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GLOBAL SOLVABILITY OF AN INVERSE PROBLEM FOR A MOORE-GIBSON-THOMPSON EQUATION WITH PERIODIC BOUNDARY AND INTEGRAL OVERDETERMINATION CONDITIONS

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Abstract This article studies the inverse problem of determining the pressure and convolution kernel in an integro-differential Moore-Gibson-Thompson equation with initial, periodic boundary and integral overdetermination conditions on the rectangular domain. By Fourier method this problem is reduced to an equivalent integral equation and on based of Banach's fixed point argument in a suitably chosen function space, the local solvability of the problem is proven. Then, the found solutions are continued throughout the entire domain of definition of the unknowns.

Keywords: Moore-Gibson-Thompson equation, initial-boundary problem, periodic boundary conditions, inverse problem, Fourier spectral method, Banach principle.

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1 Introduction

In many practical applications, parameters, which describe the properties of media, are unknown or scarcely known. The unknown parameters are related to the observed data through a forward model, such as a differential equation, that maps realizations of parameters to observable. In particular, the solvability of the inverse problems in various formulations with different overdetermination conditions for partial differential equations is extensively studied in many papers (see [1]-[4]).

The problem of determining the convolution kernel of integro-differential equations was studied in many publications [5]-[13] (see, also, the references in [5]), in which both one and multi-dimensional inverse problems with classical initial, initial-boundary and overdetermination conditions were investigated. The existence and uniqueness theorems of inverse problem solutions were proved. The works [14]-[17] are devoted to numerical methods for solving inverse problems of determining the kernel in integro-differential equations of hyperbolic type. The papers present a numerically determining the parameters of the memory function for a horizontally layered medium.

In molecular dynamics simulations, periodic boundary conditions are usually applied to calculate properties of liquids, crystals or mixtures. The periodic boundary conditions arise from many important applications in life sciences. In [18]-[20], the existence, uniqueness, and continuous data dependence of the solution were proven, and numerical solutions to the diffusion problem with periodic boundary conditions were developed.

Results on inverse problems for the linear Moore-Gibson-Thompson (MGT) equation are available in literature. In [21], [22], the authors studied the inverse problems for MGT equation in order to determine the space varying frictional damping term. In the above papers, the Lipschitz stability for the inverse problem is obtained, and a convergent algorithm for the reconstruction of the unknown coefficient is designed. The techniques used due to Carleman inequalities for wave equations and properties of the MGT equation. In the work [23], the kernel of the MGT equation with memory is classified into 3 types. Then, authors study how a memory term creates damping mechanism and how the memory causes energy decay even in the cases when the original memoryless system is unstable.

The main aim of this paper is to determine the convolution kernel of the MGT model with memory of type 1 [23].

We consider the one-dimensional integro-differential MGT equation:

$$u_{ttt}+u_{tt}-u_{xxt}-u_{xx}+\int\limits_0^t g(t-\tau)u_{xx}(x,\tau)d\tau=0,\quad (x,t)\in\Omega_T, \tag{1}$$

with initial

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x),\ u_{tt}(x,0)=\phi(x),\ x\in[0,l], \tag{2}$$

and periodic boundary conditions

$$u(0,t)=u(l,t),\quad u_x(0,t)=u_x(l,t),\quad t\in[0,T], \tag{3}$$

where $\Omega_T=\{(x,t):0<x<l,\ 0<t\le T\}$.

The problem of determining a function $u(x,t)\in C^{2,3}_{x,t}(\Omega_T)\cap C^{1,2}_{x,t}\left(\overline{\Omega}_T\right)$, that satisfies (1)-(3) with known functions $\varphi(x)$, $\psi(x)$, $\phi(x)$ and $g(t)$ will be called the *direct problem.*

In the *inverse problem,* it is required to determine the function $g(t)$ using overdetermination conditions about the solution of the direct problem (1)-(3):

$$\int\limits_0^l \eta(x)u(x,t)\,dx=h(t),\qquad t\in[0,T], \tag{4}$$

where $\eta(x)$ and $h(t)$ are given functions.

Definition 1. *The double of functions $\{u(x,t),g(t)\}$ from the class $C^{2,3}_{x,t}(\Omega_T)\cap C^{1,2}_{x,t}\left(\overline{\Omega}_T\right)\times C[0,T]$ is said to be a classical solution of problem (1)-(4), if the functions $u(x,t)$, $g(t)$ satisfy the following conditions:*

1. *the function $u(x,t)$ and its derivatives u_{ttt}, u_{tt}, u_{xxt}, u_{xx} are continuous in the domain Ω_T;*
2. *the function $g(t)$ is continuous on the interval $[0,T]$;*
3. *equation (1) and conditions (2)-(4) are satisfied in the classical sense.*

Throughout this article, the functions φ, ψ, ϕ, η and h are assumed to satisfy the following conditions:

(K1): $\varphi(x) \in C^2(0,l)$, $\varphi'''(x) \in L_2(0,l)$, $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $i = 0,1,2$;
(K2): $\psi(x) \in C^2(0,l)$, $\psi'''(x) \in L_2(0,l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = 0,1,2$;
(K3): $\phi(x) \in C^1(0,l)$, $\phi''(x) \in L_2(0,l)$, $\phi^{(i)}(0) = \phi^{(i)}(l)$, $i = 0,1$;
(K4): $\eta(x) \in C^1(0,l)$, $\eta(0) = \eta(l) = 0$, $h(t) \in C^4[0,T]$;
(K5): $\int\limits_0^l \eta(x)\varphi(x)\,dx = h(0)$, $\int\limits_0^l \eta(x)\psi(x)\,dx = h'(0)$, $\int\limits_0^l \eta(x)\phi(x)\,dx = h''(0)$;
(K6): $\varphi_0 := \int\limits_0^l \eta(x)\varphi''(x)\,dx \neq 0$.

2 Classical solvability of inverse problem

In this section, we consider the problem of determining the unknown functions $\{u(x,t), g(t)\}$ from the integro-differential equation (1) with initial-boundary condition, and additional condition.

2.1 Equivalence of inverse problem

Now, to study the main problem (1)-(4), we consider the following auxiliary inverse local initial and boundary value problem.

Lemma 2.1. *Let (K1)-(K5) be held. Then the problem of finding a classical solution of (1)-(4) is equivalent to the problem of determining the functions* $u(x,t) \in C^{2,3}_{x,t}(\Omega_T) \cap C^{1,2}_{x,t}\left(\overline{\Omega}_T\right)$ *and* $g(t) \in C[0,T]$ *satisfying (1)-(3) and*

$$h'''(t) + h''(t) - \int\limits_0^l \eta(x) u_{xx}(x,t)dx - \int\limits_0^l \eta(x) u_{xxt}(x,t)dx$$

$$+ \int\limits_0^l \eta(x) \int\limits_0^t g(t-s) u_{xx}(x,s)dsdx = 0, \quad t \in [0,T]. \tag{5}$$

Remark 1. *From Lemma* 2.1, *we know that* $(1)-(3),(5)$ *is an equivalent form of the original inverse problem* $(1)-(4)$. *So, in the next sections, we discuss* $(1)-(3),(5)$, *other than the original one.*

Proof. Let $\{u(x,t), g(t)\}$ be a solution of inverse problem (1)-(4). The solution $\{u(x,t), g(t)\}$ is also a solution to the problem (1)-(3), (5). We should show only (5). Multiplying both sides of equation (1) by a function $\eta(x)$ and integrating 0 to l with respect to x gives

$$\frac{d^3}{dt^3}\int\limits_0^l \eta(x)u(x,t)dx + \frac{d^2}{dt^2}\int\limits_0^l \eta(x)u(x,t)dx - \int\limits_0^l \eta(x)u_{xx}(x,t)dx$$

$$-\int_0^l \eta(x)u_{xxt}(x,t)dx + \int_0^l \eta(x)\int_0^t g(t-s)u_{xx}(x,s)dsdx = 0, \tag{6}$$

for all $t \in [0,T]$. Using the condition $h(t) \in C^4[0,T]$ and additional condition (4), we get

$$\frac{d^3}{dt^3}\int_0^l \eta(x)u(x,t)\,dx = h'''(t),\ \frac{d^2}{dt^2}\int_0^l \eta(x)u(x,t)\,dx = h''(t),\ t \in [0,T]. \tag{7}$$

Hence, from (6), taking into account (4) and (8), we arrive at (5).

Now we assume that $\{u(x,t), g(t)\}$ satisfies (1)-(3), (5). In order to prove that $\{u(x,t), g(t)\}$ is the solution to the inverse problem (1)-(4), it suffices to show that $\{u(x,t), g(t)\}$ satisfied (5). From (5) and (6), we obtained following Cauchy problem

$$\begin{cases} y'''(t) + y''(t) = 0, & t \in [0,T], \\ y^{(i)}(0) = 0, & i = 0,1,2, \end{cases} \tag{8}$$

where $y(t) = \int_0^l \eta(x)u(x,t)dx - h(t)$.

It is easy to see that the solution to the problem (8) is only trivial, that is, $y(t) \equiv 0$, for all $t \in [0,T]$, which implies $\int_0^l \eta(x)u(x,t)dx - h(t) \equiv 0$, $t \in [0,T]$, i.e., the condition (4) is satisfied.

2.2 Investigation of inverse problem

The use of the Fourier method for solving problem (1)-(3) leads to the spectral problem for the operator given by the differential expression and boundary conditions

$$X_n''(x) + \lambda_n^2 X_n(x) = 0, \quad x \in (0,l), \tag{9}$$

$$X_n(0) = X_n(l),\ X_n'(0) = X_n'(l),\ n = 0,1,2,... \tag{10}$$

In [24], it is known that the system of eigenfunctions

$$(1,\ X_{1n},\ X_{2n}) = (1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \ldots, \cos\lambda_n x, \sin\lambda_n x, \ldots), \tag{11}$$

where $\lambda_n = \frac{2\pi n}{l}$ $(n = 0,1,2,3,\ldots)$, is a basis for $L_2(0,l)$. That is the system of eigenfunctions of spectral problem (9), (10).

Since the system (11) form a basis in $L_2(0,l)$, we shall seek the first component classical solution $\{u(x,t), g(t)\}$ of the problem (1)-(4) in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} u_{2n}(t)\sin\lambda_n x, \tag{12}$$

where

$$u_{10}(t) = \frac{1}{\sqrt{l}}\int_0^l u(x,t)dx, \quad u_{1n}(t) = \sqrt{\frac{2}{l}}\int_0^l u(x,t)\cos\lambda_n x dx,$$

$$u_{2n}(t)=\sqrt{\frac{2}{l}}\int\limits_0^l u(x,t)\sin\lambda_n x dx,\quad n=1,2,3,\ldots$$

Then, applying the formal scheme of the Fourier method for determining of unknown coefficients $u_{10}(t)$ and $u_{in}(t)$ $(i:=1,2;n=1,2,\ldots)$ of function $u(x,t)$ from (1) and (2), we find

$$\begin{cases} u_{10}'''(t)+u_{10}''(t)=0,\quad 0\le t\le T,\\ u_{10}(0)=\varphi_{10},\quad u_{10}'(0)=\psi_{10},\quad u_{10}''(0)=\phi_{10},\end{cases}\tag{13}$$

$$\begin{cases} u_{in}'''(t)+u_{in}''(t)+\lambda_n^2 u_{in}'(t)+\lambda_n^2 u_{in}(t)=\lambda_n^2\int\limits_0^t g(t-\tau)u_{in}(\tau)d\tau,\quad 0\le t\le T,\\ u_{in}(0)=\varphi_{in},\ u_{in}'(0)=\psi_{in},\ u_{in}''(0)=\phi_{in},\quad i:=1,2,\ n=1,2,\ldots\end{cases}\tag{14}$$

where

$$f_{10}=\frac{1}{\sqrt{l}}\int\limits_0^l f(x)dx,\ f_{1n}=\sqrt{\frac{2}{l}}\int\limits_0^l f(x)\cos\lambda_n x dx,\ f_{2n}=\sqrt{\frac{2}{l}}\int\limits_0^l f(x)\sin\lambda_n x dx,$$

for $n=1,2,3,\ldots$ and $f=\varphi,\psi,\phi$.

Solving the initial problem (13) we have

$$u_{10}(t)=\varphi_{10}+t\psi_{10}+\left(e^{-t}+t-1\right)\phi_{10}.\tag{15}$$

Further, it is easy to note problems (14) are equivalent to the following Volterra-type integral equations:

$$u_{in}(t)=\frac{1}{\lambda_n^2+1}\left[\lambda_n^2 e^{-t}+\lambda_n\sin\lambda_n t+\cos\lambda_n t\right]\varphi_{in}$$

$$+\frac{1}{\lambda_n}\sin\lambda_n t\cdot\psi_{in}+\frac{1}{\lambda_n^2+1}\left(e^{-t}+\frac{1}{\lambda_n}\sin\lambda_n t-\cos\lambda_n t\right)\phi_{in}-\frac{\lambda_n^2}{\lambda_n^2+1}$$

$$\times\int\limits_0^t\int\limits_0^\tau u_{in}(s)g(\tau-s)\left(e^{\tau-t}+\frac{1}{\lambda_n}\sin\lambda_n(t-\tau)-\cos\lambda_n(t-\tau)\right)ds\,d\tau,\ n=1,2,3,\ldots.\tag{16}$$

Now, we differentiate equality (5) with respect to t and using equality (12), (16), condition (K6) after simple converting, we obtain the following integral equation for determining $g(t)$:

$$g(t)=\Phi(t)+\int_0^t g(t-\tau)G_1(\tau)d\tau+\int_0^t\int_0^\tau g(\tau-s)G_2([u],t)dsd\tau$$

$$+\int_0^t\int_0^\tau\int_0^s g(t-\tau)g(s-\sigma)G_3([u],t)d\sigma dsd\tau,\tag{17}$$

where

$$\Phi(t)=\frac{1}{\varphi_0}\left[h^{(IV)}(t)+h'''(t)+\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\lambda_n^3\left(\eta_{1n}\varphi_{1n}+\eta_{2n}\varphi_{2n}\right)\sin\lambda_n t\right]$$

$$-\frac{1}{\varphi_0}\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\lambda_n^2\left(\eta_{1n}\psi_{1n}+\eta_{2n}\psi_{2n}\right)\left(\lambda_n\sin\lambda_n t-\cos\lambda_n t\right)$$

$$-\frac{1}{\varphi_0}\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\lambda_n^2\left(\eta_{1n}\phi_{1n}+\eta_{2n}\phi_{2n}\right)\cos\lambda_n t-\sqrt{l}\eta_{10}\left(\psi_{10}+\phi_{10}\right),$$

$$G_1(t)=\frac{\sqrt{l}}{\varphi_0}\eta_{10}\left(\psi_{10}+\left(-e^{-t}+1\right)\phi_{10}\right)$$

$$+\frac{1}{\varphi_0}\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{\lambda_n^2+1}\left(\eta_{1n}\varphi_{1n}+\eta_{2n}\varphi_{2n}\right)\left[-\lambda_n^2e^{-t}+\lambda_n^2\cos\lambda_n t-\lambda_n\sin\lambda_n t\right]$$

$$+\frac{1}{\varphi_0}\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\lambda_n^2\left(\eta_{1n}\psi_{1n}+\eta_{2n}\psi_{2n}\right)\cos\lambda_n t$$

$$+\frac{1}{\varphi_0}\sqrt{\frac{l}{2}}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{\lambda_n^2+1}\left(\eta_{1n}\phi_{1n}+\eta_{2n}\phi_{2n}\right)\left(-e^{-t}+\cos\lambda_n t+\lambda_n\sin\lambda_n t\right),$$

$$G_2([u],t)=\frac{1}{\varphi_0}\sum_{n=1}^{\infty}\lambda_n^4\left(\eta_{1n}u_{1n}+\eta_{2n}u_{2n}\right)\cos\lambda_n t,$$

$$G_3([u],t)=\frac{1}{\varphi_0}\sum_{n=1}^{\infty}\frac{\lambda_n^4}{\lambda_n^2+1}\left(\eta_{1n}u_{1n}+\eta_{2n}u_{2n}\right)\left[-e^{-t}+\cos\lambda_n t+\lambda_n\sin\lambda_n t\right].$$

Thus, the solution of problem (1)-(4) was reduced to the solution of systems (12), (17) with respect to unknown functions $u(x,t)$ and $g(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to the problem (1)-(3), (5):

Lemma 2.2. *If $\{u(x,t),g(t)\}$ is any solution to problem (1)-(3), (5), then the functions $u_{10}(t)$, $u_{in}(t)$, $i=1,2$ satisfy the system of (13), (14) on interval $[0,T]$, respectively.*

Remark 2. *From Lemma* 2.2 *it follows that to prove the uniqueness of the solution of problem (1)-(3), (5), it suffices to prove the uniqueness of the solution of system (12), (17).*

Let us consider the functional space $B^3_{0,T}$ that is introduced in the work [25], so that $B^3_{0,T}$ is a set of all functions of the form (12) considered in Ω_T. Moreover, the functions $u_{1n}(t)$ $(n=0,1,2,\ldots)$, $u_{2n}(t)$ $(n=1,2,\ldots)$ contained in sums of (12) are continuously differentiable on $[0,T]$ and

$$J_T(u)=\|u_{10}(t)\|_{C[0,T]}+\left\{\sum_{n=1}^{\infty}\left(\lambda_n^3\|u_{1n}\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{n=1}^{\infty}\left(\lambda_n^3\|u_{2n}\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<\infty.$$

The norm on the set $J_T(u)$ is given by:

$$\|u\|_{B^3_{0,T}}=J_T(u).$$

Let $E^3_{0,T}$ denote the space consisting of the topological product $B^3_{0,T} \times C[0,T]$, in which the norm of the element $\{u, g\}$ defined by the formula

$$\|\{u,g\}\|_{E^3_{0,T}} = \|u\|_{B^3_{0,T}} + \|g\|_{C[0,T]}.$$

It is known that the spaces $B^3_{0,T}$ and $E^3_{0,T}$ are Banach space [26].

Now, in the space $E^3_{0,T}$ consider the operator

$$\Lambda(u,g) = \{\Lambda_1(u,g);\ \Lambda_2(u,g)\} = \{\widetilde{u};\widetilde{g}\} = \left\{ \sum_{n=1}^{\infty} \widetilde{u}_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} \widetilde{u}_{2n}(t)\sin\lambda_n x;\ \widetilde{g} \right\},$$

where the functions $\widetilde{u}_{1n}(t)$ $(n = 0, 1, 2, ...)$, $\widetilde{u}_{2n}(t)$ $(n = 1, 2, ...)$ and $\widetilde{g}(t)$ are equal to the right-hand sides of (15), (16) and (17), respectively.

Based on (15)-(17), by means of simple transformations we find

$$\|\widetilde{u}_{10}\|_{C[0,T]} \le |\varphi_{10}| + T|\psi_{10}| + T|\phi_{10}|, \tag{18}$$

$$\left(\sum_{n=1}^{\infty} \left(\lambda_n^3 \|\widetilde{u}_{in}\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \le 4 \left(\sum_{n=1}^{\infty} \left(\lambda_n^3 |\varphi_{in}| \right)^2 \right)^{\frac{1}{2}} + 2 \left(\sum_{n=1}^{\infty} \left(\lambda_n^2 |\psi_{in}| \right)^2 \right)^{\frac{1}{2}}$$

$$+ \left(\sum_{n=1}^{\infty} \left(\lambda_n |\phi_{in}| \right)^2 \right)^{\frac{1}{2}} + 2T^2 \|g\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \left(\lambda_n^3 \|u_{in}\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}},\ i = 1, 2, \tag{19}$$

$$\|\widetilde{g}\|_{C[0,T]} \le \frac{1}{|\varphi_0|} \left(\|h\|_{C^4[0,T]} + \eta_{10} \left(\psi_{10} + \phi_{10} \right) \right)$$

$$+ \frac{\sqrt{l}}{\sqrt{2}|\varphi_0|} \sum_{i=1}^{2} \left(\sum_{n=1}^{\infty} |\eta_{in}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(\lambda_n^3 |\varphi_{in}| \right)^2 \right)^{\frac{1}{2}}$$

$$+ \frac{\sqrt{2l}}{|\varphi_0|} \sum_{i=1}^{2} \left(\sum_{n=1}^{\infty} |\eta_{in}|^2 \right)^{\frac{1}{2}} \left[\left(\sum_{n=1}^{\infty} \left(\lambda_n^3 |\psi_{in}| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left(\lambda_n^2 |\phi_{in}| \right)^2 \right)^{\frac{1}{2}} \right]$$

$$+ \frac{2\sqrt{2l}}{|\varphi_0|} \|g\|_{C[0,T]} T \sum_{i=1}^{2} \left(\sum_{n=1}^{\infty} |\eta_{in}|^2 \right)^{\frac{1}{2}} \times$$

$$\times \left[\left(\sum_{n=1}^{\infty} \left(\lambda_n^2 |\varphi_{in}| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left(\lambda_n^2 |\psi_{in}| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left(\lambda_n |\phi_{in}| \right)^2 \right)^{\frac{1}{2}} \right]$$

$$+ \frac{\sqrt{l}}{|\varphi_0|} \|g\|_{C[0,T]} T^2 \sum_{i=1}^{2} \left(\sum_{n=1}^{\infty} \left(\lambda_n |\eta_{in}| \right)^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(\lambda_n^3 |u_{in}| \right)^2 \right)^{\frac{1}{2}}$$

$$+ \frac{\sqrt{l}}{|\varphi_0|} \|g\|^2_{C[0,T]} \frac{T^3}{3} \sum_{i=1}^{2} \left(\sum_{n=1}^{\infty} |\eta_{in}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(\lambda_n^2 |u_{in}| \right)^2 \right)^{\frac{1}{2}}. \tag{20}$$

Due to the fulfillment of the conditions (K1)-(K6), the series in (19) and (20) with φ_{in}, ψ_{in}, ϕ_{in}, η_{in} will be convergent. For example, for the first series on the right side (19) it will be shown as follows:

$$\left(\sum_{n=1}^{\infty}\left(\lambda_n^3|\varphi_{in}|\right)^2\right)^{\frac{1}{2}}=\left(\sum_{n=1}^{\infty}\left(\sqrt{\frac{2}{l}}\lambda_n^3\int_0^l\varphi(x)\left\{\begin{array}{c}X_{1n}(x)\\X_{2n}(x)\end{array}\right\}dx\right)^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{n=1}^{\infty}\left(\sqrt{\frac{2}{l}}\int_0^l\varphi'''(x)\left\{\begin{array}{c}X_{2n}(x)\\X_{1n}(x)\end{array}\right\}dx\right)^2\right)^{\frac{1}{2}}\leq\|\varphi'''(x)\|_{L_2(0,l)}. \tag{21}$$

For the other series, this is easily shown in the same way.

Applying estimates of the form (21) to (18)-(20), we get

$$\|\widetilde{u}\|_{B^3_{0,T}}\leq A_1(T)+C_1(T)\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}}, \tag{22}$$

$$\|\widetilde{g}\|_{C[0,T]}\leq A_2(T)+B(T)\|g\|_{C[0,T]}+C_2(T)\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}}+D(T)\|g\|^2_{C[0,T]}\|u\|_{B^3_{0,T}}, \tag{23}$$

where

$$A_1(T)=|\varphi_{10}|+T|\psi_{10}|+T|\phi_{10}|+4\|\varphi'''\|_{L_2(0,l)}+2\|\psi''\|_{L_2(0,l)}+\|\phi'\|_{L_2(0,l)};$$

$$A_2(T)=\frac{1}{|\varphi_0|}\Big(\|h\|_{C^4[0,T]}+\sqrt{l}\eta_{10}\left(\psi_{10}+\phi_{10}\right)\Big)$$

$$+\frac{1}{|\varphi_0|}4\sqrt{l}\|\eta\|_{L_2(0,l)}\left[\|\varphi'''\|_{L_2(0,l)}+\|\psi'''\|_{L_2(0,l)}+\|\phi''\|_{L_2(0,l)}\right];$$

$$B(T)=\frac{4\sqrt{2l}T}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\Big(\|\varphi''\|_{L_2(0,l)}+\|\psi''\|_{L_2(0,l)}+\|\phi'\|_{L_2(0,l)}\Big);$$

$$C_1(T)=2T^2;\quad C_2(T)=\frac{2\sqrt{l}}{|\varphi_0|}T^2\|\eta'\|_{L_2(0,l)};\quad D(T)=\frac{1}{|\varphi_0|}T^3\|\eta\|_{L_2(0,l)}.$$

It follows from the inequalities (22), (23) that

$$\|\widetilde{u}\|_{B^3_{0,T}}+\|\widetilde{g}\|_{C[0,T]}$$

$$\leq A(T)+B(T)\|g\|_{C[0,T]}+C(T)\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}}+D(T)\|g\|^2_{C[0,T]}\|u\|_{B^3_{0,T}}, \tag{24}$$

where $A(T)=A_1(T)+A_2(T)$, $C(T)=C_1(T)+C_2(T)$.

Now we can prove the following theorem.

Theorem 2.1. *Let the conditions (K1)-(K6) and*

$$(A(T)+2)B(T)<\frac{2}{3},\quad (A(T)+2)^2C(T)<\frac{2}{3},\quad (A(T)+2)^3D(T)<\frac{2}{3} \tag{25}$$

be fulfilled. Then the problem (1)-(3), (5) has a unique solution in the ball $S=S_r\Big(\|z\|_{E^3_{0,T}}\leq r=A(T)+2\Big)$ *of the space* $E^3_{0,T}$.

Proof. Let us denote $z = \{u(x,t), g(t)\}$ and rewrite the system of equations (12), (17) in the form of the following operator equation:

$$z = \Lambda z, \tag{26}$$

where $\Lambda = (\Lambda_1, \Lambda_2)^T$, Λ_1 and Λ_2 defined by the right sides of (12) and (17), respectively.

Analogously to (24) we obtain that for any $z, z_1, z_2 \in S_r$ the following estimates hold:

$$\|\Lambda z\|_{E^3_{0,T}} \leq A(T) + B(T)\|g\|_{C[0,T]} + C(T)\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}} + D(T)\|g\|^2_{C[0,T]}\|u\|_{B^3_{0,T}}$$

$$\leq A(T) + B(T)\left(A(T)+2\right) + C(T)\left(A(T)+2\right)^2 + D(T)\left(A(T)+2\right)^3, \tag{27}$$

$$\|\Lambda z_1 - \Lambda z_2\|_{E^3_{0,T}} \leq 2C(T)r\left(\|g_1-g_2\|_{C[0,T]} + \|u_1-u_2\|_{B^3_{0,T}}\right)$$

$$+B(T)\|g_1-g_2\|_{C[0,T]} + 3D(T)r^2\left(\|g_1-g_2\|_{C[0,T]} + \|u_1-u_2\|_{B^3_{0,T}}\right). \tag{28}$$

Then taking into account (25) in (27) and (28), it follows that the operator Λ acts in the ball S_r and is contracted.

Therefore, in accordance with the Banach theorem (see [27], pp. 87-97), the operator Λ has unique fixed point in the ball S_r; i.e., it is a unique solution in the ball S_r.

In this way we conclude that the function $u(x,t)$ as an element of space B^3_T is continuous and has continuous derivative $u_{xx}(x,t)$, in Ω_T.

Calculating the first derivative of (15), (16) and after simple transformations we find

$$\|u'_{10}\|_{C[0,T]} \leq |\psi_{10}| + |\phi_{10}|,$$

$$\left(\sum_{n=1}^{\infty}\left(\lambda_n^2\|u'_{1n}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq 8\left(\sum_{n=1}^{\infty}\left(\lambda_n^2|\varphi_{1n}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{n=1}^{\infty}\left(\lambda_n^2|\psi_{1n}|\right)^2\right)^{\frac{1}{2}}$$

$$+8\left(\sum_{n=1}^{\infty}\left(\lambda_n|\phi_{1n}|\right)^2\right)^{\frac{1}{2}} + 4\|g\|_{C[0,T]}T^2\left(\sum_{n=1}^{\infty}\left(\lambda_n^3\|u_{1n}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\leq 8\|\varphi''\|_{L_2(0,l)} + 4\|\psi''\|_{L_2(0,l)} + 8\|\phi'\|_{L_2(0,l)} + 4T^2\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}},\ i=1,2.$$

Similarly, we can calculate second derivative for (15), (16) as above estimate as, we get

$$\|u''_{10}\|_{C[0,T]} = |\phi_{10}|,$$

$$\left(\sum_{n=1}^{\infty}\left(\lambda_n\|u''_{1n}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\leq 8\Big(\|\varphi''\|_{L_2(0,l)} + \|\psi''\|_{L_2(0,l)} + \|\phi'\|_{L_2(0,l)}\Big) + 4T^2\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}},\ i=1,2.$$

From (13) and (14) it is easy to see that

$$\|u'''_{10}\|_{C[0,T]} = |\phi_{10}|,$$

$$\left(\sum_{n=1}^{\infty}\left(\|u'''_{in}\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq 80\Big(\|\varphi''\|_{L_2(0,l)} + \|\psi''\|_{L_2(0,l)} + \|\phi'\|_{L_2(0,l)}\Big)$$
$$+16T\Big(\|\varphi''\|_{L_2(0,l)} + \|\psi''\|_{L_2(0,l)} + \|\phi'\|_{L_2(0,l)}\Big)\|g\|_{C[0,T]}$$
$$+40T^2\|g\|_{C[0,T]}\|u\|_{B^3_{0,T}} + 8T^3\|g\|^2_{C[0,T]}\|u\|_{B^3_{0,T}}.$$

From the last relations, it is obvious that $u_{tt}(x,t)$, $u_{ttt}(x,t)$, $u_{xxt}(x,t)$, is continuous in Ω_T.

It is easy to verify that the equation (1) and conditions (2), (3), (5) are satisfied in the usual sense.

Consequently, $\{u(x,t), g(t)\}$ is a solution of $(1)-(3)$, and (5) by Lemma 2.2 it is unique.

Remark 3. *Inequality* (25) *is satisfied for sufficiently small values of* $T \in (0,\tau]$, $\tau > 0$.

Finally, from Lemma 2.1 and Theorem 2.1 immediately implies that the original problem $(1)-(4)$ has a unique classical solution.

Theorem 2.2. *Suppose that all assumptions of Theorem* 2.1 *and Lemma* 2.1 *hold. Then, for sufficiently small* T *problem* $(1)-(4)$ *has a unique solution in the ball* S_R *of the space* $E^3_{0,T}$.

3 Main result and its proof

Now we will prove the theorem of global solvability for the inverse problem.

Theorem 3.1. *Under hypotheses (K1)-(K6), there exists a unique solution* $\{u(x,t), g(t)\} \in E^3_{0,T}$ *of the inverse problem (1)-(3), (5) for any* $T > 0$.

Proof. Theorem 2.2 ensure that there exist a unique solution $\big\{\widehat{u}(x,t), \widehat{g}(t)\big\} \in E^3_{0,\tau}$ to problem $(1)-(3)$, (5) for sufficiently small $\tau > 0$. Now we show that the unique solution $\{\widehat{u}(x,t), \widehat{g}(t)\}$ in $[0,\tau]$ can be extended to a larger time interval $(\tau, 2\tau)$.

Rewrite the system of (1)-(3), (5) as follows:

$$u_{ttt}(x,t) + u_{tt}(x,t) - u_{xxt}(x,t) - u_{xx}(x,t) + \int_0^\tau \widehat{g}(t-s)\widehat{u}_{xx}(x,s)\,ds$$
$$+\int_\tau^t g(t-s)u_{xx}(x,s)\,ds = 0, \quad x \in (0,l),\ t \in (\tau,T), \tag{29}$$
$$u(x,\tau) = \widehat{u}(x,\tau),\ u_t(x,\tau) = \widehat{u}_t(x,\tau),$$
$$u_{tt}(x,\tau) = \widehat{u}_{tt}(x,\tau),\ x \in [0,l], \tag{30}$$
$$u(0,t) = u(l,t), \quad u_x(0,t) = u_x(l,t), \quad t \in [0,T], \tag{31}$$

and

$$h'''(t) + h''(t) - \int_0^l \eta(x)\Big(u_{xx}(x,t) + u_{xxt}(x,t)\Big)\,dx$$

$$+\int_0^l \eta(x)\int_0^\tau \widehat{g}(t-s)\widehat{u}_{xx}(x,s)\,ds\,dx$$

$$+\int_0^l \eta(x)\int_\tau^t g(t-s)u_{xx}(x,s)dsdx=0,\quad t\in(\tau,T). \tag{32}$$

Obviously, if we prove that there exists a solution $\{u(x,t),g(t)\}\in E^3_{\tau,T}$ with some $T\le 2\tau$, then $\{u(x,t),g(t)\}$ defined by

$$\{u(x,t),g(t)\}=\begin{cases}\{\widehat{u}(x,t),\widehat{g}(t)\}, & t\in[0,\tau],\\ \{u(x,t),g(t)\} & t\in[\tau,2\tau],\end{cases} \tag{33}$$

is a solution of the inverse problem (1)-(3) and (5) on the larger interval $[0,2\tau]$.

We repeat the Banach principle to prove the existence and uniqueness of $\{u,g\}$. Define an operator

$$\Lambda:\widetilde{S}_{\rho,T}\to E^3_{\tau,T}, \tag{34}$$

where

$$\widetilde{S}_{\rho,T}=\Big\{(u,g)\in E^3_{\tau,T}:u(x,\tau)=\widehat{u}(x,\tau),\ u_t(x,\tau)=\widehat{u}_t(x,\tau),$$
$$u_{tt}(x,\tau)=\widehat{u}_{tt}(x,\tau),\ u(0,t)=u(l,t),\ u_x(0,t)=u_x(l,t),$$
$$(x,t)\in\Omega_T,\quad \|u\|_{B^3_{\tau,T}}+\|g\|_{C[\tau,T]}\le\rho\Big\},$$

here u is the unique solution of the initial and boundary value problem (29), (30), (31). Furthermore, g is the solution of (32) in terms of u.

From (22), we know that the function $\widehat{u}(x,t)$ belongs to class $B^3_{0,\tau}$ in $t\in[0,\tau]$. Therefore, we can conclude that $\{\widehat{u}(\cdot,\tau),\widehat{u}_t(\cdot,\tau)\}\in C^2[0,l]$, and $\widehat{u}_{tt}(\cdot,\tau)\in C^1[0,l]$. In addition, we should show that $\Big\{\widehat{u}_{xxx}(\cdot,\tau),\ \widehat{u}_{txxx}(\cdot,\tau),\ \widehat{u}_{ttxx}(\cdot,\tau)\Big\}\in L_2(0,l)$. According to Theorem 2.2 the function $u(\cdot,\tau)$ the same as (12) at $t\in[\tau,T]$. Taking into account (12), we get

$$\|\widehat{u}_{xxx}(\cdot,\tau)\|_{L_2(0,l)}\le\left(\sum_{n=1}^{\infty}\left(\lambda_n^3\left(\widehat{u}_{10}+\widehat{u}_{1n}+\widehat{u}_{2n}\right)\right)^2\right)^{\frac12}$$
$$\le\delta_1\left(\|\varphi'''\|_{L_2(0,l)}+\|\psi''\|_{L_2(0,l)}+\|\phi'\|_{L_2(0,l)}\right)e^{3\|\widehat{g}\|\tau}.$$

As a result of simple calculations, we obtain

$$\|\widehat{u}_{txxx}(\cdot,\tau)\|_{L_2(0,l)}\le\left(\sum_{n=1}^{\infty}\left(\lambda_n^3\left(\widehat{u}'_{10}+\widehat{u}'_{1n}+\widehat{u}'_{2n}\right)\right)^2\right)^{\frac12}$$

$$\le\delta_2\left(1+\|\widehat{g}\|\tau e^{3\|\widehat{g}\|\tau}\right)e^{3\|\widehat{g}\|\tau}\left(\|\varphi'''\|_{L_2(0,l)}+\|\psi'''\|_{L_2(0,l)}+\|\phi''\|_{L_2(0,l)}\right)e^{3\|\widehat{g}\|\tau}.$$

As above, we get

$$\|\widehat{u}_{ttxx}(\cdot,\tau)\|_{L_2(0,l)}\le\left(\sum_{n=1}^{\infty}\left(\lambda_n^2\left(u''_{10}+u''_{1n}+u''_{2n}\right)\right)^2\right)^{\frac12}$$

$$\leq \delta_3 \left(1 + (1+\tau)\|\widehat{g}\|e^{3\|\widehat{g}\|\tau} + \|\widehat{g}\|\tau^2 e^{3\|\widehat{g}\|\tau}\left(1+\widehat{g}\tau e^{3\|\widehat{g}\|\tau}\right)\right) \times$$
$$\times \left(\|\varphi'''\|_{L_2(0,l)} + \|\psi'''\|_{L_2(0,l)} + \|\phi''\|_{L_2(0,l)}\right),$$

where δ_1, δ_2, δ_3 are arbitrary positive real constants.

Further, by (22), (29), (30) and (31) for $t \in [\tau, T]$, we have

$$\|u\|_{B^3_{\tau,T}} \leq \tau\|\widehat{u}_{10}\|_{C^2[\tau,T]} + 4\|\widehat{u}_{xxx}\|_{L_2(0,l)} + 2\|\widehat{u}_{txx}\|_{L_2(0,l)}$$

$$+\|\widehat{u}_{ttx}\|_{L_2(0,l)} + 2\tau^2\|\widehat{g}\|_{C[0,T]}\|\widehat{u}\|_{B^3_{0,T}} + 2\left(T-\tau\right)^2\|g\|_{C[\tau,T]}\|u\|_{B^3_{\tau,T}}. \tag{35}$$

Besides, similarly as (23) and (32) for $t \in [\tau, T]$:

$$\|g\|_{C[\tau,T]} \leq \frac{1}{|\varphi_0|}\Big(\|h\|_{C[\tau,T]} + \sqrt{l}|\eta_{10}|\|\widehat{u}_{10}\|_{C^2[\tau,T]}\Big)$$

$$+\frac{4\sqrt{l}}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\left(\|\widehat{u}_{xxx}(\cdot,\tau)\|_{L_2(0,l)} + \|\widehat{u}_{txxx}(\cdot,\tau)\|_{L_2(0,l)} + \|\widehat{u}_{ttxx}(\cdot,\tau)\|_{L_2(0,l)}\right)$$

$$+\frac{8\sqrt{l}\tau}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\Big(\|\varphi''\|_{L_2(0,l)} + \|\psi''\|_{L_2(0,l)} + \|\phi'\|_{L_2(0,l)}\Big)\|\widehat{g}\|_{C[0,T]}$$

$$+\frac{8\sqrt{l}(T-\tau)}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\Big(\|\widehat{u}_{xx}(\cdot,\tau)\|_{L_2(0,l)} + \|\widehat{u}_{txx}(\cdot,\tau)\|_{L_2(0,l)}\Big)\|g\|_{C[\tau,T]}$$

$$+\frac{8\sqrt{l}(T-\tau)}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\|\widehat{u}_{ttx}(\cdot,\tau)\|_{L_2(0,l)}\|g\|_{C[\tau,T]}$$

$$+\frac{2\sqrt{l}}{|\varphi_0|}\|\eta'\|_{L_2(0,l)}\left(\tau^2\|\widehat{g}\|_{C[0,T]}\|\widehat{u}\|_{B^3_{0,T}} + (T-\tau)^2\|g\|_{C[\tau,T]}\|u\|_{B^3_{\tau,T}}\right)$$

$$+\frac{1}{|\varphi_0|}\|\eta\|_{L_2(0,l)}\left(\tau^3\|\widehat{g}\|^2_{C[0,T]}\|\widehat{u}\|_{B^3_{0,T}} + (T-\tau)^3\|g\|^2_{C[\tau,T]}\|u\|_{B^3_{\tau,T}}\right). \tag{36}$$

So, taking into account (35), we get

$$\|u\|_{B^3_{\tau,T}} \leq \delta_4\left(1 + \tau + \tau^2 + (T-\tau)^2\|g\|_{C[\tau,T]}\|u\|_{B^3_{\tau,T}}\right). \tag{37}$$

On the other hand, by (36), it is true the estimate

$$\|g\|_{C[\tau,T]} \leq \delta_5\Big(1 + \tau + \tau^2 + \tau^3 + (T-\tau)\|g\|_{C[\tau,T]}$$

$$+(T-\tau)^2\|g\|_{C[\tau,T]}\|u\|_{B^3_{\tau,T}} + (T-\tau)^3\|g\|^2_{C[\tau,T]}\|u\|_{B^3_{\tau,T}}\Big). \tag{38}$$

where δ_4 and δ_5 are independent of T.

Using (37) and (38), we obtain

$$\|\Lambda(u,g)\|_{E^3_{\tau,T}} \leq \delta_4\left(1+\tau+\tau^2\right) + \delta_5\left(1+\tau+\tau^2+\tau^3\right) + \delta_5\left(T-\tau\right)\rho$$

$$+(\delta_4+\delta_5)\left(T-\tau\right)^2\rho^2 + \delta_5\left(T-\tau\right)^3\rho^3 = \delta_4 + \delta_5 + \zeta(\rho,\tau,T-\tau), \tag{39}$$

where

$$\zeta(\rho,\tau,T-\tau)=\delta_4\left(\tau+\tau^2\right)+\delta_5\left(\tau+\tau^2+\tau^3\right)$$
$$+\delta_5\left(T-\tau\right)\rho+\left(\delta_4+\delta_5\right)\left(T-\tau\right)^2\rho^2+\delta_5\left(T-\tau\right)^3\rho^3.$$

By similar calculations to (29), (30), (31) we have

$$\|\Lambda(u_1,g_1)-\Lambda(u_2,g_2)\|_{E^3_{\tau,T}}\leq\delta_5(T-\tau)\|(u_1-u_2,g_1-g_2)\|_{E^3_{\tau,T}}$$
$$+\Big(\left(\delta_4+\delta_5\right)\rho(T-\tau)^2+\delta_5\rho^2(T-\tau)^3\Big)\|(u_1-u_2,g_1-g_2)\|_{E^3_{\tau,T}}. \tag{40}$$

It is easy to see that if we choose $\rho\geq r$, then we could get $T-\tau\leq\tau$ to satisfy

$$\delta_4+\delta_5\leq\frac{\rho}{2},\quad \zeta(\rho,\tau,T-\tau)\leq\frac{\rho}{2}.$$

Furthermore noticing that (39) and (28) have the same structure, we can choose $T-\tau=\tau$ to satisfy (39), which yields $\|\Lambda(u,g)\|\leq\rho$ i.e. $\Lambda(u,g)\in\widetilde{S}_{\rho,T}$.

Additionally,

$$\|\Lambda(u_1,g_1)-\Lambda(u_2,g_2)\|_{E^3_{\tau,T}}<\|(u_1-u_2,g_1-g_2)\|_{E^3_{\tau,T}},$$

for $T=2\tau$. This estimate show that Λ is a contraction map on $\widetilde{S}_{\rho,T}$ for all $T\in(\tau,T]$. Repeating the extension process limited times, we could obtain a solution $\{u,g\}\in E^3_{0,T}$ of the inverse problem (1)-(3) and (5) for any T.

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