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THREE-COMPONENT VERSION OF THE TIKHONOV REGULARIZATION METHOD FOR OPERATOR EQUATIONS OF THE FIRST KIND

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Abstract An ill-posed problem in the form of a linear operator equation is considered. It is assumed that the solution to the equation in the one-dimensional case can be represented in the form of a sum of three components: the first component contains discontinuities, the second contains discontinuities in the derivative, and the third is continuous. To construct a stable approximate solution, the three-component Tikhonov method is used. In this case, the stabilizer is the sum of three functionals: BV_p-norm of the first component, BV_p-norm of the derivative for the second component and the norm of the Sobolev space for the third component, and each functional depends on only one component. The convergence of the sum of regularized components to the solution of the original equation is proved. In addition, piecewise uniform convergence of approximate solutions is established. The results of numerical experiments on reconstructing a three-component model solution for the Fredholm equation of the first kind are presented.

Keywords: ill-posed problem, Tikhonov regularization, non-smooth solution, total variation, subgradient method.

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1 Introduction

For ill-posed problems with a solution that has some singularities in different parts of the domain of definition, an important problem is the construction of a stabilizing functional that takes into account this information as much as possible when using variational methods of regularization of these problems. The most common case is when, along with smooth areas, the solution contains discontinuities and kinks, as well as areas with close extrema, etc.

The main approach, which first emerged in applications [1, 2] and then became the object of theoretical research [3, 4, 5], is based on representing the solution as a sum of several components. For simplicity of presentation, we will first restrict ourselves to the case of two components $u = u_1 + u_2$. Then in the Tikhonov regularization method the stabilizing functional is constructed in the form of the sum of two functionals $\Omega(u_1, u_2) = \Omega_1(u_1)+\Omega_2(u_2)$ each of which depends on only one component and takes into account its peculiarity.

Provided that there is a priori information about the presence of discontinuities and kinks in the solution of the linear operator equation,

$$Au = f \tag{1.1}$$

consider Tikhonov's regularization method in the form ($||f - f_\delta|| \le \delta$)

$$\inf\left\{\|A(u_1+u_2)-f_\delta\|^2_{L_2}+\alpha\left[\|u_1\|_{BV_p}+\|u_2^{(1)}\|_{BV_p}\right]:\ u_2(a)=0,\ u_1,u_2^{(1)}\in BV_p\right\}=\Phi_*, \tag{1.2}$$

here BV_p is a complete normed space with the norm [6]

$$\|u\|_{BV_p}=\|u\|_{L_p}+G^b_a(u),\ p>1, \tag{1.3}$$

$$G^b_a(u)=\sup\left\{\int_a^b u(x)\cdot v'(x)\,dx:\ v\in C^1_0(a,b),\ |v(x)|\le 1\right\}.$$

If in (1.3) instead of L_p is used L_1- norm is used, then the corresponding Banach space is denoted by BV [7].

It should be noted that the norm of the space BV and its smooth approximation were successfully used as a stabilizer when reconstructing non-smooth components of the solution in the multidimensional case $n \ge 2$ (see [5]). Here in the proof convergence of regularized components of approximate solutions, the theorem on the compactness of the embedding operator $J: BV \to L_p(D)$ [8] is essentially used. However, in the one-dimensional case there is no analogue of this theorem; therefore, we cannot use a similar technique to prove the convergence of approximate solutions. To investigate Tikhonov regularization (1.2), we need a strengthened version of the following statement about approximation of the function by smooth functions.

Statement 1 [7]. For any function $u \in BV$ there is a sequence of functions $u_i \in C^\infty(D)$ for which the following relations hold:

$$\lim_{i\to\infty}\|u_i-u\|_{L_1(D)}=0,\ \lim_{i\to\infty}G_D(u_i)=G_D(u), \tag{1.4}$$

where $G_D(u)$ is the total variation of the function u, given in a domain $D \subseteq \mathbb{R}^n$.

Let us establish that in the one-dimensional case the convergence of $\{u_i\}$ holds not only in L_1, but also in $L_p[a,b]$, $p > 1$..

Lemma 1. For any function $u \in BV_p$ there is a sequence $u_i \in C^\infty(a,b)$, for which the following relations holds:

$$\lim_{i\to\infty}\|u_i-u\|_{L_p}=0,\ \lim_{i\to\infty}G^b_a(u_i)=G^b_a(u). \tag{1.5}$$

Proof

From the first relation in (1.4) it follows the existence of a convergent almost everywhere subsequence

$$u_{i_k}(x)\to u(x)\ \text{(almost everywhere)}\ \ x\in[a,b]. \tag{1.6}$$

Let x_0 be the convergence point of the subsequence $\{u_{i_k}\}$, which implies that $|u_{i_k}| \le c_1$ is bounded. Since $u_{i_k} \in C^\infty$ and (1.4) holds, we have $G^b_a(u_{i_k}) = V^a_b(u_{i_k}) \to G^b_a(u)$, from the following estimate holds

$$|u_{i_k}(x)-u_{i_k}(x_0)|\le V^x_{x_0}(u_{i_k})\le V^b_a(u_{i_k})\le c_2. \tag{1.7}$$

From (1.6) and (1.7) it follows that

$$\max_{x\in[a,b]}|u_{i_k}(x)|\le c_2. \tag{1.8}$$

From the convergence of the sequence $\{u_{i_k}\}$ almost everywhere (1.6) and the boundedness almost everywhere (1.8) by the Lebesgue theorem on transition to the limit for the integral, the first relation (1.5) follows, and the second relation implies from (1.4).

2 Existence end convergence of regularized solutions

Let us show that there is an analogue of the normal solution to problem (1.1).

Theorem 1. Let the operator A acting from $L_p, (p > 1)$, into L_2 be continuous. Then there is a unique solution $(\hat{u}_1, \hat{u}_2)$ to the following problem

$$\inf \left\{ \|u_1\|_{BV_p} + \|u_2^{(1)}\|_{BV_p} : \, A(u_1 + u_2) - f = 0, \, u_2(a) = 0, \, u_1, u_2^{(1)} \in BV_p \right\} = \Psi_*. \qquad (2.1)$$

Proof

Let (u_{1k}, u_{2k}) be a minimizing sequence in problem (2.1). Then, due to its boundedness, there exist weakly convergent subsequences

$$u_1^{k_i} \to \hat{u}_1 \text{ (weakly) in } L_p, \, (u_2^{k_i})^{(1)} \to \hat{v} \text{ (weakly) in } L_p.$$

Since W_p^1 is a weakly complete space and $u_2(a) = 0$, then $\hat{v} = \hat{u}_2^{(1)}$, where $\hat{u}_2$ is an element of the space L_p. Due to the weak lower continuity of the norm L_p and the total variation G_a^b ([8], theorem 2.4), we have the relations:

$$0 \le \|A(\hat{u}_1 + \hat{u}_2) - f\| \le \liminf_{i \to \infty} \|A(u_1^{k_i} + u_2^{k_i}) - f\| = 0,$$

$$\Psi_* \le \|\hat{u}_1\|_{BV_p} + \|\hat{u}_2^{(1)}\|_{BV_p} \le \liminf_{i \to \infty} \left(\|u_1^{k_i}\|_{BV_p} + \|(u_2^{k_i})^{(1)}\|_{BV_p} \right) \le \Psi_*,$$

i. e. $(\hat{u}_1, \hat{u}_2)$ is solution to problem (2.1). Since the L_p- norm for $p > 1$ is strictly convex, and G_a^b is a convex functional, then the pair $\hat{u}_1, \hat{u}_2$ is the unique solution of problem (2.1).

Theorem 2. Let the operator A acting from $L_p[a, b]$, $p > 1$ into $L_2[a, b]$ be linear and continuous. Then:

1) for any $\alpha > 0$ problem (1.2) has a unique solution u_1^α, u_2^α;

2) for $\alpha(\delta) \to 0$, $\delta^2/\alpha(\delta) \to 0$, $\delta \to 0$ the convergence of the components holds:

$$\lim_{\delta \to 0} \|u_1^{\alpha(\delta)} - \hat{u}_1\|_{L_p} = 0, \, \lim_{\delta \to 0} \|u_2^{\alpha(\delta)} - \hat{u}_2\|_{W_p^1} = 0,$$

where $(\hat{u}_1, \, \hat{u}_2)$ is the solution of problem (2.1), hence, $\hat{u} = \hat{u}_1 + \hat{u}_2$ is the normal solution of problem (1.1) with respect to the stabilizer $\Omega(u_1, u_2) = ||u_1||_{BV_p} + ||u_2^{(1)}||$.

3) if the component $\hat{u}_1(x)$ (the derivative of $\hat{u}_2^{(1)}$ of the cjmponent $\hat{u}_2$) does not contain brakes on $[a_1, b_1] \in [a, b]$, then for $\alpha(\delta) \to 0, \delta^2/\alpha(\delta) \to 0, \delta \to 0$ the sequence

$$u_1^{\alpha(\delta)} \to \hat{u}_1, \; (u_2^{\alpha(\delta)})^{(1)} \to (\hat{u}_2)^{(1)})$$

converges uniformly on $[a_1, a_2]$.

Proof

Solvability. Let us denote the objective functional in problem (1.2) by Φ. Let (u_1^k, u_2^k) be the minimizing sequence in problem (1.2), i.e. $\Phi(u_1^k, u_2^k) \to \Phi_*$. Since each of the sequences $\{u_i^k\}(i = 1, 2)$ is bounded, the existence of a weakly convergent subsequence follows:

$$u_1^{k_i} \to \bar{u}_1 \text{ (weakly) in } L_p, \, (u_2^{k_i})^{(1)} \to \bar{v}_2 \text{ (weakly) in } L_p.$$

Then, as in the proof of Theorem 1, we can replace $\bar{v}_2$ by $\bar{u}_2^{(1)}$, where $\bar{u}_2 \in L_p$. Due to the continuity of the operator A, weak lower continuity of the L_p-norm and generalized variation, we obtain

$$\Phi_* \le \Phi(\bar{u}_1, \bar{u}_2) \le \liminf_{i \to \infty} \Phi(u_1^{k_i}, u_2^{k_i}) \le \Phi_*,$$

i.e. problem (1.2) is solvable. Since the objective functional Φ is strictly convex, the solution is unique.

Convergence in L_p. Let us redesignate $(\bar u_1, \bar u_2)$ by (u_1^α, u_2^α). We have obvious inequalities

$$\Phi(u_1^\alpha, u_2^\alpha) \le \Phi(\hat u_1, \hat u_2),$$

$$\|u_1^\alpha\|_{L_p} + G_a^b(u_1^\alpha) + \|(u_2^\alpha)^{(1)}\|_{L_p} + G_a^b((u_2^\alpha)^{(1)}) \le \|\hat u_1\|_{L_p} + G_a^b(\hat u_1) + \|(\hat u_2)^{(1)}\|_{L_p} + G_a^b((\hat u_2)^{(1)}) + \frac{\delta^2}{\alpha(\delta)}, \tag{2.2}$$

where $(\hat u_1, \hat u_2)$ is solution to problem (2.1). Under the following conditions on the parameters $\alpha_k = \alpha(\delta_k) \to 0, \delta^2/\alpha_k \to 0, \delta_k \to 0$ as k $\to \infty$ from it follows that there are weakly convergent sequences in L_p:

$$u_1^{\alpha_k} \rightharpoonup \tilde u_1 \text{ (weakly) in } L_p,\ (u_2^{\alpha_k})^{(1)} \rightharpoonup \tilde u_2 \text{ (weakly) in } L_p, \tag{2.3}$$

for which the follwing inequalities are true

$$\|A(\tilde u_1 + \tilde u_2) - f\|^2 \le \liminf_{k\to\infty} \Phi(u_1^{\alpha_k}, u_2^{\alpha_k}) \le \Phi(\hat u_1, \hat u_2) \le \lim_{\delta_k \to 0} \left(\delta_k^2 + \alpha(\delta_k)\cdot\Omega(\hat u_1, \hat u_2)\right) = 0,$$

i.e. $(\tilde u_1, \tilde u_2)$ is solution of the operator equation (1.1); here $\Omega(u_1, u_2)$ denotes the stabilizing functional in (1.2). Passing to the lower limit in inequality (2.2) at $k \to \infty$, we have ratio

$$\begin{aligned} &\|\tilde u_1\|_{L_p} + G_a^b(\tilde u_1) + \|(\tilde u_2)^{(1)}\|_{L_p} + G_a^b((\tilde u_2)^{(1)}) \\ &\le \liminf_{k\to\infty} \left(\|u_1^{\alpha_k}\|_{L_p} + G_a^b(u_1^{\alpha_k}) + \|(u_2^{\alpha_k})^{(1)}\|_{L_p} + G_a^b((u_2^{\alpha_k})^{(1)})\right) \\ &\le \|\hat u_1\|_{L_p} + G_a^b(\hat u_1) + \|(\hat u_2)^{(1)}\|_{L_p} + G_a^b((\hat u_2)^{(1)}) \end{aligned} \tag{2.4}$$

which means that equality is realized in (2.4) and $(\tilde u_1, \tilde u_2)$ coincides with the solution to problem (2.1). In addition, (2.4) implies convergence of the norms:

$$\lim_{k\to\infty} \|u_1^{\alpha_k}\|_{L_p} = \|\hat u_1\|_{L_p},\ \lim_{k\to\infty} \|(u_2^{\alpha_k})^{(1)}\|_{L_p} = \|(\hat u_2)^{(1)}\|_{L_p} \tag{2.5}$$

Combining (2.3) and (2.5), we obtain strong convergence of the components in L_p, i.e. proof of item 2 of the theorem.

Piecewise uniform convergence. By Lemma 1, for any $\bar u_1, \bar u_2^{(1)}$ there exist sequences such that

$$\lim_{k\to\infty} \|u_1^k - \bar u_1\|_{L_p} = 0,\ \lim_{k\to\infty} G_a^b(u_1^k) = G_a^b(\bar u_1), \tag{2.6}$$

$$\lim_{k\to\infty} \|(u_2^k)^{(1)} - (\bar u_2)^{(1)}\|_{L_p} = 0,\ \lim_{k\to\infty} G_a^b((u_2^k)^{(1)}) = G_a^b((\bar u_2)^{(1)}). \tag{2.7}$$

Let us first show that for any first component $\bar u_1$ there is an equivalent function $\bar{\bar u}_1$, such that $V_a^b(\bar{\bar u}_1) = G_a^b(\bar{\bar u}_1) = G_a^b(\bar u_1)$. From (2.6) (see proof of Lemma 1) it follows the existence subsequences $\{u_1^{k_i}\}$ for which $u_1^{k_i}$ (almost everywhere), $|u(x)| \le c_1, V_a^b(u_1^{k_i}) \le c_2$. Then, based on Hellys theorem, we can consider that

$$u_1^{k_i}(x) \to \bar{\bar u}_1(x)\, \forall x \in [a,b],\ V_a^b(\bar{\bar u}_1) \le c_3, \tag{2.8}$$

hence, $\bar u_1(x) = \bar{\bar u}_1(x)$, $G_a^b(\bar u_1) = G_a^b(\bar{\bar u}_1)$. From (2.6) and (2.8) we have for $u_1^{k_i} \in C^\infty[a,b]$

$$V_a^b(\bar{\bar u}_1) \le \liminf_{i\to\infty} V_a^b(u_1^{k_i}) = \lim_{i\to\infty} G_a^b(u_1^{k_i}) = G_a^b(\bar u_1) = G_a^b(\bar{\bar u}_1). \tag{2.9}$$

By property ([7], page 29) the following relation holds

$$G_a^b(\bar{u}_1) = \inf\Big\{V_a^b(g) :\ g(x) = \bar{u}_1(x)\,(\text{almost everywhere}),\ x \in [a,b]\Big\}.$$

Together with the relation (2.9) this implies

$$V_a^b(\bar{\bar{u}}_1) \le G_a^b(\bar{u}_1) = \inf\Big\{V_a^b(g) :\ g(x) = \bar{u}_1(x)\,(\text{almost everywhere}),\ x \in [a,b]\Big\} \le V_a^b(\bar{\bar{u}}_1).$$

A similarly established property is proved for the function $u_2^{(1)}$.

From what was proved above it follows that we can assume that the following equality is true: $G_a^b(u_1^{\alpha_k}) = V_a^b(u_1^{\alpha_k})\,\forall u_1^{\alpha_k},$ therefore, taking into account (2.4) we have

$$G_a^b(\hat{u}_1) = \lim_{k\to\infty} G_a^b(u_1^{\alpha_k}) = \lim_{k\to\infty} V_a^b(u_1^{\alpha_k}). \tag{2.10}$$

Taking into account the convergence of $\{u_1^{\alpha_k}\}$ in L_p proved in paragraph 1, using a similar scheme from the proof of Lemma 1 we select a pointwise convergent subsequence, which can be considered coinciding with $u_1^{\alpha_k}$

$$u_1^{\alpha_k}(x) \to \hat{u}_1(x)\,\forall x \in [a,b]. \tag{2.11}$$

On the one side,combining (2.10), (2.11), we obtain

$$V_a^b(\hat{u}_1) \le \lim_{k\to\infty} V_a^b(u_1^{\alpha_k}) = G_a^b(\hat{u}). \tag{2.12}$$

On the other hand, the following relation is valid

$$V_a^b(\hat{u}_1) \le G_a^b(\hat{u}_1) = \inf\Big\{V_a^b(g) :\ g(x) = \hat{u}_1(x)\,(\text{almost everywhere}),\ x \in [a,b]\Big\} \le V_a^b(\hat{u}_1). \tag{2.13}$$

From (2.11) – (2.13) and the results of the work ([9], Chapter 4, 1, Theorem 1, Corollary 2) the proof of item 3 of the theorem follows.

Taking (2.7) into account, the piecewise uniform convergence of regularized solutions for the derivative of the second component $\hat{u}_2^{(1)}$ is proved in a similar way.

Corollary 1. If, along with the components u_1, u_2, there is the third component, i.e. $u = u_1+u_2+u_3$, where u_3 is responsible for the smooth component, then $\|u_3\|_{W_p^n} (p > 1, n \ge 1)$ can be taken as the third stabilizing functional $\Omega_3(u_3)$ In this case, all the properties in the conclusion of Theorem 2 regarding the components u_1, u_2 are preserved, and in item 2 we additionally include the relation $\lim_{\alpha\to 0} ||u_3^\alpha - \hat{u}_3||_{W_p^n} = 0$.

3 Discrete approximation and subgradient methods

After agging the third component u_3 with the stabilizer $||u_3||^2_{w_2^1}$ in (1.2) and using a finite-difference approximation, we associate the problem (1.2) with the following sequence of finite-dimensional problems:

$$\inf\Bigg\{\|A_n(u_{1n}+u_{2n}+u_{3n}) - f_n\|^2_{l_2^n} + \alpha\Big[\|u_{1n}\|_{l_p^n} + G_n(\Delta_1 u_{1n}) + \|\Delta_1 u_{2n}/h\|_{l_p^n} + G_n(\Delta_2 u_{2n}) + \\ + ||u_{3n}||^2_{w_2^{n,1}}\Big] :\ u_{2n}(a) = 0,\ u_{1n}, u_{2n}, u_{3n} \in \mathbb{R}^n\Bigg\}, \tag{3.1}$$

where

$$||u_{1n}||l_p^n = (\sum_{i=1}^{n} h|u_{1n}^i|^p)^{1/p},\ ||G_n(\Delta_1 u_{1n}) = \sum_{i=0}^{n-1} |u_{1n}^{i+1} - u_{1n}^i|,$$

$$\|\Delta_1 u_{2n}/h\|_{l_p^n} = \left(\sum_{i=0}^{n-1} \left|(u_{2n}^{i+1} - u_{2n}^i)/h\right|^p\right)^{\frac{1}{p}},$$

$$G_n(\Delta_2 u_{2n}) = \sum_{i=1}^{n-1} \left|\left(u_{2n}^{i+1} - 2u_{2n}^i + u_{2n}^{i-1}\right)/h\right|,$$

$$||u_{3n}||^2_{w_2^{n,1}} = \sum_{i=1}^{n} h|u_{3n}^i|^2 + \sum_{i=1}^{n-1} h|(u_{3n}^i - u_{3n}^i)/h|^2.$$

When formulating the following theorem on the approximation of the solution $(\hat{u}_1^\alpha, \hat{u}_2^\alpha)$ of problem (1.2) by solutions u_{1n}, u_{2n} of finite-dimensional problems (3.1) some terms, definitions and notations relating to discrete approximation of spaces, discrete convergence of elements and operators, which are presented, for example, in [10, 11], are used. Let's use the notations "$- \rightarrow$" and "$- \rightharpoondown$" for discrete and weak discrete convergence.

Theorem 3. Let $A : L_p \to L_2$ be a linear bounded and let the discrete convergence conditions $A_n - \to A$, $A_n - \rightharpoondown A$, $f_n - \to f_\delta$ as $n \to \infty$ be fulfilled, where $\{A_n\}$ is the sequence of linear operators $A_n : l_p^n \to l_2^n$. Let $r_n : l_p^n \to L_p$ be the operaor of piecewise linear interpolation. Then the problem (3.1) has a unique solution $(\bar{u}_{1n}, \bar{u}_{2n}, \bar{u}_{3n})$ with the following properties:

$$\lim_{n\to\infty} ||r_n \bar{u}_{1n} - u_1^\alpha||_{L_p} = 0,\ \lim_{n\to\infty} ||r_n \bar{u}_{2n} - u_2^\alpha||_{L_p} = 0,$$

$$\lim_{n\to\infty} ||r_n \bar{u}_{3n} - u_3^\alpha||_{L_p} = 0,$$

where $(u_1^\alpha, u_2^\alpha, u_3^\alpha)$ is the solution of problem (1.2) after adding the stabilizer $||u_3||_{W_2^1}$.

The proof is based on the scheme outlined in [10].

Since any convex function is subdifferentiable, the the following subgradient method can be used to solve problem (1.3):

$$u_i^{k+1} = u_i^k - \gamma_k \cdot \frac{v^k}{\|v^k\|},\ v^k \in \partial\Phi(u_1^k, u_2^k, u_3^k),\ i = 1, 2, \tag{3.2}$$

where $\partial\Phi(u_1^k, u_2^k, u_3^k)$ — subdifferential of the objective function of problem (3.1) at point (u_1^k, u_2^k, u_3^k). Provided that $\gamma_k > 0$, $\sum_{k=1}^\infty \gamma_k = \infty$, $\gamma_k > 0$, $\sum_{k=1}^\infty \gamma_k^2 < \infty$, the iterative process (3.2) converges to the solution $u_{1n}^\alpha, u_{2n}^\alpha, u_{3n^\alpha}$ as $n \to \infty$[12]. For γ_k we can take, for example, $\gamma_k = 1/k$.

4 Numerical experiments

A numerical experiment was performed for problem (3.1) with using the subgradient method for a integral equation arising under continuation of a gravitational field to the depth H [13]

$$A\,u \equiv \frac{1}{\pi} \int_{-1}^{1} \frac{H}{(x-s)^2 + H^2} u(s) ds = f(x),\ H = 0.3.$$

After discrete approximation of the integral equation calculations were carried out at 201 nodes of the uniform grid.

In the experiment the model solution is the sum of three components u_1, u_2, u_3, where the function u_1 has the breaks of the first kind, u_2 has discontinuities in the derivative of the first kind, u_3 is smooth. The model solution has the form $u(x) = u_1(x) + u_2(x) + u_3(x)$, where

$$u_1(x) = \begin{cases} 1, & \text{if } -0.9 \le x \le -0.6; \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(x) = \begin{cases} -3 \cdot |x| + 0.9, & \text{if } -1 \le x < -0.75; \\ 0, & \text{otherwise.} \end{cases}$$

$$u_3(x) = \begin{cases} 3 \cdot \exp\left(-\frac{0.25^2}{0.25^2-(x-0.7)^2}\right), & \text{if } |x-0.7| < 0.25; \\ 0, & \text{otherwise.} \end{cases}$$

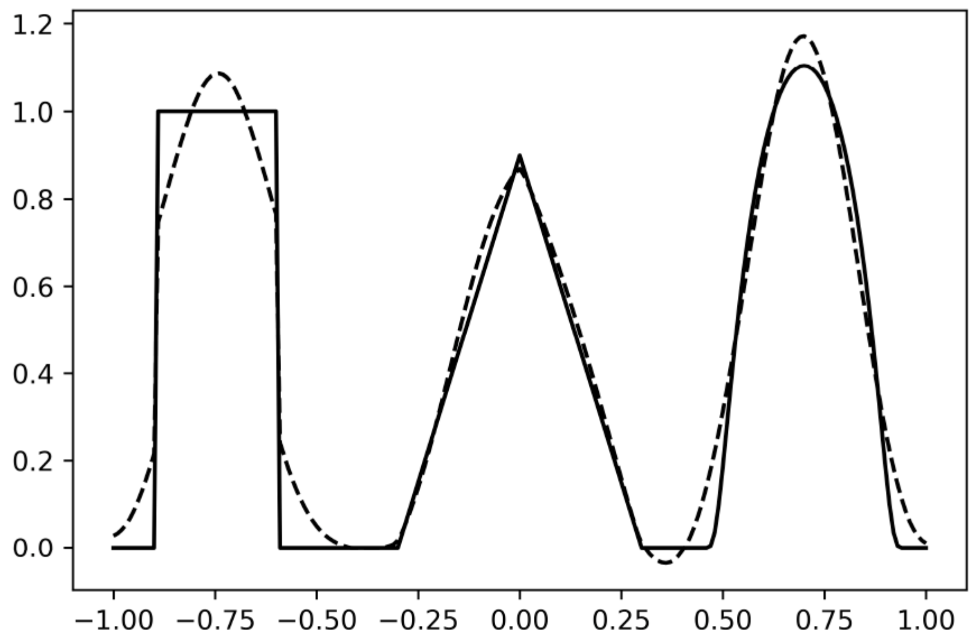

Figure 1.

Figure 1 contains the exact (solid line) and numerical (dotted line) solutions obtained for the parameters of regularization $\alpha_1 = 0.5 \cdot 10^{-4}$, $\alpha_2 = 0.5 \cdot 10^{-6}$, $\alpha_3 = 0.5 \cdot 10^{-6}$, L_p−norm with $p = 1.1$ and the right hand side $f_\delta(x)$ given with relative error $\delta = 0.012$ after $N = 10000$ iteration by the subgradient method. The relative error of the numerical solution is $\bar{\Delta} = 0.125$, and the relative residual equals to 0.04.

Conclusion

From the theoretical view point for the modified three-component Tikhonov method, the convergence of the sum of regularized components to the solution of the original equation is proved. Also, piecewise uniform convergence of approximate solutions is established. The numerical results obtained with using the subgradient method show that in the case when the solution has three types of peculiarities, the regularizing algorithm simultaneously reconstructs all three components of the solution and its subtle structure. The results of a numerical experiment show that the structure of a component with a discontinuity of the first kind is reconstructed worse than the structure of the other two components, i.e. there is oversmoothing of the first component. Maybe, it is necessary to choose more carefully the control parameters.Thus, a multicomponent version of the Tikhonov regularization method with several stabilizing functionals allows us to simultaneously restore the solution to an ill-posed problem with different types of peculiarities.

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