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IDENTIFICATION OF POSITIONS OF SEPARATED RIGID INCLUSIONS BY BOUNDARY MEASUREMENTS FOR A SIGNORINI PROBLEM

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Abstract A variational inequality describing an elastic body with a finite set of rigid inclusions is considered. The Signorini condition is imposed on a part of the boundary of the body. On the other part a homogeneous Dirichlet boundary condition is specified. The inclusions are arranged such that distance between any two inclusions is not less than a given positive number. All inclusions are located at a nonzero distance from the outer boundary. An inverse problem is investigated, which consists in identification of positions of the rigid inclusions from the measurement of displacements on an observation boundary. Continuous dependency of the solution of the forward problem on parameters of inclusions' location and rotation is established. This provides existence of a solution for the inverse identification problem.

Key words: contact problem, rigid inclusion, variational inequality, inverse problem.

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1 Introduction

Optimal control and inverse problems describing inhomogeneous bodies have been attracting the close attention of specialists. Since, on the one hand, the improvement of performance properties of composites is a crucial direction of technological progress. On the other hand, a lot of practical problems for composites imply studies in which it is necessary to identify information about internal components based on external measurements. Below we overview some mathematical results on parameter identification and its applications from the literature.

We cite the classical theory of inverse problems and its applications in mathematical physics [1, 2]. Inverse problems for the nondestructive testing of inclusions embedded in Kirchhoff–Love plates were investigated in [3, 4, 5]. A shape and topological sensitivity analysis was performed, as well as relevant numerical results were provided in [6] for an inverse problem of detection of the position of a hole in a domain. Shape optimization of rigid inclusions for elastic bodies with cracks were investigated, for example, in [7, 8]. In [9] a variational inequality was controlled by the position and size of an inhomogeneity posed in a one dimensional domain with moving boundary.

In [10, 11, 12] authors investigated the problem of determining the internal structure of an inhomogeneous body on the basis of boundary measurements. Geometric position of a deformable inclusion in frictionless unilateral contact with the matrix was identified on the basis of measurements surveyed at some sensor points on the external boundary in [13]. Based on optimization methods, a theoretical framework for identifying voids and inclusions was established in [14, 15, 16, 17], for cracks in [18, 19], and other works. The papers [20, 21] present solution of the inverse dynamic seismic problem in the frequency domain for a horizontally layered medium by determining parameters of an anisotropic layer (thin-layered pack). In the framework of variational inequalities with unilateral constraints of the Signorini type, see problems of optimal control by volume or boundary forces in [22, 23], and suitable numerical methods in [24]. Different optimality systems for strong stationarity in the case of optimal control of constrained problems can be found in [25, 26, 27]. We cite problems describing rigid and elastic inclusions in [28, 29, 30] as well as imperfect interfaces in [31, 32] which are of the great interest in the description of composite materials.

In the present paper, we study an obstacle problem for an elastic body with a finite set of bulk rigid inclusions of a prescribed number. The inclusions are mutually separated between themselves and from the boundary of the body by a strictly positive distance that is fixed a-priori. Our motivation stems from modeling of micro-defects and damage mechanisms, fibre-reinforced composites, and peridynamic models in solid mechanics, see the respective collection of works [33]. This geometric description is useful for further application of periodic as well as non-periodic homogenization techniques when the size of inclusions decreases. We refer, for example, to [34] for the mathematical modeling of discontinuous fields in a two-phase medium.

Assuming that displacements are measured on an observation part of external boundary of the body, we formulate the inverse problem of identification of positions of inclusions. For each inclusion with a prescribed shape, it is necessary to determine the location, which is characterized by the vector of parallel translation and angles of rotation relative to the reference coordinate axes. The novelty of the class of forward problems under consideration consists in two generalizations compared to the previous investigations [35, 36]. First, angles of rotations of inclusions are taken into account. The second generalization deals with the assumption that functions of external forces itself depend on locations of inclusions. Continuous dependence of the solution of the forward problem have been proved with respect to the inclusion parameters varying in a suitable compact set. Based on this result we establish existence of a solution for the inverse identification problem.

2 Family of separated inclusions in a 3D domain

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\Gamma \in C^{0,1}$, where $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j = 0, 1, 2$, $i \neq j$, and $\mathrm{meas}(\Gamma_0) > 0$, $\mathrm{meas}(\Gamma_1) > 0$, $\mathrm{meas}(\Gamma_2) > 0$. We consider a family of m simply connected subdomains $\omega_p \subset \Omega$, $p = 1, \ldots, m$, where $m \geq 1$ is some fixed natural number. We assume that these subdomains satisfy the following properties:

a) the domains ω_p have the Lipschitz boundaries $\partial\omega_p$, $p = 1, \ldots, m$;

b) the distance $dist(\overline{\omega}_p, \Gamma) \geq \delta_0$, $dist(\overline{\omega}_q, \overline{\omega}_p) \geq \delta_0$ for each $p, q = 1, \ldots, m$, $p \neq q$, where δ_0 is a sufficiently small positive number.

Let us note that the above assumptions a) and b) provide the possibility of applying trace theorems and the Poincaré inequality in corresponding Sobolev spaces. Under these assumptions, geometry of the subdomains can be arbitrary, just as their diameters can be arbitrarily small. In particular, periodically located rigid inclusions satisfying conditions a) and b) can be considered as adopted by the periodic homogenization.

We fix some inner points $x_0^p = (x_{10}^p, x_{20}^p, x_{30}^p)$ in each ω_p , $p = 1, \ldots, m$, and consider rotations of the subdomain ω_p on angles $\gamma_i^p \in (0, 2\pi]$, $i = 1, 2, 3$, around local axes

$$e_i^p = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\quad x_j = x_{j0}^p = const, \quad j \neq i\}, \quad i = 1, 2, 3.$$

For convention we compose the rotation matrix A of a sequence of rotations in the right-hand rule, that is, the rotation on $\gamma_1 \in (0, 2\pi]$ around the e_1^p-axis, after rotation on $\gamma_2 \in (0, 2\pi]$ around the e_2^p-axis which, in turn, after rotation on $\gamma_3 \in (0, 2\pi]$ around the e_3^p-axis, as follows

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma_1 & -\sin\gamma_1 \\ 0 & \sin\gamma_1 & \cos\gamma_1 \end{pmatrix} \begin{pmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 1 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 \end{pmatrix} \begin{pmatrix} \cos\gamma_3 & -\sin\gamma_3 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This case corresponds to counterclockwise rotation of vectors in the right coordinate system.

Let us denote by $\omega_p(\gamma^p)$ the transformed domains obtained by the rotations $\gamma^p = (\gamma_1^p, \gamma_2^p, \gamma_3^p)$. For each $X = (x^1, \ldots, x^m) \in \mathbb{R}^{3m}$, $\Upsilon = (\gamma^1, \ldots, \gamma^m) \in (0, 2\pi]^{3m}$ with $x^p = (x_1^p, x_2^p, x_3^p)$, $\gamma^p = (\gamma_1^p, \gamma_2^p, \gamma_3^p)$, $p = 1, \ldots, m$, we define the induced family

$$\mathcal{S}(X, \Upsilon) = \{\omega_1(x^1, \gamma^1), \ldots, \omega_m(x^m, \gamma^m)\}$$

of domains obtained by shifting of $\omega_p(\gamma^p)$, where $x^p = (x_1^p, x_2^p, x_3^p)$, $p = 1, \ldots, m$, are the translation vectors for each subdomain $\omega_p(\gamma^p)$, such that

$$\omega_p(x^p, \gamma^p) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,| \\ (x_1, x_2, x_3) = (x_1^p, x_2^p, x_3^p) + (\hat{x}_1, \hat{x}_2, \hat{x}_3), \quad (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \omega_p(\gamma^p)\}.$$

We consider a compact subset $\mathcal{A} \subset \mathcal{S}(X, \Upsilon)$ with elements satisfying the conditions:

$$\omega_p(x^p, \gamma^p) \subset \Omega, \quad dist(\overline{\omega}_p(x^p, \gamma^p), \Gamma) \geq \delta_0, \quad \forall\, p = 1, \ldots, m, \tag{1}$$

$$dist(\overline{\omega}_p(x^p, \gamma^p), \overline{\omega}_q(x^q, \gamma^q)) \geq \delta_0 \quad \forall\, p, q = 1, \ldots, m, \quad p \neq q. \tag{2}$$

As an example of the compact set satisfying (1), (2) we present appropriate balls Ω and ω_p, $p = 1, \ldots, m$. Let r_Ω be the radius of Ω, and $x^{c\Omega}$ be coordinates of its center. By x_0^p and r_p, $p = 1, \ldots, m$, we denote coordinates of centers and radii of balls ω_p, correspondingly. Then we have the following expressions for distances

$$dist(\overline{\omega}_p(x^p, \gamma^p), \overline{\omega}_q(x^q, \gamma^p)) = \|x_0^p + x^p - x_0^q - x^q\|_{\mathbb{R}^3} - r_p - r_q \geq \delta_0,$$

$$dist(\overline{\omega}_p(x^p, \gamma^p), \Gamma) = r_\Omega - r_p - \|x^{c\Omega} - x_0^p - x^p\|_{\mathbb{R}^3} \geq \delta_0.$$

In the sequel we will need the following assumption.

Assumption 2.1. *For some fixed* $p \in 1, \ldots, m$ *and an arbitrary strictly inner subdomain* $D \subset \omega_p(x^p, \gamma^p)$ *there exists a positive sufficiently small number* $\delta > 0$ *such that* $D \subset \omega_p(x^p, \gamma^p) \cap \omega_p(\hat{x}^p, \hat{\gamma}^p)$ *for all* $(\hat{x}^p, \hat{\gamma}^p) \in \mathbb{R}^6$ *such that* $\|(x^p, \gamma^p) - (\hat{x}^p, \hat{\gamma}^p)\|_{\mathbb{R}^6} < \delta$.

To formulate the model of a composite body with the family of rigid inclusions, we will use the concept of a rigid inclusion occupying an arbitrary subdomain $\mathcal{O} \subset \Omega$. In this case, the displacements in $\mathcal{O}$ should have a special structure $W|_{\mathcal{O}} = \rho$ with $\rho \in R(\mathcal{O})$, and $R(\mathcal{O})$ is the space of infinitesimal rigid displacements on $\mathcal{O}$:

$$R(\mathcal{O}) = \{\rho = (\rho_1, \rho_2, \rho_3) \quad | \; \rho^t(x) = Bx + C, \; x = (x_1, x_2, x_3)^t \in \mathcal{O}\},$$

where t denotes transposition of vectors, B is a skew-symmetric matrix which can be obtained from the rotation matrix A by formula $B = \dot{A}A^t$, and C is a vector, such that

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{3}$$

and $b_{12}, b_{13}, b_{23}, c_1, c_2, c_3 \in \mathbb{R}$, see details in [37, 38].

For the Signorini problem following next, we fix the element $(X, \Upsilon) \in \mathcal{A}$ and suppose that the domains $\omega_p(x^p, \gamma^p)$, $p = 1, \ldots, m$, refer to rigid inclusions, while the domain

$$\Omega \backslash \overline{\omega}_{\overline{1,m}}, \quad \omega_{\overline{1,m}} = \bigcup_{p=1}^{m} \omega_p(x^p, \gamma^p)$$

fits to the elastic matrix of the body.

3 Signorini problem

Denote by $W = (W_1, W_2, W_3)$ the displacement vector in the Sobolev space

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \mid \quad v = 0 \quad \text{on } \Gamma_0\}, \quad H(\Omega) = H^1_{\Gamma_0}(\Omega)^3.$$

We introduce the second order symmetric tensors describing deformation of an elastic part of the inhomogeneous body, and the corresponding stress by

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial W_i}{\partial x_j} + \frac{\partial W_j}{\partial x_i}\right), \quad \sigma_{ij}(W) = c_{ijkl}\varepsilon_{kl}(W), \quad i, j = 1, 2, 3,$$

using the convention of summation over repeated indexes, where c_{ijkl} is the given elasticity tensor, assumed to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, 3, \quad c_{ijkl} = const,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi : \; \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, 3, \quad c_0 = const, \quad c_0 > 0.$$

By the geometric assumption concerning the domain Ω and using the Korn's inequality from [39, 40], the following lower estimate holds

$$\int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(W)\, dx \geq c\|W\|^2_{H(\Omega)}, \quad \forall W \in H(\Omega), \tag{4}$$

with a constant $c > 0$ independent of W.

Remark 3.1. *The inequality* (4) *implies the equivalence of the standard norm in* $H(\Omega)$ *and the semi-norm determined by the left-hand side of* (4).

We consider the frictionless contact at the external boundary, which is modeled with the well-known Signorini condition

$$W_i\nu_i \leq 0 \quad \text{on} \quad \Gamma_1,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is an outward normal to Γ. We introduce the energy functional

$$\Pi(W, X, \Upsilon) = \frac{1}{2}\int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(W)\,dx - \int\limits_{\Omega} F_i(X, \Upsilon)W_i\,dx, \tag{5}$$

where the vector of body forces $F(X, \Upsilon) = (f_1(X, \Upsilon), f_2(X, \Upsilon), f_3(X, \Upsilon)) \in C(\Omega \times (0, 2\pi]^3; L_2(\Omega))^3$ is given. Regarding rotation, the periodicity condition

$$F(X, \hat{\Upsilon}) = F(X, \tilde{\Upsilon})$$

should hold for all $\hat{\Upsilon}$, $\tilde{\Upsilon}$ such that $|\hat{\gamma}_i^p - \tilde{\gamma}_i^p| = 2\pi$, $i = 1, 2, 3$, $p = 1, \ldots, m$. Note that in contrast to the functional considered in the earlier paper [36], the function of external loads $F(X, \Upsilon)$ depends on locations and rotations of inclusions. This assumption is certainly more justified from the point of view of physics.

For fixed $(X, \Upsilon) \in \mathcal{A}$, a Signorini problem for the composite body can be formulated as the following minimization problem (see textbooks [41, 42]):

$$\text{Find } U(X, \Upsilon) \in K(X, \Upsilon) \quad \text{such that } \Pi(U, X, \Upsilon) = \inf_{W \in K(X,\Upsilon)} \Pi(W, X, \Upsilon), \tag{6}$$

where the set of admissible displacements is defined as follows

$$K(X, \Upsilon) = \{W \in H(\Omega)\,|\quad W_i\nu_i \leq 0 \quad \text{on } \Gamma_1,$$
$$W|_{\omega_p(x^p,\gamma^p)} = \rho, \text{ where } \rho \in R(\omega_p(x^p, \gamma^p)), \quad p = 1, \ldots, m\}.$$

In the virtue of coercivity (4), the problem (6) has the unique solution $U(X, \Upsilon) \in K(X, \Upsilon)$, which satisfies the variational inequality (see [36]):

$$\int\limits_{\Omega} \sigma_{ij}(U(X, \Upsilon))\varepsilon_{ij}(W - U(X, \Upsilon))\,dx \geq \int\limits_{\Omega} F_i(X, \Upsilon)(W_i - U_i(X, \Upsilon))\,dx, \tag{7}$$

for all $W \in K(X, \Upsilon)$.

4 Inverse identification problem

In this section we formulate a class of inverse problem determining the family of rigid inclusions in the composite body. For this reason, parameters $(X, \Upsilon) \in \mathcal{A}$ are to be identified from measurements. Let us introduce the cost functional $J : \mathcal{A} \to \mathbb{R}$ representing a single boundary measurement by the following formula

$$J(X, \Upsilon) = \int\limits_{\Gamma_2} U_i(X, \Upsilon)\nu_i\,dx, \tag{8}$$

where $U(X,\Upsilon)$ is the solution of the forward problem (6).

As commonly adopted in optimal control theory, we should either minimize: find $z_1 \in \mathbb{R}$ such that

$$z_1 = \min_{(X,\Upsilon)\in\mathcal{A}} J(X,\Upsilon), \tag{9}$$

or maximize the cost: find $z_2 \in \mathbb{R}$ such that

$$z_2 = \max_{(X,\Upsilon)\in\mathcal{A}} J(X,\Upsilon). \tag{10}$$

For a given intermediate value $z \in [z_1, z_2]$, we can formulate the inverse problem as follows: find parameters $(X,\Upsilon) \in \mathcal{A}$ of three-dimensional inclusions and a displacement field $U(X,\Upsilon)$ of the body satisfying the variational inequality

$$\int\limits_{\Omega} \sigma_{ij}(U(X,\Upsilon))\varepsilon_{ij}(W - U(X,\Upsilon))\,dx \geq \int\limits_{\Omega} F_i(X,\Upsilon)(W_i - U_i(X,\Upsilon))\,dx, \tag{11}$$

for all $W \in K(X,\Upsilon)$, and the equality

$$z = J(X,\Upsilon). \tag{12}$$

Theorem 4.1. *There exist finite real numbers z_1 and z_2 solving the minimization* (9) *and the maximization* (10) *problems, with $z_1 \leq z_2$, such that for any fixed $z \in [z_1, z_2]$ the inverse problem* (11), (12) *has an exact solution.*

Proof. Let us consider the reduced function $J(X,\Upsilon)$, where $U(X,\Upsilon)$ is the solution of the variational inequality (7). Now we take into account Lemma (5.2), which is rather technical and will be proved below: the solutions $U(X,\Upsilon)$ of (7) are continuous with respect to (X,Υ) in the space $H(\Omega)$. This allows us to assert that the continuous function $J(X,\Upsilon)$ attains minimum and maximum values on compact set $\mathcal{A}$. Namely,

$$z_1 = \min_{(X,\Upsilon)\in\mathcal{A}} J(X,\Upsilon), \quad z_2 = \max_{(X,\Upsilon)\in\mathcal{A}} J(X,\Upsilon).$$

Since the set $\mathcal{A}$ is compact, by the intermediate value theorem we conclude that arbitrary $z \in [z_1, z_2]$ is attained within the relations (11) and (12). The theorem is proved. $\square$

Remark 4.1. *It is worth noting that, in virtue of Lemma* (5.2)*, the cost J in* (8) *can be taken in the form $J(X,\Upsilon) = G(U(X,\Upsilon))$, where $U(X,\Upsilon)$ is the solution of the forward problem* (6)*, for an arbitrary uniformly continuous functional $G : H(\Omega) \to \mathbb{R}$. For example, the result of Theorem* (4.1) *remains correct for the following measurements over an observation part of the boundary $\Gamma_1 \cup \Gamma_2$:*

$$J_i = \int_{\Gamma_2} U_i(X,Y)\,dx, \quad i = 1,2,3, \quad \textit{or} \quad J_4 = \int_{\Gamma_1} U_i(X,Y)\nu_i\,dx.$$

5 Auxiliary lemmas

To justify Lemma (5.2) used in the proof of Theorem (4.1), it needs first to prove the auxiliary lemma.

Lemma 5.1. *Let (X^*,Υ^*) be a fixed element of the set $\mathcal{A}\subset\mathcal{S}(X,\Upsilon)$ describing position of inclusions, and let $\{(X_n,\Upsilon_n)\}\subset\mathcal{A}$ be a sequence of elements converging to (X^*,Υ^*) in $\mathbb{R}^{6m}$ as $n\to\infty$. Then for an arbitrary feasible displacement $W\in K(X^*,\Upsilon^*)$ there exists a subsequence $\{(X_k,\Upsilon_k)\}=\{(X_{n_k},\Upsilon_{n_k})\}\subset\{(X_n,\Upsilon_n)\}$ and a sequence of functions $\{W_k\}$ such that $W_k\in K(X_k,\Upsilon_k)$, $k\in\mathbb{N}$, and $W_k\to W$ strongly in $H(\Omega)$ as $k\to\infty$.*

Proof. First we consider the obvious case when a subsequence $\{(X_{n_k},\Upsilon_{n_k})\}$ exists such that $(X_{n_k},\Upsilon_{n_k})=(X^*,\Upsilon^*)$, $k\in\mathbb{N}$. Then the assertion of the lemma holds for $W_k\equiv W$. We exclude this case from further consideration.

We will use the following notations $X^*=(x^{1,*},\dots,x^{m,*})$, $\Upsilon^*=(\gamma^{1,*},\dots,\gamma^{m,*})$, $X_n=(x^{1,n},\dots,x^{m,n})$, $\Upsilon_n=(\gamma^{1,n},\dots,\gamma^{m,n})$. Let the functions

$$\rho_p^*=(b^*_{p12}x_2+b^*_{p13}x_3+c^*_{p1},-b^*_{p12}x_1+b^*_{p23}x_3+c^*_{p2},-b^*_{p13}x_1-b^*_{p23}x_2+c^*_{p3})$$

describe infinitesimal rigid displacements in domains $\omega_p(x^{p,*},\gamma^{p,*})$, where b_{pji}, c_{pi} are some constants for $i,j\in\{1,2,3\}$, $j<i$, and $p=1,\dots,m$.

In order to construct a desired subsequence, we start with introducing a system of new domains $\hat\omega_{p,k}(x^{p,*},\gamma^{p,*})$, which depend on natural numbers $k\in\mathbb{N}$ and on the domains $\omega_p(x^{p,*},\gamma^{p,*})$, $p=1,\dots,m$. For arbitrary Lipschitz domain $\mathcal{O}\subset\Omega$ and any positive number ε, we denote by $\mathcal{O}^\varepsilon$ the extended domain

$$\mathcal{O}^\varepsilon=\{x\in\mathbb{R}^3\,|\,dist(x,\mathcal{O})<\varepsilon\}.$$

It is known [43] that there exists a positive number ε_0 small enough such that for all $0<\varepsilon<\varepsilon_0$ the domain $\mathcal{O}^\varepsilon$ would be Lipschitz. So we can choose the number ε_0 such that all extended domains $\omega_p^\varepsilon(x^{p,*},\gamma^{p,*})$, $p=1,\dots,m$, would be Lipschitz for $0<\varepsilon<\varepsilon_0$.

Let the positive number $M=\max\{1/\varepsilon_0,3/\delta_0\}$, and let us denote by $\hat\omega_{p,k}(x^{p,*},\gamma^{p,*})$, $k\in\mathbb{N}$, the Lipschitz domains defined as follows

$$\hat\omega_{p,k}(x^{p,*},\gamma^{p,*})=\omega_p^{\varepsilon_k}(x^{p,*},\gamma^{p,*}),\quad\text{where}\quad 0<\varepsilon_k=\frac{1}{M+k}<\min\Big\{\varepsilon_0,\frac{\delta_0}{3}\Big\}.$$

Obviously, we have the inclusions

$$\hat\omega_{p,1}(x^{p,*},\gamma^{p,*})\supset\ldots\supset\hat\omega_{p,k}(x^{p,*},\gamma^{p,*})\supset\ldots$$

for all $p=1,\dots,m$. On the basis of the affine functions ρ_p^*, $p=1,\dots,m$, the infinitesimal rigid displacements $\rho^*_{p,1}$ can be constructed by the following equalities:

$$\rho^*_{p,1}=(b^*_{p12}x_2+b^*_{p13}x_3+c^*_{p1},-b^*_{p12}x_1+b^*_{p23}x_3+c^*_{p2},-b^*_{p13}x_1-b^*_{p23}x_2+c^*_{p3}),$$

where $x\in\hat\omega_{p,1}(x^{p,*},\gamma^{p,*})$, $p=1,\dots,m$.

As the next step, we consider the family of auxiliary problems:

$$\text{Find an argument } Q_k \in K_k \text{ of the minimum} \quad p(Q_k) = \inf_{\chi \in K_k} p(\chi), \tag{13}$$

where $p(\chi) = \int\limits_{\Omega} \sigma_{ij}(\chi - W)\varepsilon_{ij}(\chi - W)\, dx$, and the feasible sets are

$$K_k = \{\chi \in H(\Omega) \mid \quad \chi = W \text{ on } \Gamma, \quad \chi|_{\hat{\omega}_{p,k}(x^{p,*},\gamma^{p,*})} = \rho^*_{p,1}, \quad p = 1,\dots,m\}.$$

The quadratic functional $p(\chi)$ is weakly lower semicontinuous and coercive on the space $H(\Omega)$ due to the lower estimate (4), and sets K_k are convex and closed in $H(\Omega)$, $k \in \mathbb{N}$. These properties along with the reflexivity of $H(\Omega)$ provide existence of the unique solution Q_k to the problem (13) for each fixed $k \in \mathbb{N}$, see [39]. This solution is characterized equivalently by the variational inequality

$$Q_k \in K_k, \quad \int\limits_{\Omega} \sigma_{ij}(Q_k - W)\varepsilon_{ij}(\chi - Q_k)\, dx \geq 0 \quad \forall \chi \in K_k. \tag{14}$$

Note that applying a lifting operator for the Lipschitz domain $\Omega \backslash \overline{\bigcup_{p=1}^m \hat{\omega}_{p,1}(x^{p,*},\gamma^{p,*})}$, a function $\hat{\chi} \in H(\Omega)$ can be constructed such that

$$\hat{\chi} = \rho^*_{p,1} \quad \text{on} \quad \hat{\omega}_{p1}(x^{p,*},\gamma^{p,*}), \quad p = 1,\dots,m, \quad \hat{\chi} = W \quad \text{on } \Gamma.$$

Since $\hat{\chi} \in K_k$ for all $k \in \mathbb{N}$, we substitute $\hat{\chi}$ as the test function in (14), which yields inequalities

$$\int\limits_{\Omega} \sigma_{ij}(Q_k - W)\varepsilon_{ij}(\hat{\chi})\, dx + \int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(Q_k)\, dx \geq \int\limits_{\Omega} \sigma_{ij}(Q_k)\varepsilon_{ij}(Q_k)\, dx \quad \forall k \in \mathbb{N}.$$

Taking into account the Korn inequality, from the last estimate we get the uniform upper bound:

$$\|Q_k\|_{H(\Omega)} \leq c \quad \forall k \in \mathbb{N}.$$

From the above estimate it follows that we can extract a subsequence $\{Q_{k_l}\}$ (for simplicity, we denote it by $\{Q_l\}$) which converges weakly in $H(\Omega)$ to some function $\widetilde{W}$, i.e.

$$Q_l \to \widetilde{W} \quad \text{weakly in } H(\Omega). \tag{15}$$

We will show that $\widetilde{W} = W$. For the sake of simplicity, denote the union of the domains $\omega_p\left(x^{p,*},\gamma^{p,*}\right)$ by

$$\omega^*_{\overline{1,m}} = \bigcup_{p=1}^m \omega_p\left(x^{p,*},\gamma^{p,*}\right).$$

Since $(Q_l - W) \in H^1_0\left(\Omega \backslash \overline{\omega^*}_{\overline{1,m}}\right)^3$, in the limit the inclusion $(\widetilde{W} - W) \in H^1_0\left(\Omega \backslash \overline{\omega^*}_{\overline{1,m}}\right)^3$ holds. If the functions are of the form $\chi^{\pm}_l = Q_l \pm \alpha$, where α is defined by zero extension of an arbitrary function $\tilde{\alpha} \in C^\infty_0\left(\Omega \backslash \overline{\omega^*}_{\overline{1,m}}\right)^3$ into Ω, then for sufficiently large l

we have $\chi_l^\pm \in K_{k_l}$. The sequences χ_l^+ and χ_l^- can be substituted as test functions into inequalities (14). As the result, we have

$$\int_\Omega \sigma_{ij}(Q_l - W)\varepsilon_{ij}(\alpha)\, dx = 0. \tag{16}$$

Now, we fix the function α and pass to the limit $l \to \infty$ in (16). The limiting expression takes the form

$$\int_\Omega \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)\, dx = \int_{\Omega\setminus\overline{\omega^*}_{\overline{1,m}}} \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)\, dx = 0,$$

for all $\alpha \in C_0^\infty(\Omega\setminus\overline{\omega^*}_{\overline{1,m}})^3$. In view of the density of $C_0^\infty\left(\Omega\setminus\overline{\omega^*}_{\overline{1,m}}\right)^3$ in $H_0^1\left(\Omega\setminus\overline{\omega^*}_{\overline{1,m}}\right)^3$, we deduce that $\widetilde{W} - W = 0$ in $H_0^1\left(\Omega\setminus\overline{\omega^*}_{\overline{1,m}}\right)^3$. By construction, the equality $\widetilde{W} = W$ is satisfied in the union of domains $\omega^*_{\overline{1,m}}$ and on the external boundary Γ. Consequently, $\widetilde{W} = W$ in $H(\Omega)$. Then there exists a sequence $\{Q_l\}$ such that $Q_l \in K_{k_l}$, $l \in \mathbb{N}$, and $Q_l \to W$ weakly in $H(\Omega)$ as $l \to \infty$.

We proceed to prove the strong convergence. The Mazur theorem provides the existence of a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of real numbers $\{\alpha(l)_i \,|\, i = l, \ldots, N(l)\}$ satisfying $\alpha(l)_i \geq 0$ and $\sum_{i=l}^{N(l)} \alpha(l)_i = 1$ such that the sequence $\{\hat{Q}_l\}$ defined by the convex combination

$$\hat{Q}_l = \sum_{i=l}^{N(l)} \alpha(l)_i Q_i$$

converges to W strongly in $H(\Omega)$. According to this construction, a subsequence $\{k_l\}$ of natural numbers corresponds to the subsequence $\{Q_l\}$ from (15), hence for the sequence $\{N(l)\}$ we will have the corresponding subsequence $\{k_{N(l)}\}$.

Now, we compose a subsequence $\{(X_{n_k}, \Upsilon_{n_k})\} \subset \{(X_n, \Upsilon_n)\}$ by the following procedure: for every $i \in \mathbb{N}$ take the first element of $\{(X_n, \Upsilon_n)\}$ satisfying the inequality

$$\|X_n - X^*\|_{\mathbb{R}^{3m}} + D_\Omega \|\Upsilon_n - \Upsilon^*\|_{\mathbb{R}^{3m}} < \frac{1}{M + k_{N(i)}}$$

as the element $(X_{n_i}, \Upsilon_{n_i})$ of the required subsequence $\{(X_{n_k}, \Upsilon_{n_k})\}$. Here by D_Ω we denote the following value representing the diameter of the domain Ω

$$D_\Omega = \max_{x,y\in\Gamma} \|x - y\|_{\mathbb{R}^3}.$$

In this case $\hat{Q}_k \in K(X_{n_k}, \Upsilon_{n_k})$, and we can set $\{W_{n_k}\}$ by the equations

$$W_{n_k} = \hat{Q}_k, \quad k = 1, 2, \ldots.$$

This completes the proof. □

Using Lemma (5.1) we are in a position to prove the following statement which was used in the proof of Theorem (4.1).

Lemma 5.2. *Let the paremeters $(X^*, \Upsilon^*) \in \mathcal{A}$, and $\{(X_n, \Upsilon_n)\} \subset \mathcal{A}$ be a sequence converging to (X^*, Υ^*) in $\mathbb{R}^{6m}$ as $n \to \infty$. Then $U(X_n, \Upsilon_n) \to U(X^*, \Upsilon^*)$ strongly in $H(\Omega)$ as $n \to \infty$, where $U(X_n, \Upsilon_n)$, $U(X^*, \Upsilon^*)$ are solutions to the minimization problem (6) corresponding to parameters (X_n, Υ_n), (X^*, Υ^*), respectively.*

Proof. We will prove the assertion by contradiction. Assume that there exists a number $\epsilon_0 > 0$ and a sequence $\{(X_n, \Upsilon_n)\} \subset \mathcal{A}$ such that $(X_n, \Upsilon_n) \to (X^*, \Upsilon^*)$, $\|U_n - U^*\| \geq \epsilon_0$, where $U_n = U(X_n, \Upsilon_n)$, $U^* = U(X^*, \Upsilon^*)$ are the corresponding solutions of (6).

Because $(0, 0, 0) \in K(X_n, \Upsilon_n)$ for all $n \in \mathbb{N}$, we can insert zero in (11) for fixed $n \in \mathbb{N}$. This provides the estimate

$$\int\limits_{\Omega} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)\,dx \leq \int\limits_{\Omega} F_i(X_n, \Upsilon_n)(U_n)_i\,dx, \quad \forall\, n \in \mathbb{N}. \tag{17}$$

Since $F(X, \Upsilon) \in C(\Omega \times (0, 2\pi]^3; L_2(\Omega))^3$, for $(X, \Upsilon) \in \mathcal{A}$ there exists some constant $C > 0$ such that

$$\|F(X, \Upsilon)\|_{L_2(\Omega)^3} \leq C,$$

for all $(X, \Upsilon) \in \mathcal{A}$. The last inequality together with (17) ensures the uniform estimate

$$\|U_n\|_{H(\Omega)} \leq c$$

with some constant $c > 0$ independent of $n \in \mathbb{N}$. Consequently, replacing $\{U_n\}$ by its subsequence, if necessary, we conclude that $\{U_n\}$ converges to some function $\tilde{U}$ weakly in $H(\Omega)$.

Now we show that the limit function $\tilde{U} \in K(X^*, \Upsilon^*)$. Indeed, the relation

$$U_n|_{\omega_p(x^{p,n})} = \rho_{p,n} \in R(\omega_p(x^{p,n}, \gamma^{p,n}))$$

holds for some affine functions $\rho_{p,n}$, $p = 1, \ldots, m$, and $n \in \mathbb{N}$. In accordance with the Sobolev embedding theorem, see e.g. [39], for all $p = 1, \ldots, m$ it follows

$$U_n|_{\omega_p(x^{p,*}, \gamma^{p,*})} \to \tilde{U}|_{\omega_p(x^{p,*}, \gamma^{p,*})} \quad \text{strongly in } L_2(\omega_p(x^{p,*}, \gamma^{p,*}))^3, \tag{18}$$

$$U_n|_\Gamma \to \tilde{U}|_\Gamma \quad \text{strongly in } L_2(\Gamma)^3 \quad \text{as } n \to \infty. \tag{19}$$

Choosing a subsequence, if necessary, we also get $U_n \to \tilde{U}$ a.e. on $\omega_p(x^{p,*}, \gamma^{p,*})$ for all $p = 1, \ldots, m$.

In the next step we fix a natural number p, $1 \leq p \leq m$, and an arbitrary strictly inner subdomain $D \subset \omega_p(x^{p,*}, \gamma^{p,*})$. According to Assumption (2.1) there exists a sufficiently large N such that, if $n \geq N$, then $D \subset \omega_p(x^{p,*}, \gamma^{p,*}) \cap \omega_p(x^{p,n}, \gamma^{p,n})$. Therefore, the sequence $\{\rho_{p,n}\}$ converges to $\tilde{U}$ a.e. on D as $n \to \infty$. This allows us to conclude that the numerical sequences

$$\{b^n_{pij}\}, \quad \{c^n_{ip}\}, \quad i, j = 1, 2, 3, \quad i \neq j,$$

which determine the infinitesimal rigid displacements $\rho_{p,n}$, $n = 1, 2, \ldots$ on D, are bounded in $\mathbb{R}$. We can extract subsequences (retaining notation) such that

$$b_{pij} \to b_{pij}, \quad c^n_{ip} \to c_{ip}, \quad i, j = 1, 2, 3, \quad i \neq j, \quad \text{as } n \to \infty,$$

and a subsequence $\{(X_{n_k}, \Upsilon_{n_k})\}$ such that

$$U(X_{n_k}, \Upsilon_{n_k}) \to (b_{p12}x_2+b_{p13}x_3+c_{p1}, -b_{p12}x_1+b_{p23}x_3+c_{p2}, -b_{p13}x_1-b_{p23}x_2+c_{p3}) \qquad (20)$$

a.e. on D as $k \to \infty$. Consequently, a.e. on D

$$\tilde{U} = (b_{p12}x_2 + b_{p13}x_3 + c_{p1}, -b_{p12}x_1 + b_{p23}x_3 + c_{p2}, -b_{p13}x_1 - b_{p23}x_2 + c_{p3}).$$

In the view of arbitrariness of the domain $D \subset \omega_p(x^{p,*}, \gamma^{p,*})$, the above equation holds a.e. on $\omega_p(x^{p,*}, \gamma^{p,*})$. Hence we conclude that

$$\tilde{U}|_{\omega_p(x^{p,*},\gamma^{p,*})} \in R(\omega_p(x^{p,*}, \gamma^{p,*})).$$

Since p is arbitrary, this inclusion is true for all $p = 1, \dots, m$.

We show that $\tilde{U}$ satisfies the Signorini condition in $K(X^*, \Upsilon^*)$, that is, the inequality $\tilde{U}_i\nu_i \le 0$ on Γ_1. Bearing in mind the convergence (19), if necessary, we can once again extract a convergent subsequence satisfying $U_n|_\Gamma \to \tilde{U}|_\Gamma$ a.e. on Γ. This fact allows us to pass to the limit in the inequality

$$(U_n)_i\nu_i \le 0 \quad \text{on} \quad \Gamma_1.$$

In the limit we deduce $\tilde{U}_i\nu_i \le 0$ on Γ_1, hence the limit function $\tilde{U} \in K(X^*, \Upsilon^*)$.

Our next goal is to prove that $\tilde{U} = U(X^*, \Upsilon^*)$. For this purpose we will analyze the variational inequality (11) and its limiting case. According to Lemma (5.1), for any $W \in K(X^*, \Upsilon^*)$ there exists a subsequence of parameters $\{(X_{n_k}, \Upsilon_{n_k})\} \subset \{(X_n, \Upsilon_n)\}$ and a sequence of functions $\{W_k\}$ such that $W_k \in K(X_{n_k}, \Upsilon_{n_k})$ and $W_k \to W$ strongly in $H(\Omega)$ as $k \to \infty$. Since $F(X_{n_k}, \Upsilon_{n_k}) \to F(X^*, \Upsilon^*)$ strongly in $L_2(\Omega)^3$ as $k \to \infty$, the convergent sequences $\{W_k\}$ and $\{U_n\}$ allow us to pass to the limit $k \to \infty$ in the following variational inequalities derived from (11) for the parameters $\{(X_{n_k}, \Upsilon_{n_k})\}$ and test functions $W_k \in K_k$:

$$\int_\Omega \sigma_{ij}(U_{n_k})\varepsilon_{ij}(W_{n_k} - U_{n_k})\,dx \ge \int_\Omega F_i(X_{n_k}, \Upsilon_{n_k})(W_{n_k} - U_{n_k})_i\,dx. \qquad (21)$$

The result yields

$$\int_\Omega \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})\,dx \ge \int_\Omega F_i(X^*, \Upsilon^*)(W - \tilde{U})_i\,dx \quad \forall\, W \in K(X^*, \Upsilon^*).$$

The unique solvability of this variational inequality ensures that $\tilde{U} = U^*$.

To complete the proof, it remains to establish the existence of a solution sequence $U_n = U(X_n, \Upsilon_n)$, $n = 1, 2, \dots$, converging to $U(X^*, \Upsilon^*)$ strongly in $H(\Omega)$. By substituting $W = 2U_n$ and $W = 0$ into the variational inequalities (11), we get

$$\int_\Omega \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)\,dx = \int_\Omega F_i(X_n, \Upsilon_n)(U_n)_i\,dx \quad \forall\, n \in \mathbb{N}. \qquad (22)$$

Equations (22) together with the weak convergence $U_n \to U^*$ in $H(\Omega)$ as $n \to \infty$ imply

$$\lim_{n\to\infty} \int_\Omega \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)\,dx = \lim_{n\to\infty} \int_\Omega F_i(X_n, \Upsilon_n)(U_n)_i\,dx$$

$$= \int_\Omega F_i(X^*, \Upsilon^*)U_i^*\,dx = \int_\Omega \sigma_{ij}(U^*)\varepsilon_{ij}(U^*)\,dx.$$

The equivalence of norms (see Remark (3.1)) provides $U_n \to U^*$ strongly in $H(\Omega)$ as $n \to \infty$. But this contradicts to the assumption that $\|U_n - U^*\| \geq \epsilon_0 > 0$ and proves Lemma (5.2). □

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References

[1] Lavrentiev M.M., Romanov V.G., Vasiliev V.G., *Multidimensional Inverse Problems for Differential Equations*, Springer, Berlin, Heidelberg, 1970.

[2] Hasanov Hasanoğlu A., Romanov V.G., *Introduction to Inverse Problems for Differential Equations*, Springer, Cham, 2021.

[3] Bilotta A., Morassi A., Rosset E., Turco E., Vessella S., *Numerical size estimates of inclusions in Kirchhoff–Love elastic plates*, Int. J. Solids Struct. 168. (2019), 58–72.

[4] Morassi A., Rosset E., Vessella S., *Optimal stability in the identification of a rigid inclusion in an isotropic Kirchhoff-Love plate*, SIAM J. Math. Anal. 51. (2019), 731–747.

[5] Lazarev N., *Inverse problem for cracked inhomogeneous Kirchhoff–Love plate with two hinged rigid inclusions*, Bound. Value Probl. 2021, (2021), 88.

[6] Hintermüller M., Laurain A., *Optimal shape design subject to elliptic variational inequalities*, SIAM J. Control Optim., 49. 3 (2011), 1015–1047.

[7] Shcherbakov V., *Shape optimization of rigid inclusions for elastic plates with cracks*, Z. Angew. Math. Phys. 67. (2016), 71.

[8] Rudoy E.M., *Shape sensitivity analysis of equilibrium problem for bodies with thin rigid inclusions*, J. Math. Sci. 211. (2015), 847–862.

[9] Kovtunenko V.A., Leugering G., *A shape-topological control of variational inequalities*, Eurasian Math. J., 7. 3 (2016), 41–52.

[10] Alessandrini G., Morassi A., Rosset E., *Detecting an Inclusion in an Elastic Body by Boundary Measurements*, SIAM Review, 46. 3 (2004), 477–498.

[11] Khludnev A.M., *Inverse problems for elastic body with closely located thin inclusions*, Z. Angew. Math. Phys., 70. 134 (2019).

[12] Khodadad M., Ardakani M.D., *Inclusion identification by inverse application of boundary element method, genetic algorithm and conjugate gradient method*, Am. J. Appl. Sci., 5. 9 (2008), 1158–1166.

[13] Mallardo V., Alessandri C., *Inverse problems in the presence of inclusions and unilateral constraints: a boundary element approach*, Comput. Mech., 26. (2000), 571–581.

[14] Cakoni F., Kovtunenko V.A., *Topological optimality condition for the identification of the center of an inhomogeneity*, Inverse Probl., 34. 3. (2018), 035009.

[15] Kovtunenko V.A., Kunisch K., *High precision identification of an object: optimality conditions based concept of imaging*, SIAM J.Control Optim., 52. 1. (2014), 64.

[16] Shifrin E.I., Shushpannikov P.S., *Identification of small well-separated defects in an isotropic elastic body using boundary measurements*, Int. J. Solids Struct., 50. 22-23. (2013), 3707–3716.

[17] Yu B., Hu P., Wei P. , Cao, G., Wang B., *The identification of voids and inclusions based on the parameter level set method using boundary data*, Appl. Math. Model., 112, (2022), 505–539.

[18] Ghilli D., Kunisch K., Kovtunenko V.A., *Inverse problem of breaking line identification by shape optimization*, J. Inverse Ill-Posed Probl., 28. 1. (2020), 119–135.

[19] Shifrin E.I., Kaptsov A.V., *Identification of multiple cracks in 2D elasticity by means of the reciprocity principle and cluster analysis*, Inverse Probl., 34. 1. (2018), 015009.

[20] Karchevsky A.L., Fatianov A.G., *Numerical solution of the inverse problem for a system of elasticity with the aftereffect for a vertically inhomogeneous medium*, Sib. Zh. Vychisl. Mat., 4. 3 (2001), 259–268.

[21] Karchevsky A.L., Oralbekova Zh.O., Iskakov K.T., *Solution of the inverse problem of subsurface electric exploration for horizontally stratified medium*, J. Appl. Math., 13. 3 (2013), 432121.

[22] Rademacher A., Rosin K., *Adaptive optimal control of Signorini's problem*, Comput. Optim. Appl., 70 (2018), 531–569.

[23] Bermúdez A., Saguez C., *Optimal control of a Signorini problem*, SIAM J. Control Optim., 25. 3 (1987), 576–582.

[24] Meixner A., Piersanti P., *Numerical approximation of the solution of an obstacle problem modelling the displacement of elliptic membrane shells via the penalty method*, Appl. Math. Optim., 89. 2 (2024), 45.

[25] Wachsmuth G., *Strong stationarity for optimal control of the obstacle problem with control constraints*, SIAM J. Control Optim., 24. 4 (2014), 1914–1932.

[26] Hintermüller M., Kopacka I., *Mathematical programs with complementarity constraints in function space: C-and strong stationarity and a path-following algorithm*, SIAM J. Control Optim., 20. 2 (2009), 868–902.

[27] Kovtunenko V.A., Kunisch K., *Directional differentiability for shape optimization with variational inequalities as constraints*, ESAIM: Control Optim. Calc. Var., 29. (2023), 64.

[28] Kovtunenko V.A., Itou H., Khludnev A.M., Rudoy E.M., *Non-smooth variational problems and applications*, Phil. Trans. Royal Soc. A: Math. Phys. Eng. Sci., 380. 2236. (2022), 20210364.

[29] Furtsev A.I., *Problem of equilibrium for hyperelastic body with rigid inclusion and non-penetrating crack*, Sib. Electron. Math. Rep., 21. 1 (2024), 17–40. (In Russian, English abstract).

[30] Khludnev A.M., Fankina I.V., *Noncoercive problems for elastic bodies with thin elastic inclusions*, Math. Meth. Appl. Sci., 46. 13 (2023), 14214–14228.

[31] Rudoy E., Sazhenkov S., *Variational approach to modeling of curvilinear thin inclusions with rough boundaries in elastic bodies: case of a rod-type inclusion*, Mathematics, 11. (2023) 3447.

[32] Fankina I.V., Furtsev A.I., Rudoy E.M., Sazhenkov S.A., *The homogenized quasi-static model of a thermoelastic composite stitched with reinforcing threads*, J. Comput. Appl. Math., 434. (2023), 115346.

[33] Altenbach H., Bruno G., Eremeyev V.A., Gutkin M.Yu., Müller W.H. (Eds.), *Mechanics of Heterogeneous Materials*, Springer, Cham, 2023.

[34] Kovtunenko V.A., Zubkova A.V., *Mathematical modeling of a discontinuous solution of the generalized Poisson–Nernst–Planck problem in a two-phase medium*, Kinet. Relat. Mod., 11. 1. (2018), 119–135.

[35] Lazarev N., Itou H., *Optimal location of a rigid inclusion in equilibrium problems for inhomogeneous Kirchhoff-Love plates with a crack,* Math. Mech. Solids. 24. (2019), 3743–3752.

[36] Lazarev N., Rudoy E., *Optimal location of a finite set of rigid inclusions in contact problems for inhomogeneous two-dimensional bodies*, J. Comput. Appl. Math. 403. (2022), 113710.

[37] Ciarlet P.G., *Three-Dimensional Elasticity*, Elsevier, Amsterdam, 1988.

[38] Khludnev A.M., Novotny A.A., Sokołowski J., Zochowski A., *Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions,* J. Mech. Phys. Solids. 57. (2009), 1718–1732.

[39] Khludnev A.M., Kovtunenko V.A., *Analysis of Cracks in Solids*, WIT-Press, Southampton, 2000.

[40] Hlavaček I., Haslinger J., Nečas J., Lovišek J., *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[41] Duvaut G., Lions J.-L., *Inequalities in Mechanics and Physics*, Springer, Berlin, 2013.

[42] Lions J.L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.

[43] Doktor P., *Approximation of domains with Lipschitzian boundary*, Časopis pro pěstování matematiky. 101. 3 (1976), 237–255.

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