EURASIAN JOURNAL OF MATHEMATICAL
AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 12, Issue 1 (2024) 83 – 93

RECOVERY OF THE NON-STATIONARY COMPONENT OF THE RIGHT-HAND SIDE OF THE SUBDIFFUSION EQUATION

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Abstract The paper proposes a non-iteration method for numerically solving the inverse problem of identifying the coefficient of the right-hand side of the time-dependent fractional diffusion equation. The redefinition condition for each $t \in (0, T]$ is given: the value of the function at some internal point of the domain or the integral of the desired function in the domain. The results of the numerical implementation of the proposed method on test problems with exact solutions are presented.

Key words: fractional time derivative, subdiffusion equation, inverse problem, finite difference method, decomposition method.

AMS Mathematics Subject Classification: 58J35, 65M32.

DOI: 10.32523/2306-6172-2024-12-1-83-93

1 Introduction

In some cases, diffusion as a physical process is more accurately described by a partial differential equation of fractional order. Theoretical results and numerical methods for solving inverse initial-boundary value problems for differential equations are generalized in the monographs of Isakov, Prilepko and Cannon [1] – [3]. The study of inverse problems for differential equations with fractional derivatives is in rapid development, both in theoretical terms and in their applications, and has become a tool for mathematical modeling of complex dynamic processes in various (ordinary and fractal) environments, allowing to solve various problems of analysis and synthesis, diagnostics, and the creation of new systems. The applied value of coefficient inverse problems in this process is very significant, in fact, they represent an extensive class of inverse problems.

Among the possible formulations of inverse problems with unknown coefficients in a parabolic equation, a typical one is the problem [4], in which it is necessary to find the unknown functions $u(x, t)$, and as a special inverse problem, one can single out the problem where it is necessary to find $p(t)$ in the source function $f(x, t) = p(t)\phi(x)$. Of interest here is the unknown time dependence of the source coefficient (right-hand side) with a known spatial distribution.

An additional condition is specified as a function of time at an internal point or as an integral average over the entire region. A fundamentally new method for finding the coefficient was developed in [5] for a parabolic equation. The development of a numerical algorithm for finding the lowest order coefficient in a parabolic equation is often based on the idea of transforming the equation by introducing new unknowns and moving to a linear inverse problem. This article discusses this problem and this approach as applied to anomalous diffusion. In the works [6] – [7] another direct non-iteration method for determining the lowest coefficient was developed, which is implemented at each time layer based on solving two standard grid elliptic problems. A generalization of this approach and a numerical implementation of the computational algorithm were carried out in the article [8], and the inverse problem of identifying the time-dependent source coefficient was solved.

In [9], the work proposed a new analytical algorithm for highly nonlinear reaction-diffusion equations with fractional time division. The proposed method is a combination of the Variational Iteration Method (VIM) and the Adomian Decomposition Method (ADM) and is further improved by introducing a new correction functional. The work [10] proposed discretization methods: direct recursive discretization of the Tustin operator and a direct discretization method using the Al-Alawi operator via continued fraction extension (CFE). Detailed sampling procedures are given.

The paper [11] constructed an effective numerical method for solving the inverse problem for a parabolic equation with a fractional time derivative. An implicit finite difference method is used to discretize the problem, and a conjugate gradient method is proposed for the inverse problem.

In [12], the authors solved a nonlinear inverse problem to determine the time-dependent convection coefficient in the subdiffusion equation from internal point measurements for the one-dimensional case. The existence, uniqueness, and regularity of the solution to the direct problem are proved using the fixed point theorem.

The paper [13] considered two inverse problems of recovering the coefficients of the nonstationary one-dimensional diffusion-convection equation.

The work [14] considered the diffusion equation with a fractional time derivative in a finite region. A computationally efficient implicit difference approximation is proposed for solving the time fractional diffusion equation. In this paper, we present a numerical solution to the inverse problem of determining the coefficient on the right side of the subdiffusion equation under given additional conditions. It is assumed that the desired factor of the coefficient is represented as a function that depends only on time, and the other factor is a known function that depends on the spatial variable. The spatial approximation is constructed by the finite difference method. The main features of the linear inverse problem under consideration are taken into account when choosing linearized approximations in time. Linear problems at the appropriate time level are solved by applying a special decomposition and solving two standard elliptic problems.

2 Problem statement

Let us consider the inverse initial-boundary value problem for a linear one-dimensional subdiffusion equation with a fractional time derivative of order $\alpha \in (0, 1)$, satisfying the given homogeneous Dirichlet boundary conditions and having a source function that is the product of two functions, one of which depends on time , and the other from the spatial variable. Let the value of the desired function at a given internal point of the region be specified as an additional condition.

Thus, it is required to determine the functions $u(x,t)$ and $p(t)$ from the conditions:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = \dfrac{\partial^2 u}{\partial x^2} + p(t)f(x), \qquad 0 < x < l, \qquad 0 < t \leq T, \\ u(0,t) = u(l,t) = 0, \qquad 0 < t \leq T, \\ u(x,0) = \varphi(x), \qquad 0 \leq x \leq l, \\ u(x^*,t) = \phi(t), \;\; x^* \in (0,l). \end{cases} \tag{1}$$

Here the functions $f(x),\; \varphi(x),\; 0 \leq x \leq l$ and $\phi(t),\; 0 < t \leq T$ are given.

The Caputo fractional derivative of order α is determined by the formula:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\partial u(x,s) ds}{\partial t}(t-s)^{-\alpha}, \tag{2}$$

where $\alpha \in (0,1),\; \Gamma(\cdot)$ is the gamma function.

3 Algorithm development and finite-difference analog of the problem

We use the idea of P.N. Vabishchevich proposed in the work [5]. Let $p(t)$ be differentiable and integrable, and we can represent this function as $p(t) = D_t^\alpha(\theta(t))$, where $\theta(t)$ is an auxiliary function. Let the function $f(x)$ be twice differentiable and $f(x^*) \neq 0$ at a given point, $f(x) = 0$ on the boundary of the domain. We find a solution to problem (1) in the form of

$$u(x,t) = \theta(t)f(x) + \omega(x,t). \tag{3}$$

Then we obtain the following formulation of the problem for the functions $\theta(t), \omega(x,t)$:

$$\begin{cases} D_t^\alpha(\omega(x,t)) = \theta(t)\dfrac{\partial^2 f(x)}{\partial x^2} + \dfrac{\partial^2 \omega(x,t)}{\partial x^2}, \qquad 0 < x < l, \qquad 0 < t \leq T, \\ \omega(0,t) = \omega(l,t) = 0, \qquad 0 < t \leq T, \\ \omega(x,0) = \varphi(x), \qquad 0 \leq x \leq l, \\ \omega(x^*,t) = \phi(t) - \theta(t)f(x^*), \quad 0 < t \leq T. \end{cases} \tag{4}$$

Let's move on to constructing a difference analogue of the initial-boundary value problem (4). To approximate the fractional Caputo derivative of order α in time on a uniform time grid with a step $\tau = T/M$, we use the discrete analogue of P. Zhuang and F. Liu [14]:

$$\frac{\partial^\alpha u(x,t_j)}{\partial t^\alpha} = \sigma_{\tau\alpha} \sum_{k=1}^{j} s_k(u_i^{j-k+1} - u_i^{j-k}), \quad j = \overline{1,M}, \quad i = \overline{0,N},$$

where

$$s_1 = 1, \quad s_k = k^{1-\alpha} - (k-1)^{1-\alpha}, \quad k = 2, 3, \ldots, j. \quad \sigma_{\tau\alpha} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}.$$

The same work shows that the error in the approximation of the fractional derivative is of the order of $O(\tau^{2-\alpha})$. Let's write down the sum for selecting layers:

$$D_t^{\alpha}(\omega(x,t_j)) = \sigma_{\tau\alpha}\sum_{k=1}^{j} s_k(\omega_i^{j-k+1} - \omega_i^{j-k}) = \sigma_{\tau\alpha}(\omega_i^j - \omega_i^{j-1} + \sum_{k=2}^{j} s_k(\omega_i^{j-k+1} - \omega_i^{j-k})).$$

Let us introduce the notation Φ^{j-1} for the lower layers:

$$\Phi^{j-1} = -\omega_i^{j-1} + \sum_{k=2}^{j} s_k(\omega_i^{j-k+1} - \omega_i^{j-k}).$$

We write the approximation of the second derivative in the form:

$$\frac{\partial^2 f(x)}{\partial x^2} + \frac{\partial^2 \omega(x,t)}{\partial x^2} = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + \frac{\omega_{i-1}^j - 2\omega_i^j + \omega_{i+1}^j}{h^2} = \Lambda f + \Lambda\omega^j,$$

Taking into account the introduced notations, the main equation of problem (4) will take the form:

$$\sigma_{\tau\alpha}(\omega_i^j + \Phi^{j-1}) = \theta^j \Lambda f + \Lambda\omega^j. \tag{5}$$

Using relation (3), we derive the function $\theta(t)$:

$$\theta(t) = \frac{u(x,t) - \omega(x,t)}{f(x)} = \frac{u(x^*,t) - \omega(x^*,t)}{f(x^*)} = \frac{\phi(t) - \omega(x^*,t)}{f(x^*)}, \tag{6}$$

Let us substitute the resulting expression for $\theta(t)$ into equation (5) and $x^* = x_m$:

$$\sigma_{\tau\alpha}\omega_i^j - \Lambda\omega^j + \frac{\omega_m^j \Lambda f}{f_m} = \frac{\phi^j}{f_m}\Lambda f - \sigma_{\tau\alpha}\Phi^{j-1}.$$

To exclude ω_k^j, P.N. Vabishchevich proposed to use the method of decomposition of the grid function ω_j^j in the form $\omega_i^j = y_i^j + \omega_m z_i^j$

$$\sigma_{\tau\alpha}y_i^j + \sigma_{\tau\alpha}\omega_m^j z - \Lambda y^j - \omega_m^j \Lambda z^j + \frac{\omega_m^j \Lambda f}{f_m} = \frac{\phi^j}{f_m}\Lambda f - \sigma_{\tau\alpha}\Phi^{j-1}.$$

To make the following expressions easier to read, we denote the right side of the last relation by g^{j-1}

$$(\sigma_{\tau\alpha}y_i^j - \Lambda y^j - g^{j-1}) + \omega_m^j(\sigma_{\tau\alpha}z_i^j - \Lambda z^j + \Lambda f/f_m) = 0,$$

since a sufficient condition for the sum of terms to be equal to zero is the equality of the terms in brackets to zero, we have a system of linear equations to find y_i^j, z_i^j:

$$\begin{cases} \sigma_{\tau\alpha}y_i^j - \Lambda y^j = g^{j-1}, \\ \sigma_{\tau\alpha}z_i^j - \Lambda z^j = -\Lambda f/f_m. \end{cases} \tag{6}$$

Thus, additional grid functions y_i and z_i are calculated from system (6). We define ω_m^j and ω_i^j:

$$\omega_m^j = \frac{y_m^j}{1 - z_m^j}, \quad \omega_i^j = y_i^j + \omega_m z_i^j.$$

Relation (3) will allow us to find $\theta(t)$ and $u(x,t)$. Using the assumption $p(t) = D_t^\alpha(\theta(t))$, we find the required function $p(t)$.

To demonstrate the flexibility and universality of the decomposition method, we present one more replacement:

$$\omega_i^j = y_i^j + \theta(t) z_i^j, \tag{7}$$

which we substitute into (5) and get:

$$\sigma_{\tau\alpha} y_i^j + \sigma_{\tau\alpha}\theta^j z_i^j - \Lambda y^j - \theta^j \Lambda z^j - \theta^j \Lambda f = g^{j-1}.$$

By analogy with (6), we obtain the system

$$\begin{cases} \sigma_{\tau\alpha} y_i^j - \Lambda y^j = g^{j-1} \\ \sigma_{\tau\alpha} z_i^j - \Lambda z^j = \Lambda f \end{cases} \tag{8}$$

Additional functions y and z are calculated from system (8). Now we can calculate ω_m, θ, ω

$$\omega_k = y_m + \theta^j z_m, \quad \theta^j = (\omega_m - \phi^j)/f_m.$$

The calculation results are the same.

Let us consider the possibility of applying an integral additional condition. That is, the additional condition is specified as an integral over the area:

$$\int_0^l u(x,t)dx = \phi(t).$$

We have:

$$\int_0^l u(x,t)dx = \int_0^l (\theta(t) f(x) + \omega(x,t))dx = \theta(t)\int_0^l f(x)dx + \int_0^l \omega(x,t))dx = \phi(t),$$

let's substitute from (7)

$$\theta(t)\int_0^l f(x)dx = \phi(t) - \int_0^l (y + \theta(t) z)dx = \phi(t) - \int_0^l y dx - \theta(t)\int_0^l z dx,$$

from here:

$$\theta(t) = \frac{\phi(t) - \sum_{i=0}^{N} y_i h}{\sum_{i=0}^{N} (f_i + z_i) h}.$$

4 Numerical experiments

We will carry out a numerical implementation of computational algorithms on a model problem from [15], [16] with different conditions and different values of the fractional time derivative exponent α and compare the obtained calculation results with the exact solution.

Example 1. We consider the inverse problem with a smooth initial condition (1) on the domains $x \in [0, l]$ and $t \in [0, T]$ with different $\alpha = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9$. Design parameters for space: $N = 100,\ 200;\ l = 1$, for time: $T = 1;\ M = 100,\ 200$. All input and output functions are known:

$$u(x,t) = t^2 sin(\pi x), \qquad f(x) = sin(\pi x), \qquad \varphi(x) = 0,$$

$$\phi(t) = t^2, \qquad p(t) = \frac{\Gamma(3)t^{2-\alpha}}{\Gamma(3-\alpha)} + t^2\pi^2.$$

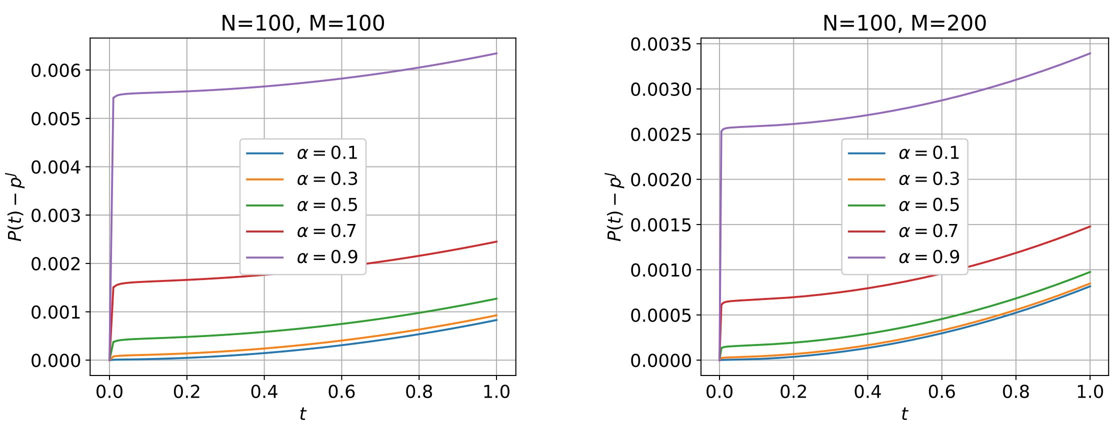

Figure 1: The error of finding the function $p(t)$ at different α and when the redefinition function $u(x^*, t) = \phi(t), \quad x^* = l/2$ at $N = 100, M = 100$. On the right at $N = 100, M = 200$.

Figs. 1 and 2 present the results of the calculation according to the proposed algorithm to illustrate the effect of grid shredding in space and time. As it turned out, the accuracy is more influenced by the grinding in time. Fig. 3 shows an example when the redefinition condition is integral over the domain. No significant improvement in the result was found. In Fig. 4, the influence of the location of the selection of the point at which the function of the additional condition is set is checked. At least for this example, the impact is minimal. The same example is in the case when the redefinition condition is set as an integral over the domain.

Example 2.

In Figs. 5 and 6, the test was carried out for the function from [16]. Interval in time and an additional condition: integral of the function over the area. An inverse problem with a smooth initial condition (1) is considered. Calculations were carried out on the same area, the grid at $\alpha = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9$, for $N = 100,\ 200;\ M =$

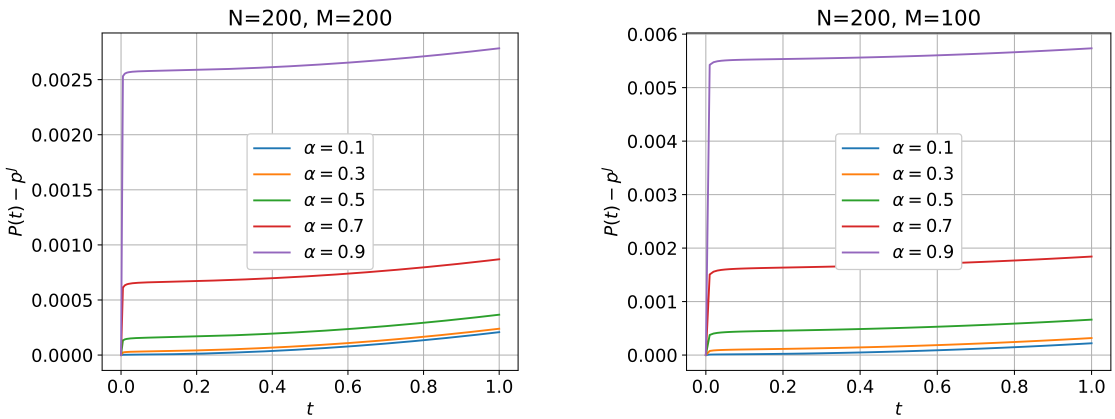

Figure 2: The error of finding the function $p(t)$ at different α and when the redefinition function $u(x^*, t) = \phi(t), \quad x^* = l/2$ at $N = 200, M = 200$. On the right at $N = 200, M = 100$.

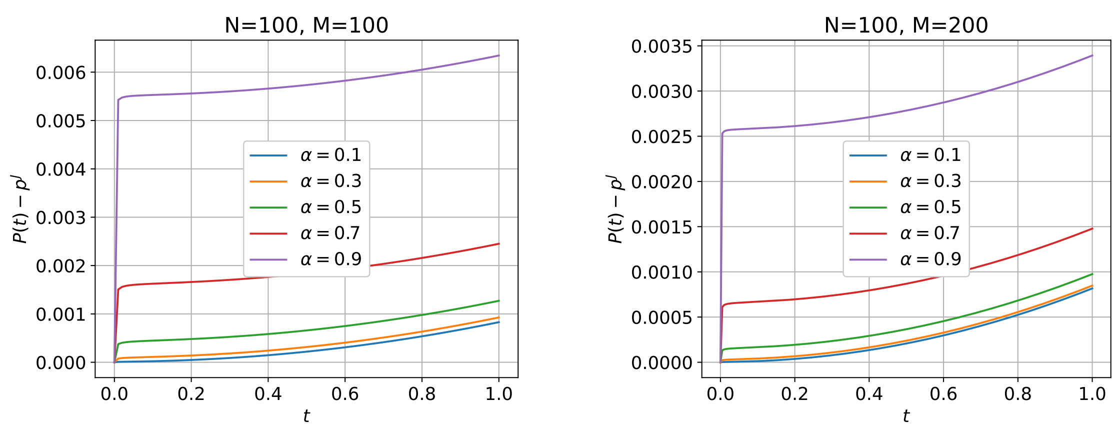

Figure 3: The error of finding the function $p(t)$ at different α and when the redefinition function $\int_0^l u(x,t)dx = \phi(t)$, at $N = 100, M = 100, 200$.

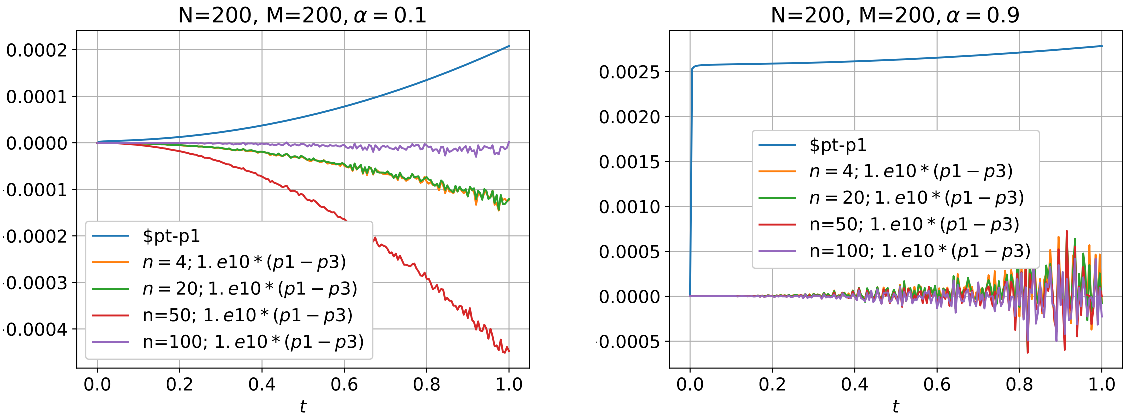

Figure 4: Error in finding the function $p(t) = pt$ for $\alpha = 0.1$ when the override function $u(x^*, t) = \phi(t), \quad x^* = n * h$ for $n = 4, 20, 50, 100, N = 200, M = 200$. p1 –found when the override function is integral. p3–found when the override function is specified at different points. On the right at $\alpha = 0.9$.

100; $M = 200$; $l = 1.$; $T = 15$. Results are shown in Fig. 5, 6. All input and output functions are known:

$$u(x,t) = sin(\pi x)sint, \qquad f(x) = sin(\pi x), \qquad \varphi(x) = 0,$$

$$\int_0^l u(x,t)dx = \phi(t) = sint, \qquad p(t) = sin(t + \alpha\pi/2) + \pi^2 sint.$$

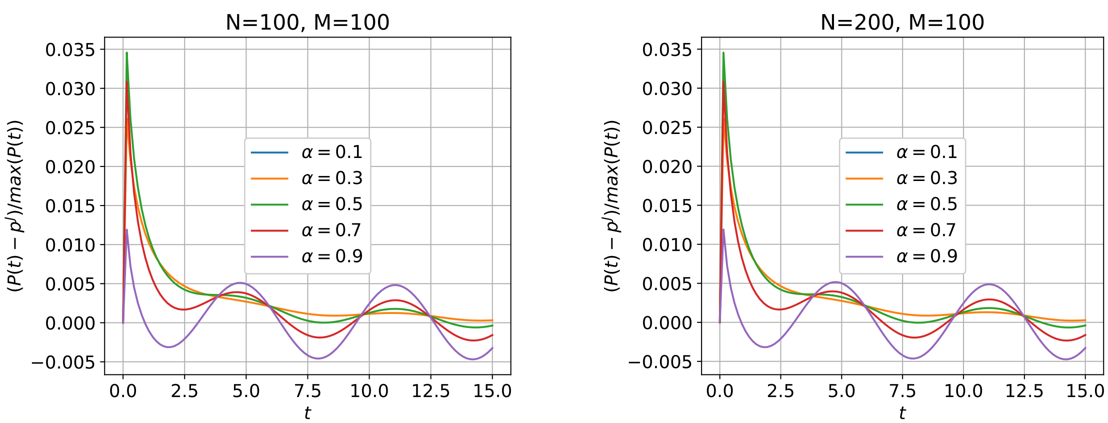

Figure 5: The error of finding the function $p(t)$ at different α and when the redefinition function $\int_0^l u(x,t)dx = \phi(t)$, at $N = 100, M =$100. On the right at $N = 200, M = 100$.

Conclusion

To numerically solve the inverse problem of identifying the factor of the right side of the subdiffusion equation, the non-iteration decomposition method proposed in [5] is

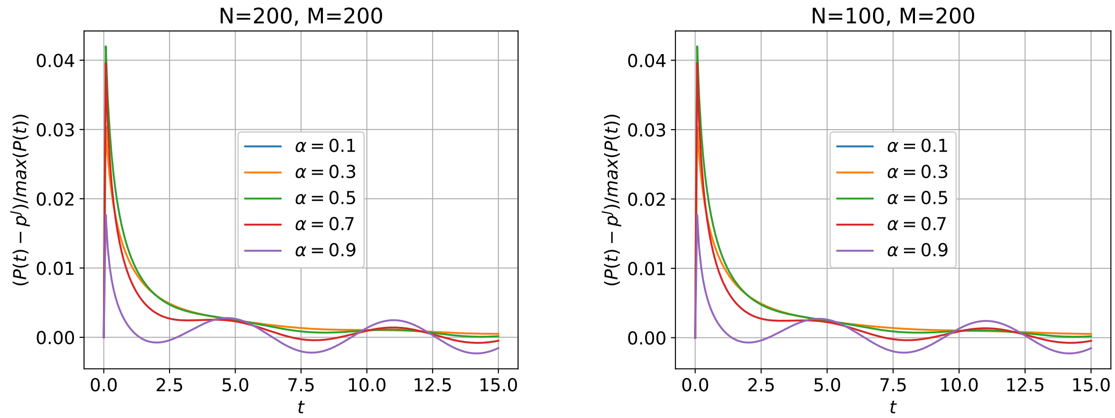

Figure 6: The error of finding the function $p(t)$ at different α and when the redefinition function $\int\limits_0^l u(x,t)dx = \phi(t)$, at $N = 200, M =$200. On the right at $N = 100, M = 200$.

used, which reduces to a system of elliptic linear differential equations. Reconstruction of the function and calculation of the desired function of the multiplier on the right side are performed with high accuracy. The results of the numerical implementation of the proposed method are presented using model examples with exact solutions on different spatial and temporal grids and the order of the fractional derivative with respect to time. Calculations were carried out for cases where the additional condition is presented as a function of time at the internal point and as an integral over the area. Calculations showed a fairly high efficiency of the proposed method.

The authors express deep gratitude to Professor V.I. Vasil'ev for constructive comments and essential advice during the research and for providing all possible support during the writing of the article.

The work is supported by of the Ministry of Education and Science of the Russian Federation, agreement dated 02/16/2023. project 075-02-2023-947

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Received 18.09.2023, revised 27.11.2023, Accepted 08.12.2023