, then this ball is a complete metric space with respect to the metric defined by the above norm.

By virtue of the condition of the lemma $\|\hat{f}\|_{C(G(T))} = \|f\|_{C[0,T]} \leqslant F$. Let $V \in \mathbf{B}_T^1$, then by virtue of (17) we obtain

$$\|V\|_{C(G(T))} = \|V - \hat{f} + \hat{f}\|_{C(G(T))} \leqslant \|V - \hat{f}\|_{C(G(T))} + \|\hat{f}\|_{C(G(T))} \leqslant 1 + F. \tag{18}$$

Using (16), (18), we get

$$\begin{aligned} \|A_1 V - \hat{f}\|_{C(G(T))} &= \sup_{(z,t)\in G(T)} |A_1(V(z,t)) - f(t-z)| \\ &\leqslant \frac{1}{2} \sup_{(z,t)\in G(T)} \iint\limits_{D_1(z,t)} \big[|p(\xi)| + |\bar{q}(\xi)||V(\xi,\tau)|^{m-1}\big]|V(\xi,\tau)|\, d\xi d\tau \\ &\leqslant \big[p_0 + q_0(1+F)^{m-1}\big](1+F)\frac{T^2}{8}. \end{aligned} \tag{19}$$

It follows from the formulas (18) and (19), if the condition

$$T_1' = T_1'(F) = \min\big\{T,\ \sqrt{8}(p_0(1+F) + q_0(1+F)^m)^{-1/2}\big\}$$

is satisfied, then the inequality $\|A_1 V - \hat{f}\|_{C(G(T_1'))} \leqslant 1$ holds, i. e. $A_1 V \in B_{T_1'}^1$. Thus, A_1 maps the ball $\mathbf{B}_{T_1'}^1$ into itself.

Let us show that operator A_1, defined by the equality (16), is compressive for a sufficiently small $T > 0$.

For $V \in B_T^1$ the inequality

$$V(z,t) \geqslant f(t-z) - |V(z,t) - f(t-z)| \geqslant 0, \quad (z,t) \in G(T),$$

holds. Therefore, $|V(z,t)| = V(z,t)$, and we can omit the modulus sign in the integrand expression.

Let $V_1(z,t), V_2(z,t) \in \mathbf{B}_T^1$ and we write the difference $V_1^m(z,t) - V_2^m(z,t)$ as follows:

$$V_1^m(z,t) - V_2^m(z,t) = m \int\limits_{V_2(z,t)}^{V_1(z,t)} s^{m-1}\, ds,$$

and replace the variable in the integral

$$\begin{gathered} s = V_1(z,t)s' + V_2(z,t)(1-s'), \quad ds = (V_1(z,t) - V_2(z,t))\, ds', \\ s = V_2(z,t) \ \Rightarrow\ s' = 0, \quad s = V_1(z,t) \ \Rightarrow\ s' = 1. \end{gathered}$$

as a result, we get

$$V_1^m(z,t) - V_2^m(z,t) = (V_1(z,t) - V_2(z,t))R_m(V_1(z,t), V_2(z,t)), \tag{20}$$

where

$$R_m(V_1(z,t),V_2(z,t)) = m\int\limits_0^1 \left[V_1(z,t)s' + V_2(z,t)(1-s')\right]^{m-1} ds'. \qquad (21)$$

Then using equations (18), (20) we conclude

$$|R_m(V_1(z,t),V_2(z,t))| \leqslant m(1+F)^{m-1}, \qquad (22)$$

$$\left|V_1^m(z,t) - V_2^m(z,t)\right| \leqslant |V_1(z,t) - V_2(z,t)|m(1+F)^{m-1}. \qquad (23)$$

Let us estimate the difference

$$\|A_1V_1 - A_1V_2\|_{C(G(T))} = \sup_{(z,t)\in G(T)} |A_1V_1(z,t) - A_1V_2(z,t)|$$

$$\leqslant \frac{1}{2}\sup_{(z,t)\in G(T)} \iint\limits_{D_1(z,t)} \Big(|p(\xi)|\big|V_1(\xi,\tau) - V_2(\xi,\tau)\big| + |\bar q(\xi)|\big|V_1^m(\xi,\tau) - V_2^m(\xi,\tau)\big|\Big)\, d\xi d\tau$$

$$\leqslant \frac{1}{2}\sup_{(z,t)\in G(T)} \iint\limits_{D_1(z,t)} \Big(|p(\xi)| + |\bar q(\xi)||R_m(V_1(z,t),V_2(z,t))|\Big)\big|V_1(\xi,\tau) - V_2(\xi,\tau)\big|\, d\xi d\tau$$

$$\leqslant \underbrace{\frac{T^2}{8}\left[p_0 + q_0 m(1+F)^{m-1}\right]}_{=:\rho_1} \|V_1 - V_2\|_{C(G(T))}. \qquad (24)$$

It follows from (24) that for $\rho_1 \in (0,1)$ we can choose

$$T_1'' = T_1''(F,\rho_1) = \min\left\{T,\ \sqrt{8\rho_1}\big(p_0 + q_0 m(1+F)^{m-1}\big)^{-1/2}\right\}$$

so that

$$\|A_1V_1 - A_1V_2\|_{C(G(T_1''))} \leqslant \rho_1\|V_1 - V_2\|_{C(G(T_1''))}.$$

Then A_1 is a compressive mapping on the set $B^1_{T_1}$, where $T_1 = \min\left\{T_1', T_1''\right\}$. Then, according to Banach Compressive Mappings Principle, there exists a unique solution of the operator equation (15) in $\mathbf{B}_{T_1}$. Lemma 3.1 has been proved. □

Theorem 3.1. *When the conditions of Lemma* 3.1 *are fulfilled, the function* $V(z,t) \in C^2(G(T_1))$.

Proof. By virtue of Lemma 3.1, equation (14) can be written in the form

$$V(z,t) = f(t-z) + \frac{1}{2}\iint\limits_{D(z,t)} \left[p(z)V(z,t) + \bar q(z)V^m(z,t)\right] d\xi d\tau. \qquad (25)$$

We introduce new function $\Psi(z,t) = [p(z)V(z,t) + \bar q(z)V^m(z,t)]$ and rewrite the equation (25) in the form of a sum of repeated integrals

$$V(z,t) = f(t-z) + \frac{1}{2}\int\limits_0^{(t-z)/2} d\xi \int\limits_{t-z-\xi}^{t-z+\xi} \Psi(\xi,\tau)\, d\tau$$

$$+\frac{1}{2}\int\limits_{(t-z)/2}^{z} d\xi \int\limits_{\xi}^{t-z+\xi} \Psi(\xi,\tau)\,d\tau + \frac{1}{2}\int\limits_{z}^{(z+t)/2} d\xi \int\limits_{\xi}^{t+z-\xi} \Psi(\xi,\tau)\,d\tau. \quad (26)$$

Differentiating (26) by variables t and z, we find

$$\frac{\partial V}{\partial t}(z,t) = f'(t-z) + \frac{1}{2}\int\limits_{0}^{z} \Psi(\xi, t-z+\xi)\,d\xi$$

$$+\frac{1}{2}\int\limits_{z}^{(z+t)/2} \Psi(\xi, z+t-\xi)\,d\xi - \frac{1}{2}\int\limits_{0}^{(t-z)/2} \Psi(\xi, t-z-\xi)\,d\xi; \quad (27)$$

$$\frac{\partial V}{\partial z}(z,t) = -f'(t-z) - \frac{1}{2}\int\limits_{0}^{z} \Psi(\xi, t-z+\xi)\,d\xi$$

$$+\frac{1}{2}\int\limits_{0}^{(t-z)/2} \Psi(\xi, t-z-\xi)\,d\xi + \frac{1}{2}\int\limits_{z}^{(z+t)/2} \Psi(\xi, z+t-\xi)\,d\xi. \quad (28)$$

Since the expressions standing in the right parts of the equalities (27) and (28) are continuous functions, then the expressions standing in the left parts of these equalities are also continuous functions, therefore, V_t, V_z belong to $C(G(T))$.

Differentiating equality (27), (28) by variable t, we get

$$\frac{\partial^2 V}{\partial t^2}(z,t) = f''(t-z) + \frac{1}{4}\Psi\left(\frac{z+t}{2}, \frac{z+t}{2}\right) - \frac{1}{4}\Psi\left(\frac{t-z}{2}, \frac{t-z}{2}\right)$$

$$+\frac{1}{2}\int\limits_{0}^{z} \Psi_t(\xi, t-z+\xi)\,d\xi + \frac{1}{2}\int\limits_{z}^{(z+t)/2} \Psi_t(\xi, z+t-\xi)\,d\xi$$

$$-\frac{1}{2}\int\limits_{0}^{(t-z)/2} \Psi_t(\xi, t-z-\xi)\,d\xi;$$

$$\frac{\partial^2 V}{\partial t \partial z}(z,t) = -f''(t-z) + \frac{1}{4}\Psi\left(\frac{z+t}{2}, \frac{z+t}{2}\right) + \frac{1}{4}\Psi\left(\frac{t-z}{2}, \frac{t-z}{2}\right)$$

$$-\frac{1}{2}\int\limits_{0}^{z} \Psi_t(\xi, t-z+\xi)\,d\xi + \frac{1}{2}\int\limits_{z}^{(z+t)/2} \Psi_t(\xi, z+t-\xi)\,d\xi$$

$$+\frac{1}{2}\int\limits_{0}^{(t-z)/2} \Psi_t(\xi, t-z-\xi)\,d\xi; \quad (29)$$

$$\frac{\partial^2 V}{\partial z^2}(z,t) = f''(t-z) - \Psi(z,t) + \frac{1}{4}\Psi\left(\frac{z+t}{2}, \frac{z+t}{2}\right) - \frac{1}{4}\Psi\left(\frac{t-z}{2}, \frac{t-z}{2}\right)$$

$$+\frac{1}{2}\int\limits_0^z \Psi_t(\xi,t-z+\xi)\,d\xi-\frac{1}{2}\int\limits_0^{(t-z)/2}\Psi_t(\xi,t-z-\xi)\,d\xi+\frac{1}{2}\int\limits_z^{(z+t)/2}\Psi_t(\xi,z+t-\xi)\,d\xi.$$

Since $\Psi(z,t)$, $\Psi_t(z,t)$ are continuous functions in $G(T)$, it follows from the above equations that all second derivatives of the function $V(z,t)$ are also continuous in $G(T)$. Therefore, $V(z,t)\in C^2(G(T_1))$. Theorem 3.1 has been proved. □

Corollary 3.1. *The above analysis of the forward problem allows us to draw certain conclusions about the properties of the function $\mu(t)$, which is used in the inverse problem. From the fact that the function $V(z,t)$ belongs to the space $C^2(G(T_1))$ it follows that $\mu\in C^1(G(T_1))$, and from the equality (28) for $t=z=0$ we get $\mu(0)=-f'(0)$.*

4 An investigation of the inverse problem

Inverse problem. Let T be a given positive number, $\mu(t)$ be a given function, $t\in[0,T]$. Find function $\bar q(z)$ in equation (9) from given information (12) about solution $V(z,t)$ to the forward problem (9)–(11).

Theorem 4.1. *Let functions $f(t)$ and $\mu(t)$ satisfy the conditions*

$$f\in C^2[0,T],\quad \mu\in C^1[0,T],$$
$$1=f(0)\leqslant f(t),\quad \mu(0)+f'(0)=0,\quad f'(t)+\mu(t)\geqslant 0,\quad t\in[0,T].$$

Then there exist a positive number $T_0\leqslant T$ and unique function $\bar q(z)\in C[0,T_0/2]$, such that the solution to problem (9)–(11) satisfies condition (12) for $t\leqslant T_0$.

Proof. Rewrite equation (14) in the form of a sum of repeated integrals

$$V(z,t)=f(t-z)+\frac{1}{2}\int\limits_0^{(t-z)/2}d\xi\int\limits_{t-z-\xi}^{t-z+\xi}\Phi(\xi,\tau)\,d\tau$$
$$+\frac{1}{2}\int\limits_{(t-z)/2}^{z}d\xi\int\limits_{\xi}^{t-z+\xi}\Phi(\xi,\tau)\,d\tau+\frac{1}{2}\int\limits_z^{(z+t)/2}d\xi\int\limits_{\xi}^{t+z-\xi}\Phi(\xi,\tau)\,d\tau. \quad (30)$$

Differentiating equation (30) by z, we get

$$\frac{\partial V}{\partial z}(z,t)=-f'(t-z)-\frac{1}{2}\int\limits_0^z\Phi(\xi,t-z+\xi)\,d\xi$$
$$+\frac{1}{2}\int\limits_0^{(t-z)/2}\Phi(\xi,t-z-\xi)\,d\xi+\frac{1}{2}\int\limits_z^{(z+t)/2}\Phi(\xi,z+t-\xi)\,d\xi. \quad (31)$$

From (31) at $z = 0$ we get

$$\mu(t) = -f'(t) + \int\limits_0^{t/2} \Phi(\xi, t-\xi)\, d\xi. \tag{32}$$

Differentiate equality (32) by the variable t, we find

$$\mu'(t) + f''(t) = \frac{1}{2}\Phi(t/2, t/2) + \int\limits_0^{t/2} \Phi_t(\xi, t-\xi)\, d\xi. \tag{33}$$

From (13) to by virtue of the condition of the theorem 4.1 we have $V(z,z) = f(+0) = 1$. Therefore

$$\Phi(z,z) = p(z)V(z,z) + \bar{q}(z)|V(z,z)|^{m-1}V(z,z) = p(z) + \bar{q}(z). \tag{34}$$

Let us replace $t/2 = z$ in equation (33). Then we get

$$\mu'(2z) = -f''(2z) + \frac{1}{2}\Phi(z, z+0) + \frac{1}{2}\int\limits_0^{z} \Phi'_t(\xi, 2z-\xi)\, d\xi.$$

It follows from this equality by virtue of (34) that

$$\bar{q}(z) = \bar{q}^0(z) - \int\limits_0^{z} \Phi'_t(\xi, 2z-\xi)\, d\xi, \quad 0 \leqslant z \leqslant T/2, \tag{35}$$

where

$$\bar{q}^0(z) = 2[f''(2z) + \mu'(2z)] - p(z). \tag{36}$$

Let us write out alternative equations for the functions $V(z,t)$ and $V_t(z,t)$. To do this, we apply the d'Alembert's formula to the following problem (9), (11), (12)

$$\begin{gathered} V_{zz} - V_{tt} = -\big[p(z)V + \bar{q}(z)|V|^{m-1}V\big] = -\Phi(z,t), \\ V|_{z=0} = f(t), \quad t \in (0,T], \\ V_z|_{z=0} = \mu(t), \quad t \in [0,T]. \end{gathered} \tag{37}$$

As a result, we get

$$V(z,t) = V^0(z,t) - \frac{1}{2}\iint\limits_{D_2(z,t)} \big[p(z)V(\xi,\tau) + \bar{q}(z)|V(\xi,\tau)|^{m-1}V(\xi,\tau)\big]\, d\tau d\xi, \tag{38}$$

where $D_2(z,t) = \{(\xi,\tau) \mid 0 \leqslant \xi \leqslant z,\ t - z + \xi \leqslant \tau \leqslant z + t - \xi\}$,

$$V^0(z,t) = \frac{f(t+z) + f(t-z)}{2} + \frac{1}{2}\int\limits_{t-z}^{t+z} \mu(\tau)\, d\tau. \tag{39}$$

The condition $f'(t)+\mu(t)\geqslant 0$ for $t\in[0,T]$ implies that

$$f(t+z)-f(t-z)+\int\limits_{t-z}^{t+z}\mu(\tau)d\tau\geqslant 0,\quad (z,t)\in G(T),$$

so

$$\begin{aligned}V^0(z,t)&=\frac{f(t+z)+f(t-z)}{2}+\frac{1}{2}\int\limits_{t-z}^{t+z}\mu(\tau)\,d\tau\\&=\underbrace{\frac{f(t+z)-f(t-z)}{2}+\frac{1}{2}\int\limits_{t-z}^{t+z}\mu(\tau)\,d\tau}_{\geqslant 0}+f(t-z)\geqslant f(t-z)\geqslant 1,\quad (z,t)\in G(T).\end{aligned}\tag{40}$$

Denote by $\gamma_0=\max\limits_{(z,t)\in G(T)}V^0(z,t)$. Then from (39), (40) we get

$$1<V^0(z,t)\leqslant\gamma_0,\quad (z,t)\in G(T),\tag{41}$$

and denote by $\max\limits_{(z,t)\in[0,T/2]}|\bar q^0(z)|=q_0/2$. We shall assume that $\bar q(z)$ belongs to the set

$$\Big\{\bar q(z)\in C[0,T/2]\mid \|\bar q-\bar q^0\|_{C[0,T/2]}\leqslant q_0/2\Big\}.$$

Then the following estimate holds: $\|\bar q\|_{C[0,T/2]}\leqslant q_0$.

To further prove the theorem 4.1 we shall prove the following Lemma.

Lemma 4.1. *Let $p(z)\leqslant p_0$ and $\bar q(z)\leqslant q_0$ for $z\in[0,T/2]$. Then there exists a number T_2, $0<T_2\leqslant T$, such that the function $V(z,t)$ is the unique solution of the forward problem* (37)*, and it is non-negative and continuous in the domain $G(T_2)$.*

Proof. Lemma 4.1 is proved similarly to Lemma 3.1. Rewrite the equation (38) in operator form: $V=A_2V$, where

$$A_2V(z,t)=V^0(z,t)-\frac{1}{2}\iint\limits_{D_2(z,t)}\big[p(z)V(\xi,\tau)+\bar q(z)|V(\xi,\tau)|^{m-1}V(\xi,\tau)\big]\,d\tau d\xi.\tag{42}$$

Consider the closed ball

$$\mathbf{B}^2_T:=\{V(z,t)\in C(G(T))\mid \|V-V^0\|_{C(G(T))}\leqslant 1\},\tag{43}$$

which is a complete metric space with respect to the metric defined by the norm $\|\cdot\|_{C(G(T))}$.

Let $V(z,t)\in\mathbf{B}^2_T$, then by virtue of (41) we have

$$\|V\|_{C(G(T))}=\|V-V^0+V^0\|_{C(G(T))}\leqslant\|V-V^0\|_{C(G(T))}+\|V^0\|_{C(G(T))}\leqslant 1+\gamma_0.\tag{44}$$

In addition, using the inequality (40), we have

$$V(z,t) \geqslant V_0(z,t) - |V(z,t) - V_0(z,t)| \geqslant f(t-z) - 1 \geqslant 0, \quad (z,t) \in G(T). \tag{45}$$

Using (42), (44), we get

$$\begin{aligned}\|A_2V - V^0\|_{C(G(T))} &= \sup_{(z,t)\in G(T)} |A_2V(z,t) - V_0(z,t)| \\ &\leqslant \frac{1}{2} \sup_{(z,t)\in G(T)} \iint\limits_{D_2(z,t)} \left[|p(\xi)||V(\xi,\tau)| + |\bar{q}(\xi)||V(\xi,\tau)|^m\right] d\xi d\tau \\ &\leqslant \left[p_0(1+\gamma_0) + q_0(1+\gamma_0)^m\right]\frac{T^2}{8}.\end{aligned} \tag{46}$$

From the formulas (46) it follows that if

$$T_2' = T_2'(\gamma_0) = \min\left\{T_1,\ \sqrt{8}(p_0(1+\gamma_0) + q_0(1+\gamma_0)^m)^{-1/2}\right\}$$

the inequality $\|A_2V - V^0\|_{C(G(T_2'))} \leqslant 1$, is fulfilled, i. e. $A_2V \in \mathbf{B}^2_{T_2'}$. Thus, A_2 maps the ball $\mathbf{B}^2_{T_2'}$ into itself.

Let us show that the operator A_2, defined by the equality (43), is compressive for a sufficiently small $T > 0$.

Let $V_1(z,t)$ and $V_2(z,t) \in \mathbf{B}^2_T$. By virtue of (45) $|V(z,t)| = V(z,t)$ for $V(z,t) \in \mathbf{B}^2_T$, therefore, we can omit the modulus sign in the integrand expression. Using (23) we estimate the difference

$$\begin{aligned}\|A_2V_1 - A_2V_2\|_{C(G(T))} &= \sup_{(z,t)\in G(T)} |A_2V_1(z,t) - A_2V_2(z,t)| \\ \leqslant \frac{1}{2} \sup_{(z,t)\in G(T)} &\iint\limits_{D_2(z,t)} \Big(|p(\xi)|\big|V_1(\xi,\tau) - V_2(\xi,\tau)\big| + |\bar{q}(\xi)|\big|V_1^m(\xi,\tau) - V_2^m(\xi,\tau)\big|\Big) d\xi d\tau \\ \leqslant \frac{1}{2} \sup_{(z,t)\in G(T)} &\iint\limits_{D_2(z,t)} \Big(|p(\xi)| + |\bar{q}(\xi)||R_m(V_1(z,t), V_2(z,t))|\Big)\big|V_1(\xi,\tau) - V_2(\xi,\tau)\big| d\xi d\tau \\ &\leqslant \underbrace{\frac{T^2}{8}\left[p_0 + q_0m(1+\gamma_0)^{m-1}\right]}_{=:\rho_2} \|V_1 - V_2\|_{C(G(T))}.\end{aligned} \tag{47}$$

It follows from (47) that for $\rho_2 \in (0,1)$ we can choose

$$T_2'' = T_2''(\gamma_0, \rho_2) = \min\left\{T_1,\ \sqrt{8\rho_2}\big(p_0 + q_0m(1+\gamma_0)^{m-1}\big)^{-1/2}\right\}$$

so that

$$\|A_2V_1 - A_2V_2\|_{C(G(T_1''))} \leqslant \rho_2\|V_1 - V_2\|_{C(G(T_1''))}.$$

Then A_2 is a compressive mapping on the set $\mathbf{B}^2_{T_2}$, where $T_2 = \min\left\{T_2', T_2''\right\}$. Then, according to Banach Compressive Mappings Principle, there exists a unique solution of the operator equation $V = A_2V$ in $\mathbf{B}_{T_2}$. Lemma 4.1 has been proved. □

By virtue of Lemma 4.1, equation (38) can be written as follows:

$$V(z,t) = V^0(z,t) - \frac{1}{2}\int\limits_0^z d\xi \int\limits_{t-z+\xi}^{t+z-\xi} \big[p(z)V(\xi,\tau) + \bar{q}(z)V^m(\xi,\tau)\big]\, d\tau d\xi,$$
$$(z,t) \in G(T_2). \quad (48)$$

Differentiate (48) by the variable t, we find

$$V_t(z,t) = \frac{\partial V^0}{\partial t}(z,t) - \frac{1}{2}\int\limits_0^z \Big(\big[p(\xi)V(\xi,t+z-\xi) + \bar{q}(\xi)V^m(\xi,t+z-\xi)\big] - \big[p(\xi)V(\xi,t-z+\xi) + \bar{q}(\xi)V^m(\xi,t-z+\xi)\big]\Big)\, d\xi.$$

Denote $V_t(z,t) = W(z,t)$. As a result, we get

$$W(z,t) = W^0(z,t) - \frac{1}{2}\int\limits_0^z \Big(\big[p(\xi)V(\xi,t+z-\xi) + \bar{q}(\xi)V^m(\xi,t+z-\xi)\big] - \big[p(\xi)V(\xi,t-z+\xi) + \bar{q}(\xi)V^m(\xi,t-z+\xi)\big]\Big)\, d\xi, \quad (z,t) \in G(T_2), \quad (49)$$

where

$$W^0(z,t) = \frac{\partial V_0}{\partial t}(z,t) = \frac{f'(t+z) + f'(t-z)}{2} + \frac{\mu(t+z) - \mu(t-z)}{2}. \quad (50)$$

Denote $\|W^0\|_{C(G(T_2))} = \gamma_1/2$.

Rewrite the equation (35) in form

$$\bar{q}(z) = \bar{q}^0(z) - \int\limits_0^z \big[p(\xi) + m\bar{q}(\xi)V^{m-1}(\xi,2z-\xi)\big] W(\xi,2z-\xi)\, d\xi, \quad z \in [0, T_2/2]. \quad (51)$$

We have obtained a closed system of equations (51), (48), (49):

$$\bar{q}(z) = \bar{q}^0(z) - \int\limits_0^z \big[p(\xi) + m\bar{q}(\xi)V^{m-1}(\xi,2z-\xi)\big] W(\xi,2z-\xi)\, d\xi,$$
$$V(z,t) = V^0(z,t) - \frac{1}{2}\int\limits_0^z d\xi \int\limits_{t-z+\xi}^{t+z-\xi} \big[p(z)V(\xi,\tau) + \bar{q}(z)V^m(\xi,\tau)\big]\, d\tau, \quad (52)$$
$$W(z,t) = W^0(z,t) - \frac{1}{2}\int\limits_0^z \Big(\big[p(\xi)V(\xi,t+z-\xi) + \bar{q}(\xi)V^m(\xi,t+z-\xi)\big] - \big[p(\xi)V(\xi,t-z+\xi) + \bar{q}(\xi)V^m(\xi,t-z+\xi)\big]\Big)\, d\xi.$$

In equalities (52) $(z,t)\in G(T_2)$. Further, for convenience of writing, we will omit the index 2, assuming $T_2 = T$.

Let us define the vector functions

$$\begin{gathered}\mathbf{g} = (g_1, g_2, g_3), \quad \mathbf{g}^0(z,t) = \big(g_1^0(z), g_2^0(z,t), g_3^0(z,t)\big),\\ g_1(z) = \bar{q}(z), \quad g_2(z,t) = V(z,t), \quad g_3(z,t) = W(z,t),\\ g_1^0(z) = \bar{q}^0(z), \quad g_2^0(z,t) = V^0(z,t), \quad g_3^0(z,t) = W^0(z,t)\end{gathered} \tag{53}$$

and rewrite the equations (52) in operator form

$$\mathbf{g} = \widehat{\mathbf{A}}\mathbf{g}, \tag{54}$$

where the operator $\widehat{\mathbf{A}} = (\widehat{A}_1, \widehat{A}_2, \widehat{A}_3)$ is defined as follows:

$$\begin{gathered}\widehat{A}_1\mathbf{g} = g_1^0(z) - \int_0^z \big[p(\xi) + m g_1(\xi) g_2^{m-1}(\xi, 2z-\xi)\big] g_3(\xi, 2z-\xi)\, d\xi,\\ \widehat{A}_2\mathbf{g} = g_2^0(z,t) - \frac{1}{2}\int_0^z d\xi \int_{t-z+\xi}^{t+z-\xi} \big[p(\xi) g_2(\xi,\tau) + g_1(\xi) g_2^m(\xi,\tau)\big]\, d\tau,\\ \widehat{A}_3\mathbf{g} = g_3^0(z,t) - \frac{1}{2}\int_0^z \Big(\big[p(\xi) g_2(\xi, t+z-\xi) + g_1(\xi) g_2^m(\xi, t+z-\xi)\big]\\ - \big[p(\xi) g_2(\xi, t-z+\xi) + g_1(\xi) g_2^m(\xi, t-z+\xi)\big]\Big)\, d\xi.\end{gathered} \tag{55}$$

Denote by $\mathbf{C}(G(T)) = C[0, T/2] \times C(G(T)) \times C(G(T))$ space of continuous vector functions with norm

$$\|\mathbf{g}\|_{\mathbf{C}(G(T))} = \max\big\{\|g_1\|_{C[0,T/2]}, \max_{k=2,3} \|g_k\|_{C(G(T))}\big\}. \tag{56}$$

Since $\mathbf{g}^0 \in \mathbf{C}(G(T))$, then all vector functions defined in (53) are elements of $\mathbf{C}(G(T))$. We introduce in this Banach space the closed set

$$\begin{aligned}\mathbf{R}_T := \big\{\mathbf{g} \in \mathbf{C}(G(T)) \mid \big\|g_1 - g_1^0\big\|_{C[0,T/2]} \leqslant q_0/2,&\\ \big\|g_2 - g_2^0\big\|_{C(G(T))} \leqslant 1,\ \big\|g_3 - g_3^0\big\|_{C(G(T))} \leqslant \gamma_1/2\big\}.&\end{aligned} \tag{57}$$

The following estimates

$$g_2(z,t) \geq 0, \quad \|g_1\|_{C[0,T/2]} \leqslant q_0, \quad \|g_2\|_{C(G(T))} \leqslant 1+\gamma_0, \quad \|g_3\|_{C(G(T))} \leqslant \gamma_1 \tag{58}$$

hold true on this set.

Using (55), (58), we can write the estimates for $|\widehat{A}_k\mathbf{g} - g_k^0|$, $k = 1, 2, 3$,

$$|\widehat{A}_1\mathbf{g} - g_1^0| \leqslant \int_0^z \big[|p(\xi)| + m|g_1(\xi)||g_2(\xi, 2z-\xi)|^{m-1}\big] |g_3(\xi, 2z-\xi)|\, d\xi$$

$$\leqslant \frac{T}{2}\big[p_0 + mq_0(1+\gamma_0)^{m-1}\big]\gamma_1,$$

$$|\widehat{A}_2\mathbf{g} - g_2^0| \leqslant \frac{1}{2}\int\limits_0^z d\xi \int\limits_{t-z+\xi}^{t+z-\xi} \big[|p(\xi)| + |g_1(\xi)||g_2(\xi,\tau)|^{m-1}\big]|g_2(\xi,\tau)|\,d\tau$$

$$\leqslant \frac{T^2}{8}\big[p_0(1+\gamma_0) + q_0(1+\gamma_0)^m\big], \quad (59)$$

$$|\widehat{A}_3\mathbf{g} - g_3^0| \leqslant \frac{1}{2}\int\limits_0^z \Big(\big[|p(\xi)| + |g_1(\xi)|g_2(\xi, t+z-\xi)|^{m-1}\big]|g_2(\xi, t+z-\xi)|$$
$$+ \big[|p(\xi)| + |g_1(\xi)|g_2(\xi, t-z+\xi)|^{m-1}\big]|g_2(\xi, t-z+\xi)|\Big)\,d\xi$$
$$\leqslant \frac{T}{2}\big[p_0(1+\gamma_0) + q_0(1+\gamma_0)^m\big].$$

From the formulas (59) it follows that when

$$T_0' = T_0'(q_0,\gamma_0,\gamma_1) = \min\Big\{T_2,\ q_0\big[\big(p_0 + mq_0(1+\gamma_0)^{m-1}\big)\gamma_1\big]^{-1},$$
$$\sqrt{8}\big[p_0(1+\gamma_0) + q_0(1+\gamma_0)^m\big]^{-1/2},\ \gamma_1\big[p_0(1+\gamma_0) + q_0(1+\gamma_0)^m\big]^{-1}\Big\}$$

the following inequalities hold

$$\|\widehat{A}_1\mathbf{g} - g_1^0\|_{C[0,T_0'/2]} \leqslant q_0/2, \quad \|\widehat{A}_2\mathbf{g} - g_2^0\|_{C(G(T_0'))} \leqslant 1, \quad \|\widehat{A}_3\mathbf{g} - g_3^0\|_{C(G(T_0'))} \leqslant \gamma_1/2.$$

Thus, $\widehat{\mathbf{A}}$ maps $\mathbf{R}_{T_0'}$ into itself.

Now we demonstrate that the operator $\widehat{\mathbf{A}}$, defined by the equality (54), is compressive for a sufficiently small $T > 0$.

Let

$$\mathbf{g}^1 = (g_1^1, g_2^1, g_3^1), \quad \mathbf{g}^2 = (g_1^2, g_2^2, g_3^2), \quad \mathbf{g}^k \in \mathbf{R}_T, \quad k = 1, 2.$$

Note that similarly to (20), (21) we can write

$$\begin{gathered}\big(g_2^1(z,t)\big)^m - \big(g_2^2(z,t)\big)^m = \big(g_2^1(z,t) - g_2^2(z,t)\big)R_m\big(g_2^1(z,t), g_2^2(z,t)\big),\\ R_m\big(g_2^1(z,t), g_2^2(z,t)\big) = m\int\limits_0^1 \big[g_2^1(z,t)s' + g_2^2(z,t)(1-s')\big]^{m-1}\,ds'.\end{gathered} \quad (60)$$

Moreover, by virtue of (58) similarly to (22) we have

$$\begin{gathered}\big|R_m\big(g_2^1(z,t), g_2^2(z,t)\big)\big| \leqslant m(1+\gamma_0)^{m-1},\\ \big|g_2^1(z,t)\big)^m - \big(g_2^2(z,t)\big)^m\big| \leqslant \big|g_2^1(z,t) - g_2^2(z,t)\big|\big|R_m\big(g_2^1(z,t), g_2^2(z,t)\big)\big|.\end{gathered} \quad (61)$$

Using the formulas (23), (55), (60), (61), we estimate the differences $|\widehat{A}_k\mathbf{g}^1 - \widehat{A}_k\mathbf{g}^2|$, $k = \overline{1,3}$,

$$
\begin{aligned}
|\widehat{A}_1\mathbf{g}^1-\widehat{A}_1\mathbf{g}^2| \leqslant \int\limits_0^z \Big[&|p(\xi)|\big|g_3^1(\xi,2z-\xi)-g_3^2(\xi,2z-\xi)\big| \\
&+ m\big|g_1^1(\xi)-g_1^2(\xi)\big|\big|g_2^1(\xi,2z-\xi)\big|^{m-1}\big|g_3^1(\xi,2z-\xi)\big|\big] \\
+m\big|g_1^2(\xi)\big|\big|g_2^1(\xi,2z-\xi)-g_2^2(\xi,2z-\xi)\big|&\big|R_{m-1}\big(g_2^1(\xi,2z-\xi),g_2^2(\xi,2z-\xi)\big)\big|\big|g_3^1(\xi,2z-\xi)\big| \\
&+ m\big|g_1^2(\xi)\big|\big|g_2^2(\xi,2z-\xi)\big|^{m-1}\big|g_3^1(\xi,2z-\xi)-g_3^2(\xi,2z-\xi)\big|\Big]\,d\xi \\
\leqslant \frac{T}{2}\big[p_0+m(1+\gamma_0)^{m-1}\gamma_1&+mq_0(m-1)(1+\gamma_0)^{m-2}\gamma_1+mq_0(1+\gamma_0)^{m-1}\big]\|\mathbf{g}^1-\mathbf{g}^2\|_{\mathbf{C}(G(T))} \\
= \frac{T}{2}&\underbrace{\Big[p_0+m(1+\gamma_0)^{m-2}\big[(1+\gamma_0)(\gamma_1+q_0)+q_0(m-1)\gamma_1\big]\Big]}_{=:\alpha_1}\|\mathbf{g}^1-\mathbf{g}^2\|_{\mathbf{C}(G(T))},
\end{aligned}
$$

$$
\begin{aligned}
|\widehat{A}_2\mathbf{g}^1-\widehat{A}_2\mathbf{g}^2| \leqslant \frac{1}{2}\int\limits_0^z d\xi \int\limits_{t-z+\xi}^{t+z-\xi} \Big(&|p(\xi)|\big|g_2^1(\xi,\tau)-g_2^2(\xi,\tau)\big| + \big|g_1^1(\xi)-g_1^2(\xi)\big|\big|g_2^1(\xi,\tau)\big|^m \\
&+ \big|g_1^2(\xi)\big|\big|g_2^1(\xi,\tau)-g_2^2(\xi,\tau)\big|\big|R_m\big(g_2^1(\xi,\tau),g_2^2(\xi,\tau)\big)\big|\Big)\,d\tau \\
&\leqslant \frac{T^2}{8}\underbrace{\big[p_0+(1+\gamma_0)^m+mq_0(1+\gamma_0)^{m-1}\big]}_{=:\alpha_2}\|\mathbf{g}^1-\mathbf{g}^2\|_{\mathbf{C}(G(T))}, \quad (62)
\end{aligned}
$$

$$
\begin{aligned}
|\widehat{A}_3\mathbf{g}^1-\widehat{A}_3\mathbf{g}^2| \leqslant \frac{1}{2}\int\limits_0^z \Big(&|p(\xi)|\big|g_2^1(\xi,t+z-\xi)-g_2^2(\xi,t+z-\xi)\big| + \big|g_1^1(\xi)-g_1^2(\xi)\big|\big|g_2^1(\xi,t+z-\xi)\big|^m \\
+ \big|g_1^2(\xi)\big|g_2^1(\xi,t+z-\xi)&-g_2^2(\xi,t+z-\xi)\big|\big|R_m\big(g_2^1(\xi,t+z-\xi),g_2^2(\xi,t+z-\xi)\big)\big| \\
+ |p(\xi)|\big|g_2^1(\xi,t-z+\xi)&-g_2^2(\xi,t-z+\xi)\big| + \big|g_1^1(\xi)-g_1^2(\xi)\big|\big|g_2^1(\xi,t-z+\xi)\big|^m \\
+ \big|g_1^2(\xi)\big|g_2^1(\xi,t-z+\xi)&-g_2^2(\xi,t-z+\xi)\big|\big|R_m\big(g_2^1(\xi,t-z+\xi),g_2^2(\xi,t-z+\xi)\big)\big|\Big)\,d\xi \\
&\leqslant \frac{T}{2}\underbrace{\big[p_0+(1+\gamma_0)^m+mq_0(1+\gamma_0)^{m-1}\big]}_{=:\alpha_3}\|\mathbf{g}^1-\mathbf{g}^2\|_{\mathbf{C}(G(T))}.
\end{aligned}
$$

Let $\rho \in (0,1)$ and we choose T_0'' from the condition

$$
T_0'' = T_0''(q_0,\gamma_0,\gamma_1) = \min\left\{T_2,\ \frac{2\rho}{\alpha_1},\ \sqrt{\frac{8\rho}{\alpha_2}},\ \frac{2\rho}{\alpha_3}\right\},
$$

then $\|\widehat{\mathbf{A}}V_1-\widehat{\mathbf{A}}V_2\|_{C(G(T_0''))} \leqslant \rho\|V_1-V_2\|_{\mathbf{C}(G(T_0''))}$. Thus, $\widehat{\mathbf{A}}$ is a compressive mapping on the set $\mathbf{R}_{T_0}$, where $T_0 = T_0(q_0,\gamma_0,\gamma_1) = \min\{T_0', T_0''\}$. Then there exists a unique solution of the operator equation (54) in $\mathbf{R}_{T_0}$. Theorem 4.1 has been proved. □

5 A stability estimate of the solution to the inverse problem

Let $T_0 = T_0(q_0, \gamma_0, \gamma_1)$ is the number defined in theorem 4.2. Define the class of functions

$$\mathcal{Q}(q_0) = \big\{\bar{q} \in C[0, T_0/2] \mid \|\bar{q} - \bar{q}^0\|_{C[0,T/2]} \leqslant q_0/2\big\}.$$

Here the number q_0 and the function $\bar{q}^0(z)$ are defined in the previous section (see (36)). We also define the class $\mathcal{F}(\gamma_0, \gamma_1)$ consisting of functions $f(t), \mu(t)$ for which the conditions of the theorem 4.1 are fulfilled and, in addition, for the functions $V^0(z,t)$ and $W^0(z,t)$ defined by formulas (39) and (50), respectively, the conditions

$$\max_{(z,t)\in G(T_0)} V^0(z,t) \leqslant \gamma_0, \qquad \max_{(z,t)\in G(T_0)} |W^0(z,t)| \leqslant \gamma_1/2$$

hold.

Theorem 5.1. *Let the functions $\bar{q}_k \in \mathcal{Q}(q_0)$, $k = 1, 2$, be solutions of inverse problem (9)–(12) with data $(f_k, \mu_k) \in \mathcal{F}(\gamma_0, \gamma_1)$ for $k = 1, 2$. Then there exists a positive number $C = C(q_0, \gamma_0, \gamma_1, T_0)$ such that the following estimate*

$$\|\bar{q}_1 - \bar{q}_2\|_{C[0,T_0/2]} \leqslant C\big(\|\mu_1 - \mu_2\|_{C^1[0,T_0]} + \|f_1 - f_2\|_{C^2[0,T_0]}\big). \tag{63}$$

is valid.

Proof. Rewrite equations (52) as follows:

$$\bar{q}(z) = 2[f''(2z) + \mu'(2z)] - p(z) - \int_0^z \big[p(\xi) + m\bar{q}(\xi)V^{m-1}(\xi, 2z-\xi)\big]W(\xi, 2z-\xi)\, d\xi,$$

$$V(z,t) = \frac{f(t+z) + f(t-z)}{2} + \frac{1}{2}\int_{t-z}^{t+z} \mu(\tau)\, d\tau - \frac{1}{2}\int_0^z d\xi \int_{t-z+\xi}^{t+z-\xi} \big[p(z)V(\xi,\tau) + \bar{q}(z)V^m(\xi,\tau)\big]\, d\tau, \tag{64}$$

$$W(z,t) = \frac{f'(t+z) + f'(t-z)}{2} + \frac{\mu(t+z) + \mu(t-z)}{2} - \frac{1}{2}\int_0^z \Big(\big[p(\xi)V(\xi, t+z-\xi) + \bar{q}(\xi)V^m(\xi, t+z-\xi)\big] - \big[p(\xi)V(\xi, t-z+\xi) + \bar{q}(\xi)V^m(\xi, t-z+\xi)\big]\Big)\, d\xi.$$

Let us define the functions

$$\widetilde{q} = \bar{q}_1 - \bar{q}_2, \quad \widetilde{V}_1 = V_1 - V_2, \quad \widetilde{W}_1 = W_1 - W_2, \quad \widetilde{\mu} = \mu_1 - \mu_2, \quad \widetilde{f} = f_1 - f_2,$$

and, using (23), we write the equations (64) in terms of functions $\tilde q$, $\widetilde V$, $\widetilde W$, $\widetilde\mu$, $\widetilde f$. As a result, we obtain the following relations

$$\widetilde q(z) = 2[\widetilde f''(2z) + \widetilde\mu'(2z)] - \int_0^z \big[p(\xi)\widetilde W(\xi, 2z-\xi) + m\widetilde q(\xi) V_1^{m-1}(\xi, 2z-\xi) W_1(\xi, 2z-\xi)$$
$$+ m\bar q_2(\xi)\widetilde V(\xi, 2z-\xi) R_{m-1}(V_1(\xi, 2z-\xi), V_2(\xi, 2z-\xi)) W_1(\xi, 2z-\xi)$$
$$+ m\bar q_2(\xi) V_2^{m-1}(\xi, 2z-\xi)\widetilde W(\xi, 2z-\xi)\big]\, d\xi, \quad (65)$$

$$\widetilde V(z,t) = \frac{\widetilde f(t+z) + \widetilde f(t-z)}{2} + \frac{1}{2}\int_{t-z}^{t+z} \widetilde\mu(\tau)\, d\tau$$
$$- \frac{1}{2}\int_0^z d\xi \int_{t-z+\xi}^{t+z-\xi} \big[p(z)\widetilde V(\xi,\tau) + \widetilde q(z) V_1^m(\xi,\tau)$$
$$+ \bar q_2(z)\widetilde V(\xi,\tau) R_m(V_1(\xi,\tau), V_2(\xi,\tau))\big]\, d\tau, \quad (66)$$

$$\widetilde W(z,t) = \frac{\widetilde f'(t+z) + \widetilde f'(t-z)}{2} + \frac{\widetilde\mu(t+z) + \widetilde\mu(t-z)}{2}$$
$$- \frac{1}{2}\int_0^z \Big(\big[p(\xi)\widetilde V(\xi, t+z-\xi) + \widetilde q(\xi) V_1^m(\xi, t+z-\xi)$$
$$+ \bar q_2(\xi)\widetilde V(\xi, t+z-\xi) R_m(V_1(\xi, t+z-\xi), V_2(\xi, t+z-\xi))\big]$$
$$- \big[p(\xi)\widetilde V(\xi, t-z+\xi) + \widetilde q(\xi) V_1^m(\xi, t-z+\xi)$$
$$+ \bar q_2(\xi)\widetilde V(\xi, t-z+\xi) R_m(V_1(\xi, t-z+\xi), V_2(\xi, t-z+\xi))\big]\Big)\, d\xi. \quad (67)$$

Denote $\psi(z) = \max\{|\widetilde q(z)|, \max_t |\widetilde V(z,t)|, \max_t |\widetilde W(z,t)|\}$. Using (58), from (65)–(67) we can write

$$|\widetilde q(z)| \leqslant 2\big(\|\widetilde f\|_{C^2[0,T]} + \|\widetilde\mu\|_{C^1[0,T]}\big) + \int_0^z \big[p_0 + m(1+\gamma_0)^{m-1}\gamma_1$$
$$+ mq_0(m-1)(1+\gamma_0)^{m-2}\gamma_1 + mq_0(1+\gamma_0)^{m-1}\big]\psi(\xi)\, d\xi,$$

$$|\widetilde V(z,t)| \leqslant \|\widetilde f\|_{C[0,T]} + T\|\widetilde\mu\|_{C[0,T]} + \frac{T}{2}\int_0^z \big[p_0 + (1+\gamma_0)^m + q_0 m(1+\gamma_0)^{m-1}\big]\psi(\xi)\, d\xi,$$
$$(68)$$
$$|\widetilde W(z,t)| \leqslant \|\widetilde f\|_{C^1[0,T]} + \|\widetilde\mu\|_{C[0,T]} + \int_0^z \big[p_0 + (1+\gamma_0)^m + q_0 m(1+\gamma_0)^{m-1}\big]\psi(\xi)\, d\xi.$$

From (68) we get

$$\psi(z) \leqslant C_1\big(\|\widetilde{f}\|_{C^2[0,T_0]} + \|\widetilde{\mu}\|_{C^1[0,T_0]}\big) + C_2 \int\limits_0^z \psi(z)\, d\xi, \tag{69}$$

where $C_1 = \max\{2, T_0\}$,

$$C_2 = \max\Big\{\big[p_0 + m(1+\gamma_0)^{m-1}\gamma_1 + mq_0(m-1)(1+\gamma_0)^{m-2}\gamma_1 + mq_0(1+\gamma_0)^{m-1}\big], \\ \big[p_0 + (1+\gamma_0)^m + q_0 m(1+\gamma_0)^{m-1}\big]\max\{1, T_0/2\}\Big\}.$$

Using Gronwall's inequality, we obtain the estimate

$$|\psi(z)| \leqslant C_1\big(\|\widetilde{f}\|_{C^2[0,T_0]} + \|\widetilde{\mu}\|_{C^1[0,T_0]}\big)e^{zC_2}, \quad z \in [0, T_0/2]. \tag{70}$$

The required stability estimate (63) follows from (70) with $C = C_1 e^{C_2 T_0/2}$. Theorem 5.1 has been proved. □

Corollary 5.1. *The uniqueness of the solution to inverse problem* (9)–(12) *follows from estimate* (63).

Acknowledgement

This work was performed within the state assignment of the Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Science, project no. FWNF-2022-0009.

References

[1] Lindblad H., Tao T. *Asymptotic decay for a one-dimensional nonlinear wave equation*, Analysis of PDEs (2011); arXiv:1011.0949v5 [math.AP] 25 May 2011; http://arxiv.org/abs/1011.0949v5

[2] Laurent C. *On stabilization and control for the critical Klein—Gordon equation on a 3-D compact manifold*, J. Functional Analysis, Vol. 260 (2011), p. 1304–1368.

[3] Joly R., Laurent C. *Stabilization for the semilinear wave equation with geometric control condition*, Analysis of PDEs (2013); arXiv:1205.2459v4 [math.AP] 2 Dec 2013; https://doi.org/10.48550/arXiv.1205.2459

[4] Yang S. *Global behaviors of defocusing semilinear wave equations*, Analysis of PDEs (2019); arXiv:1908.00606 [math.AP] 1 Aug 2019; http://arxiv.org/abs/1908.00606v1

[5] D'Ancona P. *On the supercritical defocusing NLW outside a ball*, Analysis of PDEs (2020); arXiv:1912.13216v2 [math.AP] 28 Jul 2020; http://arxiv.org/abs/1912.13216v2

[6] Wei D., Yang Sh. *Asymptotic decay for defocusing semilinear wave equations in* $\mathbb{R}^{n+1}$, Analysis of PDEs (2020); arXiv:2003.12264v1 [math.AP] 27 Mar 2020; http://arxiv.org/abs/2003.12264v1

[7] Dai W. *The global behaviors for defocusing wave equations in two dimensional exterior region*, Analysis of PDEs (2023); arXiv:2301.07935v2 [math.AP] 20 Sep 2023; https://doi.org/10.1007/s00229-023-01511-5

[8] Matus P.P., Lemeshevsky S.V. *Coefficient stability of solutions to difference schemes approximating mixed problems for semilinear hyperbolic equations*, Dokl. NAS Belarus, Vol. 64, No 2 (2020), p. 135–143 (in Russin); https://doi.org/10.29235/1561-8323-2020-64-2-135-143

[9] Lassas M., Uhlmann G., Wang Y.*Inverse problems for semilinear wave equations on Lorentzian manifolds*, Comm. Math. Phys., Vol. 360 (2018), p. 555–609; arXiv:1606.06261

[10] Wang Y., Zhou T. *Inverse problems for quadratic derivative nonlinear wave equations*, Comm. PDE, Vol. 44, No. 11 (2019), p. 1140–1158;

[11] Hintz P., Uhlmann G., Zhai J. *An inverse boundary value problem for a semilinear wave equation on Lorentzian manifolds*, Internat. Math. Res. Notices (2021); arXiv:2005.10447

[12] Barreto A.S., Stefanov P. *Recovery of a general nonlinearity in the semilinear wave equation*, Analysis of PDEs (2021); arXiv:2107.08513v1 [math.AP] 18 Jul 2021

[13] Chen X., Lassas M., Oksanen L., Paternain G.P. *Detection of Hermitian connections in wave equations with cubic non-linearity*, J. Eur. Math. Soc. Vol. 24, No. 7, (2022), p. 2191–2232.

[14] Barreto A.S., Uhlmann G., Wang Y. *Inverse scattering for critical semilinear wave equations*, Pure and Applied Analysis, Vol. 4 (2022), No. 2, p. 191–23.

[15] Lassas M., Liimatainen T., Potenciano-Machado L., Tyni T. *Uniqueness and stability of an inverse problem for a semi-linear wave equation*, J. Diff. Equ., Vol. 337 (2022), p. 395–435.

[16] Barreto A.S., Stefanov P. *Recovery of a cubic non-linearity in the wave equation in the weakly non-linear regime*, Commun. Math. Phys. Vol. 392, p. 25–53 (2022).

[17] Romanov V.G., Bugueva T.V. *Inverse problem for a nonlinear wave equation*, J. Appl. Ind. Math., Vol. 16, No. 2 (2022), p. 333–348; DOI: 10.1134/S1990478922020132

[18] Romanov V.G. *An inverse problem for a semilinear wave equation*, Dokl. Math., Vol. 105, No. 3 (2022), p. 166–170; DOI: 10.1134/S1064562422030097

[19] Romanov V.G., Bugueva T.V. *The problem of determining the coefficient of the nonlinear term in a quasilinear wave equation*, J. Appl. Ind. Math., Vol. 16, No. 3 (2022), p. 550–562; DOI: 10.1134/S1990478922030188

[20] Romanov V.G., Bugueva T.V. *The problem of determining the coefficient multiplying a power-law gradient nonlinearity in a semilinear wave equation*, J. Appl. Ind. Math., Vol. 17, No. 2 (2023), p. 370–384; DOI: 10.1134/S1990478923020151

[21] Romanov V.G. *An inverse problem for the wave equation with nonlinear damping*, Sib. Math. J., Vol. 64, No 3 (2023), p. 670–685; DOI: 10.1134/S003744662303014X

V. G. Romanov,
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,
Email: romanov@math.nsc.ru

T. V. Bugueva
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,

Novosibirsk State University,
Novosibirsk 630090, ul. Pirogova 1, Russia,
Email: bugueva@math.nsc.ru

Received 14.03.2024, revised 15.04.2024, Accepted 16.04.2024