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NEW STABILITY NUMBER OF THE TIMOSHENKO SYSTEM WITH ONLY MICROTEMPERATURE EFFECTS AND WITHOUT THERMAL CONDUCTIVITY

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Abstract In this work, we study the asymptotic behavior of a thermoelastic Timoshenko system with dissipation due only to microtemperature effects and no thermal diffusivity. Under an appropriate new assumption about the coefficients of the system and by using the energy method, we prove that the unique dissipation due to microtemperatures is strong enough to stabilize the system exponentially.

Key words: Exponential decay; Timoshenko system; microtemperature dissipation; energy method; exponential stability.

AMS Mathematics Subject Classification: 35B37, 35L55, 74D05, 93D15.

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1 Introduction

In recent decades, research on Timoshenko type systems has been studied by fairly a large number of researchers, and an increasing interest has been developed to determine the asymptotic behavior by using several different dampings (frictional, structural, viscoelastic) linear, nonlinear, with and without coupling with a heat equation have been treated see ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and [14]). Recently, an essential problem was addressed about the minimal dissipation required to get exponential decay. In this regard, the propagation velocities of waves (also known as the stability number) are crucial in proving the stability results. In 1920, Timoshenko [15] developed a model with two basic evolution equations given by

$$\rho A\varphi_{tt} - S_x = 0, \qquad \rho I\psi_{tt} - M_x + S = 0. \tag{1}$$

When the transverse shear strain is significant, this term is commonly used to describe the vibration of a beam, where x is the distance along the center line of the beam structure, t is the time, ρ is the density, A is the cross-sectional area, I is a cross section's moment of inertia, M is the bending moment, S is the shear force. The variables φ and ψ are the transversal displacement of the beam and the rotational angle of the beam, respectively. The constitutive equations are:

$$S = \kappa AG(\varphi_x + \psi), \qquad M = EI\psi_x. \tag{2}$$

Here κ is a constant which depends upon the shape of the cross section, E is the Young's modulus of elasticity, and G is the modulus of rigidity. Now, for simplicity, we note $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k = \kappa AG$, $b = EI$, and then substituting (2) into (1), we obtain the following system

$$\begin{aligned} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0. \end{aligned} \tag{3}$$

For the Timoshenko system with effective thermoelastic dissipation on shear force, we have, in addition to (1)

$$\rho_3\theta_t + q_x + \gamma\left(\varphi_x + \psi\right)_t = 0, \tag{4}$$

where q represents the heat flow and θ is the temperature difference. The constitutive equations in this case are as follows

$$S = k(\varphi_x + \psi) + \gamma\theta, \quad M = b\psi_x - \gamma\theta, \quad q = -\beta\theta_x, \tag{5}$$

where the coupling terms γ and β are assumed to be positive. The combination of (1),(4), and (5) yields

$$\begin{aligned} &\rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \gamma\theta_x = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta = 0, \\ &\rho_3\theta_t - \beta\theta_{xx} + \gamma\left(\varphi_x + \psi\right)_t = 0. \end{aligned} \tag{6}$$

One of the first studies concerning system (6) was by Rivera and Racke [16]. They proved that the system is exponentially stable if and only if the wave speeds are equal. Otherwise, the system is stable in a polynomial manner.

On the other hand, and unlike Timoshenko's system, the study of the asymptotic behavior of porous elastic systems with microtemperature effects has attracted the attention of many researchers, and several results have been established. For example, Casas and Quintanilla in [17], considered a one-dimensional system of micromorphic elastic solids with the usual thermal effects. More precisely, they studied the following system

$$\begin{aligned} &\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, && \text{in } (0,1)\times(0,\infty), \\ &J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, && \text{in } (0,1)\times(0,\infty), \\ &c\theta_t = k\theta_{xx} - \gamma u_{tx} - l\varphi_t - k_1 w_x, && \text{in } (0,1)\times(0,\infty), \\ &\delta w_t = k_4 w_{xx} - d\varphi_{tx} - k_2 w - k_3\theta_x, && \text{in } (0,1)\times(0,\infty). \end{aligned} \tag{7}$$

They employed the semi-group method to demonstrate the solutions' exponential stability independent of the wave propagation speeds, and their methodology was based on the arguments provided in Liu and Zheng's book [18]. A thermoelastic system was recently examined by Dridi and Djebabla [19] under the influence of temperature and micro-temperatures. They specifically took into account the following system

$$\begin{aligned} &\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, && \text{in } (0,1)\times(0,\infty), \\ &J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta + \beta\varphi_t, && \text{in } (0,1)\times(0,\infty), \\ &c\theta_t = -\gamma u_{tx} - l\varphi_t - k_1 w_x, && \text{in } (0,1)\times(0,\infty), \\ &\alpha w_t = k_3 w_{xx} - d\varphi_{tx} - k_2 w - k_3\theta_x, && \text{in } (0,1)\times(0,\infty). \end{aligned}$$

By using the multipliers method, the authors proved the exponential stability in the case of zero thermal conductivity and without any condition placed on the system's coefficients. For other studies see ([20]-[21]). Later, Saci and Djebabla improved this result in [22]. Specifically, they proved that a unique dissipation given by the microtemperatures is strong enough to produce exponential stability for the case $\beta = 0$, if

$$\chi = \chi_0 - \frac{\gamma^2}{c\rho} = 0$$

holds, where

$$\chi_0 = \frac{\mu}{\rho} - \frac{\delta}{J}.$$

In this study, we look at a Timoshenko-type one-dimensional linear system made up of four partial differential equations: Two hyperbolic equations are linked to two parabolic equations that represent the difference in temperature and microtemperature. In other terms, investigate the Timoshenko system given by:

$$\begin{array}{ll} \rho_1 u_{tt} = k\left(u_x + \varphi\right)_x - \gamma\theta_x, & \text{in } (0,1)\times(0,\infty), \\ \rho_2 \varphi_{tt} = b\varphi_{xx} - k\left(u_x + \varphi\right) - \gamma w_x + \gamma\theta & \text{in } (0,1)\times(0,\infty), \\ \rho_3 \theta_t = -k_1 w_x - \gamma\left(u_x + \varphi\right)_t, & \text{in } (0,1)\times(0,\infty), \\ w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - \gamma\varphi_{tx}, & \text{in } (0,1)\times(0,\infty), \end{array} \tag{8}$$

with the following initial conditions

$$\begin{array}{ll} u\left(x,0\right) = u^0\left(x\right),\ u_t\left(x,0\right) = u^1\left(x\right),\ \varphi\left(x,0\right) = \varphi^0\left(x\right), & x \in (0,\,1), \\ \varphi_t\left(x,0\right) = \varphi^1\left(x\right),\ w\left(x,0\right) = w^0\left(x\right),\ \theta\left(x,0\right) = \theta^0\left(x\right), & x \in (0,\,1), \end{array} \tag{9}$$

here u^0, u^1, φ^0, φ^1, w^0, θ^0 are given functions,
and the boundary conditions

$$\begin{array}{ll} u_x\left(0,t\right) = u_x\left(1,t\right) = \varphi\left(0,t\right) = \varphi\left(1,t\right) = 0, & t > 0, \\ \theta\left(0,t\right) = \theta\left(1,t\right) = w_x\left(0,t\right) = w_x\left(1,t\right) = 0, & t > 0. \end{array} \tag{10}$$

The physical significance of the positive values $k_1,\ k_2,\ k_3$ and b is widely established. The first two equations create the Timoshenko system, which is connected to the third equation, a heat equation, and the fourth, a damped micro-heat equation. More specifically, we are interested in the study of a Timoshenko system (8) with just microtemperature dissipation and without thermal diffusivity, and we offer a new stability number provided by

$$\chi_2 = \chi_1 - \frac{\rho_2\gamma^2}{\rho_3}, \tag{11}$$

where

$$\chi_1 = b\rho_1 - k\rho_2, \tag{12}$$

proving that this unique dissipation is strong enough to drive the system to the equilibrium state in an exponential manner if $\chi_2 = 0$ holds. Meanwhile, from the first equation of (8) and the boundary conditions (10), we get

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = 0,\ \forall\, t \geq 0, \tag{13}$$

and therefore

$$\int_0^1 u(x,t)dx = t\int_0^1 u^1(x)dx + \int_0^1 u^0(x)dx,\ \forall\, t \geq 0.$$

Consequently, if we set

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u^1 dx - \int_0^1 u^0 dx,\ t \geq 0,\ x \in [0,1]\,,$$

we get

$$\int_0^1 \overline{u}(x,t)dx = 0,\ t \geq 0.$$

Now, from the fourth equation of (8) and the boundary conditions (10), we obtain

$$\frac{d}{dt}\int_0^1 w(x,t)dx + k_3\int_0^1 w(x,t)dx = 0,\quad \forall\, t \geq 0, \tag{14}$$

thus

$$\int_0^1 w(x,t)dx = \left(\int_0^1 w^0 dx\right) e^{-tk_3},$$

so, if we put

$$\overline{w}(x,t) = w(x,t) - \left(\int_0^1 w^0 dx\right) e^{-tk_3},\ t \geq 0,\ x \in [0,1]\,,$$

we arrive at

$$\int_0^1 \overline{w}(x,t)dx = 0,\quad t \geq 0,$$

and $(\overline{u},\ \varphi, \theta,\ \overline{w})$ satisfies the same equations in (8)-(10). In what follows we will work with $\overline{u}$ and $\overline{w}$ but, for convenience, we write u and w instead of $\overline{u}$ and $\overline{w}$. The following describes the spirit of this manuscript: The existence and uniqueness result and its proof are briefly summarized in Section 2, and then in Section 3, we show that the exponential stability depends on a new relationship between the system's coefficients.

2 Existence and Uniqueness

In this section, we give a brief summary of the existence and uniqueness result for problem (8)-(10) using the semigroup theory. For more details, we refer the reader to [23]. We will use the following standard $L^2(0,1)$ space with the scalar product and the norm denoted by

$$\langle u, v\rangle_{L^2} = \int_0^1 u.v dx, \qquad \|u\|_{L^2}^2 = \int_0^1 u^2 dx,$$

respectively. So, if we denote $U = (u, v, \varphi, \psi, \theta, w)^T$, where $v = u_t$ and $\psi = \varphi_t$. Then, system (8)-(10) can be rewritten as follows

$$\begin{cases} U_t + \mathcal{A}U = 0,\ t > 0, \\ U\,(x,0) = U_0\,(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0)^T, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{k}{\rho_1}(u_x+\varphi)_x - \frac{\gamma}{\rho_1}\theta_x \\ \psi \\ \frac{b}{\rho_2}\varphi_{xx} - \frac{k}{\rho_2}(u_x+\varphi) - \frac{\gamma}{\rho_2}w_x + \frac{\gamma}{\rho_2}\theta \\ -\frac{\gamma}{\rho_3}v_x - \frac{\gamma}{\rho_3}\psi - \frac{k_1}{\rho_3}w_x \\ k_2 w_{xx} - k_3 w - k_1\theta_x - \gamma\psi_x \end{pmatrix}, \tag{15}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H}=H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2(0,1) \times H^1_*(0,1),$$

such that

$$\begin{aligned} H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ L^2_*(0,1) &= \left\{ \varphi \in L^2(0,1) : \int_0^1 \varphi(x)\,dx = 0 \right\}. \end{aligned}$$

For any $U = (u, v, \varphi, \psi, \theta, w)^T \in \mathcal{H}$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{w})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} \left\langle U, \tilde{U} \right\rangle_{\mathcal{H}} &= \rho_1 \int_0^1 v\tilde{v}dx + \rho_2 \int_0^1 \psi\tilde{\psi}dx + k\int_0^1 (u_x+\varphi)(\tilde{u}_x+\tilde{\varphi})\,dx \\ &+ b\int_0^1 \varphi_x\tilde{\varphi}_x dx + \rho_3\int_0^1 \theta\tilde{\theta}dx + \int_0^1 w\tilde{w}dx. \end{aligned} \tag{16}$$

The domain of $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) = \big\{ & U \in \mathcal{H} \mid u \in H^2_*(0,1) \cap H^1_*(0,1);\ v \in H^1_*(0,1); \\ & \varphi \in H^2(0,1) \cap H^1_0(0,1);\ \psi \in H^1_0(0,1);\ \theta \in H^1_0(\Omega); \\ & w \in H^2_*(0,1) \cap H^1_*(0,1) \big\}, \end{aligned}$$

where

$$H^2_*(0,1) = \left\{ \Psi \in H^2(0,1) : \Psi_x(0) = \Psi_x(1) = 0 \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$. So, from inner product (16), we have

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -k_3 \int_0^1 w^2 dx - k_2 \int_0^1 w_x^2 dx \leq 0, \tag{17}$$

from where it follows that the operator $\mathcal{A}$ is dissipative. Next, we prove that the operator $\mathcal{A}$ is surjective. So that for a given $K = (h_1, h_2, h_3, h_4, h_5, h_6)^T \in \mathcal{H}$, we prove that there exists a unique $U \in D(\mathcal{A})$ such that

$$(I - \mathcal{A})U = K. \tag{18}$$

Now, we can give the following well-posedness result.

Theorem 2.1. *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem* (8)-(10). *Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+,\ D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

3 Stability result

In this section, we use the multipliers method to estimate the energy of system (8)-(10). To achieve our goal we state and prove the following lemmas.

Lemma 3.1. *Let* (u, φ, θ, w) *be a solution of* (8)-(10). *Then, the energy functional* $E(t)$, *defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1 u_t^2 + \rho_2 \varphi_t^2 + \rho_3 \theta^2 + w^2 + b\varphi_x^2 + k(u_x + \varphi)^2\right) dx, \tag{19}$$

satisfies

$$E'(t) = -k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx \le 0. \tag{20}$$

Proof. Multiplying $(8)_1$, $(8)_2$, $(8)_3$, $(8)_4$ by u_t, φ_t, θ, w respectively, integrating by parts over $(0,1)$, using the boundary conditions and summing them up, we obtain (20). □

Lemma 3.2. *Let*

$$I_1(t) = -\rho_1 \int_0^1 u_t u dx, \ t > 0, \tag{21}$$

and let (u, φ, θ, w) *be a solution of* (8)-(10). *Then, we have*

$$\begin{aligned} I_1'(t) \le& -\rho_1 \int_0^1 u_t^2 dx + \frac{5k}{2}\int_0^1 (u_x+\varphi)^2 dx + 2k\int_0^1 \varphi_x^2 dx \\ &+ \frac{\gamma^2}{2k}\int_0^1 \theta^2 dx, \ t > 0. \end{aligned} \tag{22}$$

Proof. Direct computation, using equation $(8)_1$ and then integrating by parts, we find

$$\begin{aligned} I_1'(t) &= -\rho_1 \int_0^1 u_t^2 dx + k\int_0^1 (u_x+\varphi) u_x dx - \gamma \int_0^1 \theta u_x dx \\ &\le -\rho_1 \int_0^1 u_t^2 dx + \frac{k}{2}\int_0^1 (u_x+\varphi)^2 dx + k\int_0^1 u_x^2 dx \\ &+ \frac{\gamma^2}{2k}\int_0^1 \theta^2 dx. \end{aligned}$$

By using Young's, poincaré's inequalities and the fact that $\int_0^1 u_x^2 dx \le 2\int_0^1 (u_x+\varphi)^2 dx + 2\int_0^1 \varphi_x^2 dx$, we obtain (22). □

Next, we introduce the multiplier m given by the solution of the following Dirichlet problem

$$-m_{xx} = \varphi_x, \quad m(0) = m(1) = 0$$

Lemma 3.3. *The functional I_2 defined by*

$$I_2 = \rho_1 \int_0^1 u_t m dx + \rho_2 \int_0^1 \varphi_t \varphi dx, \quad t > 0, \tag{23}$$

satisfies, for any $\varepsilon_1 > 0$, the following estimate

$$I_2' \leq -\frac{b}{2} \int_0^1 \varphi_x^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + \left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_1}\right) \int_0^1 \varphi_t^2 dx + \frac{\gamma^2}{2b} \int_0^1 w_x^2 dx, \quad t > 0. \tag{24}$$

Proof. By differentiating the functional I_2, we get

$$I_2' = -b \int_0^1 \varphi_x^2 dx + \rho_2 \int_0^1 \varphi_t^2 dx + \rho_1 \int_0^1 u_t m_t dx - \gamma \int_0^1 w_x \varphi dx. \tag{25}$$

Using Young's and Poincaré's inqualities, we get

$$\rho_1 \int_0^1 u_t m_t dx \leq \varepsilon_1 \int_0^1 u_t^2 dx + \frac{\rho_1^2}{4\varepsilon_1} \int_0^1 \varphi_t^2 dx, \tag{26}$$

$$-\gamma \int_0^1 w_x \varphi dx \leq \frac{b}{2} \int_0^1 \varphi_x^2 dx + \frac{\gamma^2}{2b} \int_0^1 w_x^2 dx. \tag{27}$$

Substituting (26)-(27) into (25), we conclude (24). □

Lemma 3.4. *Assume that $\chi_2 = 0$, then the functional*

$$I_3(t) = \rho_2 \int_0^1 \varphi_t (u_x + \varphi) dx + \left(\rho_2 + \frac{\rho_2 \gamma^2}{k \rho_3}\right) \int_0^1 \varphi_x u_t dx - \frac{\rho_2 \gamma}{k} \int_0^1 \varphi_t \theta dx, \quad t > 0, \tag{28}$$

satisfies the following estimate

$$I_3'(t) \leq -\frac{k}{2} \int_0^1 (u_x + \varphi)^2 dx + \left(\frac{5\gamma^2}{4k} + \frac{k_1^2 \rho_2}{4\rho_3 k}\right) \int_0^1 w_x^2 dx + \frac{4\gamma^2}{k} \int_0^1 \theta^2 dx + \left(\rho_2 + \frac{2\rho_2 \gamma^2}{\rho_3 k}\right) \int_0^1 \varphi_t^2 dx - \frac{1}{k}\chi_2 \int_0^1 \varphi_x u_{tt} dx, \quad t > 0. \tag{29}$$

Proof. By differentiating the functional $I_3(t)$ with respect to t , integrating by parts over $(0,1)$. Then, using $(8)_1$, $(8)_2$ and $(8)_3$, we obtain

$$
\begin{aligned}
I_3'(t) &= -b\int_0^1 \varphi_x (u_x+\varphi)_x\,dx - k\int_0^1 (u_x+\varphi)^2\,dx - \gamma\int_0^1 w_x (u_x+\varphi)\,dx \\
&+ \rho_2\int_0^1 \varphi_t^2 dx + \gamma\int_0^1 \theta (u_x+\varphi)\,dx + \rho_2\int_0^1 \varphi_t u_{xt} dx - \frac{\gamma^2}{k}\int_0^1 \theta^2 dx \\
&+ \left(\rho_2 + \frac{\rho_2\gamma^2}{k\rho_3}\right)\int_0^1 \varphi_x u_{tt} dx + \left(\rho_2 + \frac{\rho_2\gamma^2}{k\rho_3}\right)\int_0^1 \varphi_{tx} u_t dx \\
&- \frac{\gamma b}{k}\int_0^1 \varphi_{xx}\theta dx + \gamma\int_0^1 (u_x+\varphi)\,\theta dx + \frac{\gamma^2}{k}\int_0^1 w_x\theta dx \\
&+ \frac{k_1\rho_2\gamma}{\rho_3 k}\int_0^1 \varphi_t w_x dx + \frac{\rho_2\gamma^2}{\rho_3 k}\int_0^1 \varphi_t u_{xt} dx + \frac{\rho_2\gamma^2}{\rho_3 k}\int_0^1 \varphi_t^2 dx,
\end{aligned}
$$

using equation $(8)_1$, we arrive at

$$
\begin{aligned}
I_3'(t) &= -\frac{b\rho_1}{k}\int_0^1 \varphi_x u_{tt} dx - \frac{b\gamma}{k}\int_0^1 \varphi_x\theta_x dx - k\int_0^1 (u_x+\varphi)^2\,dx \\
&+ \rho_2\int_0^1 \varphi_t^2 dx - \gamma\int_0^1 w_x (u_x+\varphi)\,dx + \gamma\int_0^1 \theta (u_x+\varphi)\,dx \\
&+ \rho_2\int_0^1 \varphi_t u_{xt} dx + \frac{\rho_2\gamma^2}{\rho_3 k}\int_0^1 \varphi_t^2 dx + \left(\rho_2 + \frac{\rho_2\gamma^2}{k\rho_3}\right)\int_0^1 \varphi_{tx} u_t dx \\
&+ \left(\rho_2 + \frac{\rho_2\gamma^2}{k\rho_3}\right)\int_0^1 \varphi_x u_{tt} dx + \frac{\gamma^2}{k}\int_0^1 w_x\theta dx - \frac{\gamma b}{k}\int_0^1 \varphi_{xx}\theta dx \\
&+ \gamma\int_0^1 (u_x+\varphi)\,\theta dx - \frac{\gamma^2}{k}\int_0^1 \theta^2 dx + \frac{k_1\rho_2\gamma}{\rho_3 k}\int_0^1 \varphi_t w_x dx \\
&+ \frac{\rho_2\gamma^2}{\rho_3 k}\int_0^1 \varphi_t u_{xt} dx.
\end{aligned}
\tag{30}
$$

The relation (30) is reduced to

$$
\begin{aligned}
I_3'(t) &= -\frac{1}{k}\left(b\rho_1 - k\rho_2 - \frac{\rho_2\gamma^2}{\rho_3}\right)\int_0^1 \varphi_x u_{tt} dx - k\int_0^1 (u_x+\varphi)^2\,dx \\
&- \gamma\int_0^1 w_x (u_x+\varphi)\,dx + 2\gamma\int_0^1 \theta (u_x+\varphi)\,dx + \frac{\gamma^2}{k}\int_0^1 w_x\theta dx \\
&+ \left(\rho_2 + \frac{\rho_2\gamma^2}{\rho_3 k}\right)\int_0^1 \varphi_t^2 dx - \frac{\gamma^2}{k}\int_0^1 \theta^2 dx + \frac{k_1\rho_2\gamma}{\rho_3 k}\int_0^1 \varphi_t w_x dx.
\end{aligned}
\tag{31}
$$

Now, by using Young's inequality, we obtain

$$-\gamma\int_0^1 w_x\left(u_x+\varphi\right)dx\le\frac{k}{4}\int_0^1\left(u_x+\varphi\right)^2dx+\frac{\gamma^2}{k}\int_0^1 w_x^2dx,$$
$$2\gamma\int_0^1\theta\left(u_x+\varphi\right)dx\le\frac{k}{4}\int_0^1\left(u_x+\varphi\right)^2dx+\frac{4\gamma^2}{k}\int_0^1\theta^2dx,$$
$$\frac{\gamma^2}{k}\int_0^1 w_x\theta dx\le\frac{\gamma^2}{4k}\int_0^1 w_x^2dx+\frac{\gamma^2}{k}\int_0^1\theta^2dx,$$
$$\frac{k_1\rho_2\gamma}{\rho_3 k}\int_0^1\varphi_t w_x dx\le\frac{k_1^2\rho_2}{4\rho_3 k}\int_0^1 w_x^2dx+\frac{\rho_2\gamma^2}{\rho_3 k}\int_0^1\varphi_t^2dx.$$

By substituting the previous inequalities into (31), the result follows. □

Lemma 3.5. *Let*

$$I_4(t)=\rho_3\int_0^1\left(\int_0^x w\left(y\right)dy\right)\theta dx,\ t>0, \tag{32}$$

satisfies, for any $\varepsilon_2>0$, the following estimate

$$\begin{aligned}I_4'(t)\le&-\frac{k_1\rho_3}{2}\int_0^1\theta^2dx+\varepsilon_2\int_0^1 u_t^2dx+\varepsilon_2\int_0^1\varphi_t^2dx+\frac{k_2^2\rho_3}{k_1}\int_0^1 w_x^2dx\\&+\left(k_1+\frac{k_3^2\rho_3}{k_1}+\frac{\gamma^2}{2\varepsilon_2}\right)\int_0^1 w^2dx-\gamma\rho_3\int_0^1\varphi_t\theta dx,\ t>0.\end{aligned} \tag{33}$$

Proof. By differentiating $I_4(t)$ with the use of $(8)_3$ and $(8)_4$, we obtain

$$\begin{aligned}I_4'(t)=&\rho_3\int_0^1\left(\int_0^x w_t\left(y\right)dy\right)\theta dx+\rho_3\int_0^1\left(\int_0^x w\left(y\right)dy\right)\theta_t dx\\=&k_2\rho_3\int_0^1\left(\int_0^x w_{xx}\left(y\right)dy\right)\theta dx-k_3\rho_3\int_0^1\left(\int_0^x w\left(y\right)dy\right)\theta dx\\&-k_1\rho_3\int_0^1\left(\int_0^x\theta_x\left(y\right)dy\right)\theta dx-\gamma\rho_3\int_0^1\left(\int_0^x\varphi_{tx}\left(y\right)dy\right)\theta dx\\&-k_1\int_0^1\left(\int_0^x w\left(y\right)dy\right)w_x dx-\gamma\int_0^1\left(\int_0^x w\left(y\right)dy\right)u_{xt}dx\\&-\gamma\int_0^1\left(\int_0^x w\left(y\right)dy\right)\varphi_t dx,\end{aligned}$$

by integrating by parts and using the fact that $\int_0^1 w\left(x\right)dx=0$, we arrive at

$$\begin{aligned}I_4'\left(t\right)=&-k_1\rho_3\int_0^1\theta^2dx-k_3\rho_3\int_0^1\left(\int_0^x w(y)dy\right)\theta dx\\&-\gamma\int_0^1\left(\int_0^x w\left(y\right)dy\right)\varphi_t dx+k_2\rho_3\int_0^1 w_x\theta dx\\&-\gamma\rho_3\int_0^1\varphi_t\theta dx+k_1\int_0^1 w^2dx+\gamma\int_0^1 wu_t dx.\end{aligned} \tag{34}$$

By using Young's inequality, we obtain

$$k_2\rho_3\int_0^1 w_x\theta dx \leq \frac{k_2^2\rho_3}{k_1}\int_0^1 w_x^2 dx + \frac{k_1\rho_3}{4}\int_0^1 \theta^2 dx, \tag{35}$$

$$-k_3\rho_3\int_0^1\left(\int_0^x w dy\right)\theta dx \leq \frac{k_3^2\rho_3}{k_1}\int_0^1 w^2 dx + \frac{k_1\rho_3}{4}\int_0^1 \theta^2 dx, \tag{36}$$

$$\gamma\int_0^1 w u_t dx \leq \varepsilon_2\int_0^1 u_t^2 dx + \frac{\gamma^2}{4\varepsilon_2}\int_0^1 w^2 dx, \tag{37}$$

$$-\gamma\int_0^1\left(\int_0^x w dy\right)\varphi_t dx \leq \varepsilon_2\int_0^1 \varphi_t^2 dx + \frac{\gamma^2}{4\varepsilon_2}\int_0^1 w^2 dx. \tag{38}$$

Estimate (33) follows by substituting (35), (36), (37) and (38) into (34).

Lemma 3.6. *Let (u,φ,θ,w) be a solution of (8)-(10). Then the functional*

$$\begin{aligned} I_5(t) = {} & \rho_2\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_t dx + \rho_2 k_1\int_0^1 \theta\varphi + \frac{\gamma\rho_2 k_1}{2\rho_3}\int_0^1 \varphi^2 dx \\ & - \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u\varphi_x dx, \ \ t>0, \end{aligned} \tag{39}$$

satisfies, for any $\varepsilon_3>0$, the following estimate

$$\begin{aligned} I_5'(t) \leq {} & -\frac{\gamma\rho_2}{2}\int_0^1 \varphi_t^2 dx + \varepsilon_3\int_0^1 (u_x+\varphi)^2 dx + \varepsilon_3\int_0^1 \varphi_x^2 dx \\ & + \varepsilon_3\int_0^1 \theta^2 dx + \left(\frac{\rho_2 k_3^2}{\gamma} + \frac{b^2}{2\varepsilon_3} + \frac{k^2}{4\varepsilon_3} + \frac{\gamma^2}{4\varepsilon_3} + \gamma\right)\int_0^1 w^2 dx \\ & + \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u_x\varphi_t dx + \left(\frac{\rho_2 k_2^2}{\gamma} + \frac{\rho_2^2 k_1^4}{2\rho_3^2\varepsilon_3}\right)\int_0^1 w_x^2 dx, \quad t>0. \end{aligned} \tag{40}$$

Proof. By differentiating $I_5(t)$, we find

$$\begin{aligned} I_5'(t) = {} & b\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_{xx} dx - k\int_0^1\left(\int_0^x w(y)\,dy\right)(u_x+\varphi)\,dx \\ & - \gamma\int_0^1\left(\int_0^x w(y)\,dy\right)w_x dx + \gamma\int_0^1\left(\int_0^x w(y)\,dy\right)\theta dx \\ & + k_2\rho_2\int_0^1 w_x\varphi_t dx - k_3\rho_2\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_t dx - k_1\rho_2\int_0^1 \theta\varphi_t dx \\ & - \gamma\rho_2\int_0^1 \varphi_t^2 dx - \frac{\rho_2 k_1^2}{\rho_3}\int_0^1 w_x\varphi dx - \frac{\rho_2 k_1\gamma}{\rho_3}\int_0^1 u_{xt}\varphi dx \\ & - \frac{\rho_2 k_1\gamma}{\rho_3}\int_0^1 \varphi_t\varphi dx + \rho_2 k_1\int_0^1 \theta\varphi_t dx + \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 \varphi_t\varphi dx \\ & - \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u_t\varphi_x dx - \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u\varphi_{tx} dx. \end{aligned}$$

Thus

$$
\begin{aligned}
I_5'(t) &= b\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_{xx}dx - k\int_0^1\left(\int_0^x w(y)\,dy\right)(u_x+\varphi)\,dx \\
&\quad -\gamma\int_0^1\left(\int_0^x w(y)\,dy\right)w_x dx + \gamma\int_0^1\left(\int_0^x w(y)\,dy\right)\theta dx \\
&\quad + k_2\rho_2\int_0^1 w_x\varphi_t dx - k_3\rho_2\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_t dx \\
&\quad -\gamma\rho_2\int_0^1\varphi_t^2 dx - \frac{\rho_2 k_1^2}{\rho_3}\int_0^1 w_x\varphi dx - \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u\varphi_{tx}dx.
\end{aligned}
$$

Now, by integrating by parts and using again the fact that $\int_0^1 w(x)\,dx = 0$, we arrive at

$$
\begin{aligned}
I_5'(t) &= -\gamma\rho_2\int_0^1\varphi_t^2 dx + \rho_2 k_2\int_0^1 w_x\varphi_t dx - b\int_0^1 w\varphi_x dx \\
&\quad - k_3\rho_2\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_t dx - k\int_0^1\left(\int_0^x w(y)\,dy\right)(u_x+\varphi)\,dx \\
&\quad +\gamma\int_0^1 w^2 dx + \gamma\int_0^1\left(\int_0^x w(y)\,dy\right)\theta dx - \frac{\rho_2 k_1^2}{\rho_3}\int_0^1 w_x\varphi dx \\
&\quad + \frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u_x\varphi_t dx.
\end{aligned} \tag{41}
$$

By using Young's, Cauchy Swarchaz's and Poincaré's inequalities, we arrive at

$$
\rho_2 k_2\int_0^1 w_x\varphi_t dx \le \frac{\gamma\rho_2}{4}\int_0^1\varphi_t^2 dx + \frac{\rho_2 k_2^2}{\gamma}\int_0^1 w_x^2 dx, \tag{42}
$$

$$
-k_3\rho_2\int_0^1\left(\int_0^x w dy\right)\varphi_t dx \le \frac{\gamma\rho_2}{4}\int_0^1\varphi_t^2 dx + \frac{\rho_2 k_3^2}{\gamma}\int_0^1 w^2 dx, \tag{43}
$$

$$
-b\int_0^1 w\varphi_x dx \le \frac{\varepsilon_3}{2}\int_0^1\varphi_x^2 dx + \frac{b^2}{2\varepsilon_3}\int_0^1 w^2 dx, \tag{44}
$$

$$
-k\int_0^1\left(\int_0^x w(y)\,dy\right)(u_x+\varphi)\,dx \le \varepsilon_3\int_0^1(u_x+\varphi)^2 dx + \frac{k^2}{4\varepsilon_3}\int_0^1 w^2 dx, \tag{45}
$$

$$
\gamma\int_0^1\left(\int_0^x w(y)\,dy\right)\theta dx \le \varepsilon_3\int_0^1\theta^2 dx + \frac{\gamma^2}{4\varepsilon_3}\int_0^1 w^2 dx, \tag{46}
$$

$$
-\frac{\rho_2 k_1^2}{\rho_3}\int_0^1 w_x\varphi dx \le \frac{\varepsilon_3}{2}\int_0^1\varphi_x^2 dx + \frac{\rho_2^2 k_1^4}{2\rho_3^2\varepsilon_3}\int_0^1 w_x^2 dx, \tag{47}
$$

which yeilds the desired result by inserting the relations (42)-(47) into (41). □

In order to deal with the annoying terms $\frac{\gamma\rho_2 k_1}{\rho_3}\int_0^1 u_x\varphi_t dx$ and $-\gamma\rho_3\int_0^1\theta\varphi_t dx$, we prove the following lemma.

Lemma 3.7. *Let (u,φ,θ,w) be a solution of (8)-(10). Then the functional*

$$
I_6(t) = \frac{\rho_2}{2}\int_0^1\varphi_t^2 dx + \frac{b}{2}\int_0^1\varphi_x^2 dx + \frac{k}{2}\int_0^1\varphi^2 dx,\ t>0, \tag{48}
$$

satisfies, for any $\varepsilon_3 > 0$*, the following estimate*

$$I_6'(t) \leq \varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\gamma^2}{4\varepsilon_3} \int_0^1 w_x^2 dx + \gamma \int_0^1 \theta \varphi_t dx - k \int_0^1 u_x \varphi_t dx, \ t > 0. \tag{49}$$

Proof. A differentiation of $I_6(t)$ and an inegration by parts over $(0,1)$, we get

$$\begin{aligned} I_6'(t) = &-b \int_0^1 \varphi_x \varphi_{tx} dx - k \int_0^1 (u_x + \varphi) \varphi_t dx - \gamma \int_0^1 w_x \varphi_t dx \\ &+ \gamma \int_0^1 \theta \varphi_t dx + b \int_0^1 \varphi_{tx} \varphi_x dx + k \int_0^1 \varphi_t \varphi dx, \end{aligned}$$

thus,

$$I_6'(t) = -k \int_0^1 u_x \varphi_t dx - \gamma \int_0^1 w_x \varphi_t dx + \gamma \int_0^1 \theta \varphi_t dx.$$

Then (49) easily follows owing to Young's inequality. □

We are now ready to state and prove the following exponential stability result. We define for N, N_1, N_2 and $N_3 > 0$ the following functional

$$\begin{aligned} \mathcal{L}(t) = &NE(t) + I_1(t) + N_1 I_2(t) + N_2 I_3(t) + \tfrac{1}{\rho_3} N_3 I_4(t) \\ &+ N_3 \tfrac{k\rho_3}{\gamma \rho_2 k_1} I_5(t) + N_3 I_6(t), \quad t \geq 0. \end{aligned} \tag{50}$$

We can simply establish that the functional energy E is equivalent to the functional $\mathcal{L}$ using the Young, Poincaré, and Cauchy-Schwarz inequalities, that is, for two positive constants κ_1 and κ_2,

$$\kappa_1 E(t) \leq \mathcal{L}(t) \leq \kappa_2 E(t), \quad \forall t \geq 0. \tag{51}$$

Theorem 3.1. *Let* $(u, \varphi, \ \theta, w)$ *be a solution of the problem determined by system* (8)*, initial conditions* (9) *and boundary conditions* (10) *and that the coefficients of the system satisfy the condition* $\chi_2 = 0$*. Then,* $(u, \varphi, \ \theta, w)$ *decays exponentially, i.e., there exist two positive constants* λ_1*,* λ_2 *such that*

$$E(t) \leq \lambda_1 E(0) e^{-\lambda_2 t}, \quad \forall t \geq 0. \tag{52}$$

Proof. By differentiating equation (50), then recalling equations (20), (22), (24), (29), (33), (40) and (49), we get

$$\begin{aligned} \mathcal{L}'(t) \leq &- C_w \int_0^1 w^2 dx - C_{w_x} \int_0^1 w_x^2 dx - C_{\varphi_t} \int_0^1 \varphi_t^2 dx - C_{(u_x+\varphi)} \int_0^1 (u_x + \varphi)^2 dx \\ &- C_{\varphi_x} \int_0^1 \varphi_x^2 dx - C_{u_t} \int_0^1 u_t^2 dx - C_\theta \int_0^1 \theta^2 dx, \quad \forall t \geq 0, \end{aligned} \tag{53}$$

where

$$\begin{aligned}
&C_{w_x} = Nk_2 - N_1\tfrac{\gamma^2}{2b} - N_2\left(\tfrac{5\gamma^2}{4k} + \tfrac{k_1^2\rho_2}{4\rho_3 k}\right) - N_3\tfrac{k_2^2}{k_1} - N_3\left(\tfrac{kk_2^2\rho_3}{\gamma^2 k_1} + \tfrac{\rho_2 kk_1^3}{2\gamma\rho_3\varepsilon_3}\right) - N_3\tfrac{\gamma^2}{4\varepsilon_3},\\
&C_w = Nk_3 - N_3\left(\tfrac{k_1}{\rho_3} + \tfrac{k_3^2}{k_1} + \tfrac{\gamma^2}{2\rho_3\varepsilon_2}\right) - \tfrac{N_3\rho_3 k}{\rho_2\gamma k_1}\left(\tfrac{\rho_2 k_3^2+\gamma^2}{\gamma} + \tfrac{2b^2+k^2+\gamma^2}{4\varepsilon_3}\right),\\
&C_{\varphi_t} = N_3\tfrac{k\rho_3}{2k_1} - N_3\tfrac{\varepsilon_2}{\rho_3} - N_3\varepsilon_3 - N_1\left(\rho_2 + \tfrac{\rho_1^2}{4\varepsilon_1}\right) - N_2\left(\rho_2 + \tfrac{2\rho_2\gamma^2}{\rho_3 k}\right),\\
&C_\theta = N_3\tfrac{k_1}{2} - \tfrac{\gamma^2}{2k} - N_2\tfrac{4\gamma^2}{k} - N_3\tfrac{k\rho_3\varepsilon_3}{\gamma\rho_2 k_1},\\
&C_{\varphi_x} = N_1\tfrac{b}{2} - N_3\tfrac{k\rho_3\varepsilon_3}{\gamma\rho_2 k_1} - 2k,\\
&C_{u_t} = \rho_1 - N_1\varepsilon_1 - \tfrac{N_3}{\rho_3}\varepsilon_2,\\
&C_{(u_x+\varphi)} = N_2\tfrac{k}{2} - N_3\tfrac{k\rho_3\varepsilon_3}{\gamma\rho_2 k_1} - \tfrac{5k}{2}.
\end{aligned} \tag{54}$$

Now, by setting $\varepsilon_2 = \frac{\rho_1\rho_3}{2N_3}$ and $\varepsilon_3 = \frac{1}{N_3}$, we obtain

$$\begin{aligned}
&C_{w_x} = Nk_2 - N_1\tfrac{\gamma^2}{2b} - N_2\left(\tfrac{5\gamma^2}{4k} + \tfrac{k_1^2\rho_2}{4\rho_3 k}\right) - N_3\tfrac{k_2^2}{k_1} - N_3\left(\tfrac{kk_2^2\rho_3}{\gamma^2 k_1} + \tfrac{N_3\rho_2 kk_1^3}{2\gamma\rho_3}\right) - N_3^2\tfrac{\gamma^2}{4},\\
&C_w = Nk_3 - N_3\left(\tfrac{k_1}{\rho_3} + \tfrac{k_3^2}{k_1} + \tfrac{N_3\gamma^2}{\rho_3^2\rho_1}\right) - \tfrac{N_3^2\rho_3 k}{\rho_2\gamma k_1}\left(\tfrac{\rho_2 k_3^2+\gamma^2}{\gamma N_3} + \tfrac{2b^2+k^2+\gamma^2}{4}\right),\\
&C_{\varphi_t} = N_3\tfrac{k\rho_3}{2k_1} - N_1\left(\rho_2 + \tfrac{\rho_1^2}{4\varepsilon_1}\right) - N_2\left(\rho_2 + \tfrac{2\rho_2\gamma^2}{\rho_3 k}\right) - \tfrac{\rho_1}{2} - 1,\\
&C_\theta = N_3\tfrac{k_1}{2} - N_2\tfrac{4\gamma^2}{k} - \tfrac{k\rho_3}{\gamma\rho_2 k_1} - \tfrac{\gamma^2}{2k},\\
&C_{\varphi_x} = N_1\tfrac{b}{2} - \tfrac{k\rho_3}{\gamma\rho_2 k_1} - 2k,\\
&C_{u_t} = \tfrac{\rho_1}{2} - N_1\varepsilon_1,\\
&C_{(u_x+\varphi)} = N_2\tfrac{k}{2} - \tfrac{k\rho_3}{\gamma\rho_2 k_1} - \tfrac{5k}{2}.
\end{aligned}$$

Now, all these terms on the right-hand side of (53) become negative if we select our parameters appropriately. First, we choose N_1 large enough so that

$$N_1\frac{b}{2} - \frac{k\rho_3}{\gamma\rho_2 k_1} - 2k > 0,$$

and N_2 large enough such that

$$N_2\frac{k}{2} - \frac{k\rho_3}{\gamma\rho_2 k_1} - \frac{5k}{2} > 0.$$

Now, we pick ε_1 so small so that

$$\varepsilon_1 < \frac{\rho_1}{2N_1}.$$

Next, we select N_3 large enough so that

$$N_3\frac{k\rho_3}{2k_1} - N_1\left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_1}\right) - N_2\left(\rho_2 + \frac{2\rho_2\gamma^2}{\rho_3 k}\right) - \frac{\rho_1}{2} > 1$$

and

$$N_3\frac{k_1}{2} - N_2\frac{4\gamma^2}{k} - \frac{k\rho_3}{\gamma\rho_2 k_1} - \frac{\gamma^2}{2k} > 0.$$

Finally, we choose N large enough such that

$$Nk_2 - N_1\frac{\gamma^2}{2b} - N_2\left(\frac{5\gamma^2}{4k} + \frac{k_1^2\rho_2}{4\rho_3 k}\right) - N_3\frac{k_2^2}{k_1} - N_3\left(\frac{kk_2^2\rho_3}{\gamma^2 k_1} + \frac{N_3\rho_2 kk_1^3}{2\gamma\rho_3}\right) - N_3^2\frac{\gamma^2}{4} > 0,$$

$$Nk_3 - N_3\left(\frac{k_1}{\rho_3} + \frac{k_3^2}{k_1} + \frac{N_3\gamma^2}{\rho_3^2\rho_1}\right) - \frac{N_3^2\rho_3 k}{\rho_2\gamma k_1}\left(\frac{\rho_2 k_3^2+\gamma^2}{\gamma N_3} + \frac{2b^2+k^2+\gamma^2}{4}\right) > 0.$$

So, we arrive at

$$\begin{aligned}\mathcal{L}'(t) &\leq -\beta_1 \int_0^1 \left(u_t^2 + \varphi_t^2 + \theta^2 + w^2 + (u_x + \varphi)^2 + \varphi_x^2\right) dx \\ &\leq -\beta_2 E(t),\end{aligned}$$

for some positive constants β_1 and β_2. Having in mind the remark on the equivalence of $E(t)$ and $\mathcal{L}(t)$, we infer that

$$\mathcal{L}'(t) \leq -d_1 \mathcal{L}(t),\ t \geq 0, \tag{55}$$

where $d_1 = \frac{\beta_2}{\kappa_1} > 0$. A simple integration of (55) gives

$$\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-d_1 t},\ t \geq 0,$$

which yields the desired result (52) by using the other side of the equivalence relation again. □

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