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NUMERICAL HOMOGENIZATION OF THE GAS FILTRATION PROBLEM

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Abstract The non-stationary gas flow equation in homogeneous and heterogeneous media is considered. Areas of this type are often encountered when considering gas production processes from a gas-bearing reservoir. The heterogeneous structure of the reservoir can strongly affect the extraction processes. For the numerical solution of the problem under consideration, we construct an approximation of the equations on a coarse grid using the numerical averaging method. The method allows us to solve the problem using less computational power and in a shorter time. A numerical comparison of the results of solving the model problem is carried out in a two-dimensional domain for the cases of linear and nonlinear variants of the equations, as well as in a homogeneous and heterogeneous medium. The finite element solution on a fine mesh was taken as a reference solution. The computational realization was performed using the FEniCS library. The construction of the geometric computational domain was performed in the program Gmsh. Paraview and the Matplotlib library in Python were used for data visualization.

Key words: numerical homogenization, gas filtration problem, gas flow equation, finite element method, heterogeneous media, homogeneous media.

AMS Mathematics Subject Classification: 65K05, 76M50.

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1 Introduction

The quest to understand and manipulate the transport of gases through porous media has been a topic of immense significance in various scientific and industrial domains. From the extraction of hydrocarbons from reservoir rocks to the design of advanced filtration systems, the ability to predict and optimize gas flow through porous materials plays a pivotal role. In this pursuit, the concept of homogenization has emerged as a powerful mathematical and computational tool, allowing researchers and engineers to bridge the gap between microscale porous structures and macroscopic transport phenomena.

To appreciate the significance of numerical homogenization in the gas filtration problem, one must first understand the complexity of gas flow through porous media. Porous materials, ranging from natural geological formations to engineered filtration media, exhibit a hierarchical structure characterized by a multitude of interconnected pores. The behavior of gases within these intricate pore networks is inherently influenced by various factors, such as pore size, shape, connectivity, surface chemistry etc. [1, 11].

At the macroscopic level, the transport of gas through porous media is governed by Darcy's law, which relates the flux of gas to the pressure gradient [2, 10]. However, the application of Darcy's law is not always straightforward due to the heterogeneity and complexity of real-world porous structures. Traditional approaches, which rely on modeling on a very fine grid to resolve all heterogeneities in porous media, are computationally expensive and often impractical for large-scale simulations.

Homogenization, a mathematical technique rooted in the theory of partial differential equations (PDEs), offers an elegant solution to this problem [12, 13, 14, 15]. The fundamental idea behind homogenization is to replace the intricate microscale details of porous media with effective macroscopic properties, simplifying the mathematical description of gas flow [16]. This simplification enables the development of computationally efficient models that capture the essential behavior of gases in porous media without explicitly simulating the pore-scale flow.

In the context of gas filtration, numerical homogenization allows researchers to determine effective permeability tensors, which characterize the macroscopic gas transport properties of the porous material. These tensors encapsulate the influence of microscale pore geometry and are vital for predicting gas flow in heterogeneous media. By employing numerical techniques such as finite element analysis (FEA) [9] or finite difference methods [8], researchers can derive these effective properties, transforming complex pore-scale problems into manageable macroscopic simulations.

In the paper, we present an algorithm based on the numerical homogenization method for gas filtration problems in heterogeneous perforated media [7, 3]. The effective characteristics of the heterogeneous medium are calculated in local domains to describe the process on a coarse grid. The gas filtration model is a nonlinear problem, and to resolve the nonlinearity, we use the Picard iteration method [6, 5, 4]. In this work, we present the numerical experiments for several problems with the different physical properties.

The paper is organized as follows: Section 2 discusses the mathematical formulations and concepts necessary to understand gas flow in porous media. The process of solving the problem using the finite element method is given in Section 3. Section 4 is devoted to describing the numerical averaging method. Section 5 describes the tests carried out, shows the final result, and analyzes the data obtained. In Section 6, we will summarize the main results of this paper and emphasize the importance of the numerical averaging method to deepen our understanding of the advantages of implementing mathematical models and solving problems with this method.

2 Problem formulation

The gas filtration equation (gas flow continuity equation) is represented in general form as follows (we use the conditions of an ideal gas):

1. continuity equation

$$m\frac{\partial \rho}{\partial t} + div(\rho v) = 0, \tag{1}$$

2. equation of state

$$\rho = \frac{P}{R\ T}, \tag{2}$$

3. Darcy's law

$$v = -\frac{k}{\mu} grad(P), \tag{3}$$

$$x \in \Omega, \quad x = (x_1, x_2), \quad 0 < t \leq T_{max}. \tag{4}$$

We use the next notations in the mathematical model:

- ρ - gas density,
- P - gas pressure,
- R - universal gas constant = 520,
- T - temperature,
- v - flow velocity,
- k - reservoir permeability coefficient,
- μ - gas viscosity coefficient.

The differential equation of gas filtration (1)-(4) is a mathematical model of a whole class of parabolic equations, and, in general, an infinite number of solutions can be obtained by integration [17].

As a geometric condition in the investigated problem, a slice of a gas-bearing reservoir (plane) with producing wells represented as holes on the plane is shown in Figure 1:

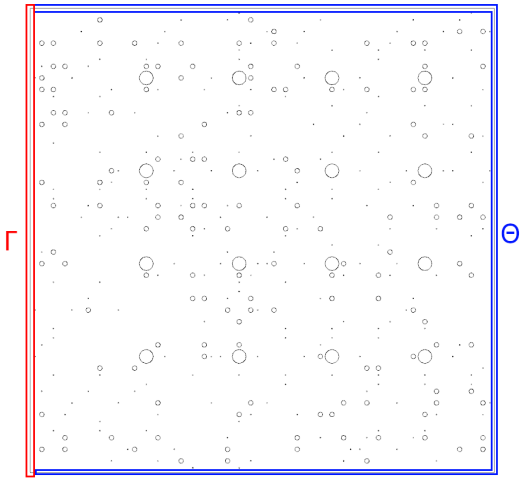


Figure 1: Reservoir geometry Ω with boundary Γ and Θ.

We supplement the equation with the following initial condition:

$$P(x, 0) = P_s, \quad 0 \leq x \leq l, \tag{5}$$

and we apply the next boundary condition:

$$-\frac{\partial P}{\partial n}\bigg|_C = C_g, \quad x \in \mathrm{C}, \qquad \frac{\partial P}{\partial n}\bigg|_\Theta = 0, \quad x \in \Theta,$$
$$P(x,t) = P_n, \quad x \in \Gamma. \tag{6}$$

In our implementation, we use the following notations:

- P_s - initial pressure in reservoir,
- P_n - reservoir boundary pressure,
- l - length/width of reservoir,
- C_g - well production constant,
- C - well boundary,
- Γ - left reservoir boundary,
- Θ - remaining boundary.

3 Fine grid approximation

Next, we consider the fine grid approximation of the problem (1)-(6). We use the finite element method with the Picard iteration method to resolve nonlinearity. We define an unstructured triangular fine grid M_h, to take into account all perforations. The problem is time-dependent, then, we note n as a number of time layers, τ as the time step, and $T_{max} = n\tau$ as the final time. We consider N_f as the number of elements in a fine grid.

Next, we present a fine grid approximation using the finite element method. Before obtaining the finite element approximation of the model, we derive a variational formulation of the problem. Therefore, we first define the standard Hilbert space of scalar functions $L_2(\Omega)$ with the scalar product and the obtained norm in this form

$$(P, v) = \int_\Omega P(x)v(x)dx, \quad \|P\| = (P, P)^{1/2}.$$

We define the Sobolev space $H^1(\Omega)$ of functions v such that $v^2 = |\nabla v^2|$ is finitely integrable in the domain of Ω [18]. After these steps, we can already define the spaces for the test

$$V = v \in H^1(\Omega) : v(x) = 0, \quad x = (x_1, x_2), \quad x \in \Gamma,$$

and trial functions

$$\hat{V} = P \in H^1(\Omega) : P(x,t) = P_s, \quad x = (x_1, x_2), \quad x \in \Gamma.$$

Firstly, to solve the equation (1)-(4), we integrate the equation over the domain Ω and multiply the test function v:

$$\int_\Omega \frac{m}{R\ T}\frac{\partial P}{\partial t} v\ dx = \int_\Omega \nabla \cdot \Big(\frac{k}{\mu\ R\ T} P \nabla P\ v\Big) dx.$$

Next, we use integration by parts with the given boundary conditions, and for approximation by time, we use an implicit time-difference scheme

$$\int_\Omega \frac{m}{R\ T}\frac{P^{n+1}-P^n}{\tau}v\ dx + \int_\Omega \frac{k}{\mu\ R\ T}P^{n+1}\nabla P^{n+1}\nabla v\ dx + \int_C C_g v\ ds = 0.$$

The problem is nonlinear, and we use the Picard iteration method to resolve nonlinearity, and the variational formulation will take the following form:

$$\int_\Omega \frac{m}{R\ T}\frac{P^{n+1}_{j+1}-P^n}{\tau}dx + \int_\Omega \frac{k}{\mu\ R\ T}P^{n+1}_j\nabla P^{n+1}_{j+1} + \int_C C_g v\ ds = 0,$$

where j denotes the Picard iteration number. Now let's introduce new variables to facilitate further calculations:

$$C_l = \frac{m}{R\ T}, \quad C_r = \frac{k}{\mu\ R\ T}.$$

Now we can write the variational form of the problem (1)-(6):

$$C_l\int_\Omega \frac{P^{n+1}_{j+1}-P^n}{\tau}v\ dx + C_r\int_\Omega P^{n+1}_j\nabla P^{n+1}_{j+1}\nabla v\ dx + \int_C C_g v\ ds = 0, \tag{7}$$

where $P \in \hat{V}$. We use the following stopping condition for the Picard iterations:

$$\|P^{n+1}_{j+1} - P^{n+1}_j\| < \varepsilon,$$

where $\varepsilon = 10^{-3}$ is the relative tolerance for convergence. And finally we divide the obtained statement into bilinear

$$a(P,v) = C_l\int_\Omega \frac{P^{n+1}_{j+1}}{\tau}vdx + C_r\ P^{n+1}_j\int_\Omega \nabla P^{n+1}_{j+1}\nabla v\ dx,$$

and linear forms

$$L(v) = C_l\int_\Omega \frac{P^n}{\tau}v\ dx - \int_C C_g\ v\ ds.$$

4 Numerical homogenization

The method consists of finding the effective coefficients on a coarse grid by solving independent local problems for each section of the region grid by solving independent local problems for each section of the domain (coarse grid cells)[13, 14, 19]. Let the computational domain be partitioned into triangular elements, where M_H and M_h - are the coarse-scale and fine-scale computational meshes. In order to calculate the effective coefficients used in solving the problem on the coarse grid, it is necessary to numerically solve local problems in subareas K_j, where j is the number of the triangular element of the coarse grid, $M_H = \cup_j K_j (j = \overline{1, N}, N_H$, - number of coarse mesh elements). Let's

write the approximation of the equation by the finite element method on a coarse mesh M_H

$$\int_\Omega c^* \frac{P_{j+1}^{n+1} - P^n}{\tau} v \, dx - \int_\Omega (k^* C_r \, P_j^{n+1} \nabla P_{j+1}^{n+1}, \nabla v) dx - \int_C C_g \, v \, ds = 0, \;\; \forall v \in \hat{V}. \quad (8)$$

Here, the coefficients c^* and k^* are the effective coefficients, which are defined in cells K_j, $j = \overline{1, N}$.
To calculate the effective characteristics, we will solve the following local problems in the subdomain K_j:

$$\begin{gathered} -\nabla \cdot (k^* C_r \nabla \Phi^m) = 0, \;\; x = (x_1, x_2) \in K_j, \\ \Phi^m = x_m, \;\; x \in \partial K_j, \;\; m = 1, 2. \end{gathered} \quad (9)$$

The effective coefficient k^* will be calculated as follows:

$$k^* = \begin{bmatrix} k_{11}^* & k_{12}^* \\ k_{21}^* & k_{22}^* \end{bmatrix}, \quad (10)$$

where the tensor components have the form

$$\begin{aligned} k_{11}^* &= \frac{1}{|K_j|} \int_{K_j} C_r \frac{\partial \Phi_1}{\partial x_1} dx, \;\; k_{12}^* = \frac{1}{|K_j|} \int_{K_j} C_r \frac{\partial \Phi_1}{\partial x_2} dx, \\ k_{21}^* &= \frac{1}{|K_j|} \int_{K_j} C_r \frac{\partial \Phi_2}{\partial x_1} dx, \;\; k_{22}^* = \frac{1}{|K_j|} \int_{K_j} C_r \frac{\partial \Phi_2}{\partial x_2} dx. \end{aligned} \quad (11)$$

To calculate the average coefficient c^* we will use the average value over the local domain

$$c^* = \frac{1}{|K_j|} \int_{K_j} C_l dx. \quad (12)$$

The calculated coefficients c^* and k^* will be used in solving the problem (8) on a coarse-scale grid. Thus, the computational algorithm for solving the problem can be presented as follows:

- construct a large-scale computational grid M_H;
- in each local subdomain K_j, compute the effective characteristics c^* and k^* by solving problem (9) and expressions (10)-(12);
- solve problem (8) on a coarse grid M_H using the calculated coefficients.

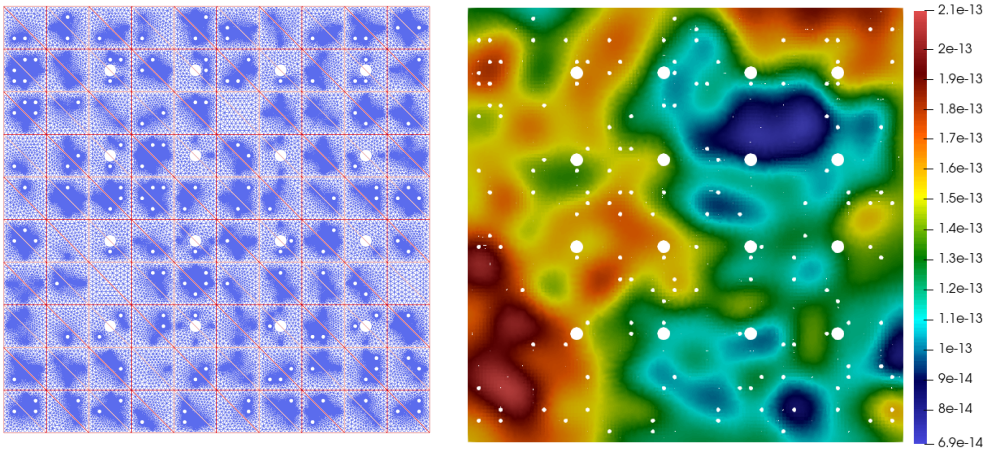
5 Numerical results

In this section, we present a numerical experiment with the proposed algorithm of numerical homogenization for the gas filtration problem. Due to the fact that the problem under consideration is a model problem and serves to prove the work of the method, we can directly modify the coefficients C_r, C_l, and C_g obtained earlier.

We will consider several cases of the problem:

- *Case 1.* Linear equation with homogeneous coefficients of a two-dimensional problem with $\Omega = [0, 10000]^2$, $C_g = 3.826 * 10^{-7}$;
- *Case 2.* Linear equation with heterogeneous coefficients of a two-dimensional problem with $\Omega = [0, 10000]^2$, $C_g = 1.913 * 10^{-9}$;
- *Case 3.* Nonlinear equation with homogeneous coefficients of a two-dimensional problem with $\Omega = [0, 10000]^2$, $C_g = 3.826 * 10^{-9}$;
- *Case 4.* Nonlinear equation with heterogeneous coefficients of a two-dimensional problem with $\Omega = [0, 10000]^2$, $C_g = 1.913 * 10^{-15}$.

The following parameters were used for the numerical study of the error of the considered method: m - 0.255, R - 520.0, T - 283.0, k - 0.203 * 10^{-12} (for *Case 1* and *Case 3*), μ - 0.02 * 10^{-3}, P_s - 7.4 * 10^3, P_n - 7.4 * 10^6. For linear problems, *Case 1* and *Case 2* we use average pressure value over computational domain Ω $P_{avg} = \frac{P_n + P_s}{2}$ instead of P_j^{n+1} in (7), (8).



(a) (b)

Figure 2: (a) Computational mesh (blue - fine grid, red - coarse grid) (b) Heterogeneous properties k for *Case 2* and *Case 4*.

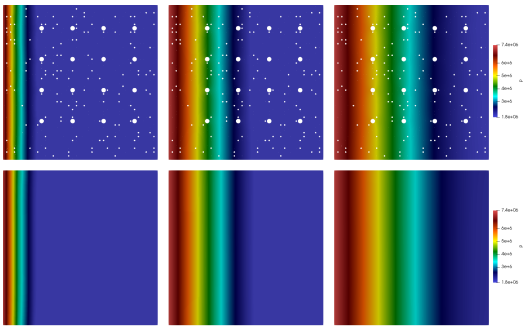


Figure 3: Pressure distribution at different moments of time t = 1, 5 and 10. First line: solution for *Case 1* on a fine grid (number of unknowns N_f= 80844). Second line: solution of the problem using the numerical homogenization method on a coarse grid (number of unknowns N_c = 121).

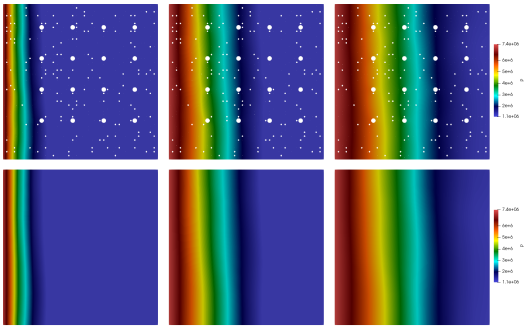


Figure 4: Pressure distribution at different moments of time t = 1, 5 and 10. First line: solution for *Case 2* on a fine grid (number of unknowns N_f= 80844). Second line: solution of the problem using the numerical homogenization method on a coarse grid (number of unknowns N_c = 121).

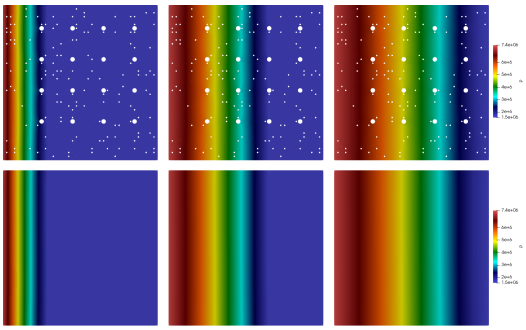


Figure 5: Pressure distribution at different moments of time t = 1, 5 and 10. First line: solution for *Case 3* on a fine grid (number of unknowns N_f= 80844). Second line: solution of the problem using the numerical homogenization method on a coarse grid (number of unknowns N_c = 121).

To prove the correctness of the proposed algorithm, we compare the coarse grid solution with the fine grid solution obtained by the finite element method. A comparison

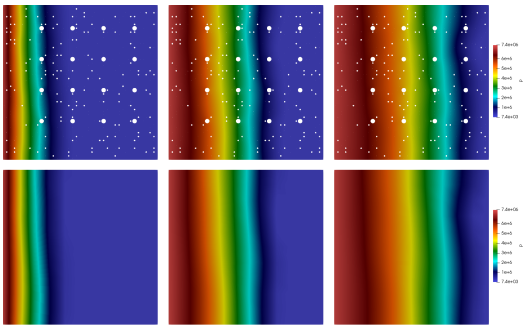


Figure 6: Pressure distribution at different moments of time t = 1, 5 and 10. First line: solution for *Case 4* on a fine grid (number of unknowns N_f= 80844). Second line: solution of the problem using the numerical homogenization method on a coarse grid (number of unknowns N_c = 121).

of the relative error in L_2 norm was made using the following formula:

$$||e||_{L_2} = \frac{\sqrt{\int_\Omega |P_c - P_f|^2 dx}}{\sqrt{\int_\Omega |P_f|^2 dx}},$$

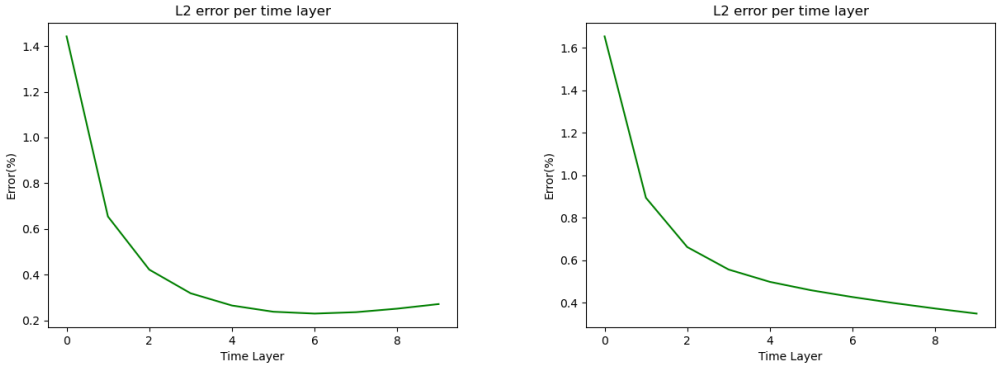
where P_c and P_f are solutions on coarse and fine grids (reference solution). For the reference solution, we take a fine grid with 80844 vertices and 157700 cells; for the homogenized solution, we take a coarse grid with 121 vertices and 100 cells.

The calculations were performed using the FEniCS library [20]. The construction of the geometric computational domain was performed in the program Gmsh [21]. Visualization of the results obtained during the solutions is performed in the Paraview [22] program, as well as using the Matplotlib [23] library for Python [24]. The results of calculations on fine and coarse grids are presented in Figures 3-6.

Time layer	A	B	C	D
1	1.442	1.654	1.266	1.537
5	0.265	0.497	0.517	0.750
10	0.271	0.348	0.325	0.563

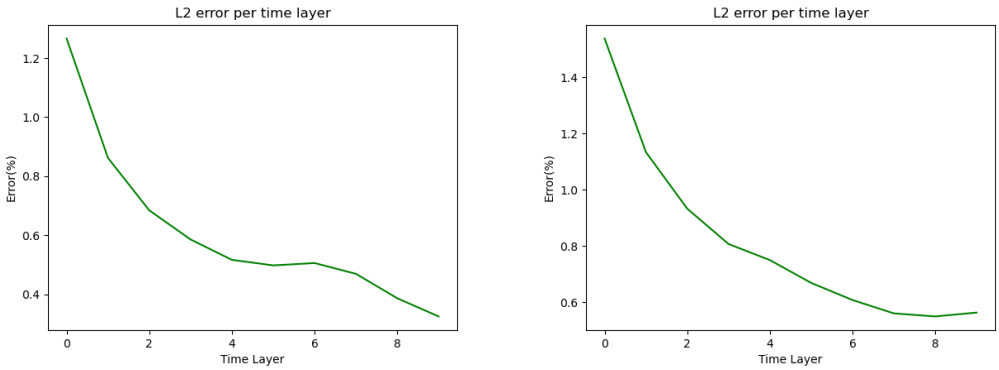
Table 1: L2 errors at the time layer: A - *Case 1*, B - *Case 2*, C - *Case 3*, D - *Case 4*.

The pressure distribution is presented at different moments of time (1-st, 5-th, and 10-th time layers) for the cases: *Case 1* at the Figure 3, *Case 2* at the Figure 4), *Case 3* at the Figure 5, *Case 4* at the Figure 6. The figures at the top line show the pressure distribution in fine grid calculations with a large number of unknowns of the system, while the figures at the bottom line show the solution on a coarse grid using the numerical homogenization method. From the above results, we can see that the solution on the coarse grid is almost equal to the solution on the fine grid. We can see the full picture by looking at the Table 1, where we present a relative error in L_2



(a) (b)

Figure 7: L2 error in percent for solutions: (a) *Case 1* (b) *Case 2*.



(a) (b)

Figure 8: L2 error in percent for solutions: (a) *Case 3* (b) *Case 4*.

norm in different time layers for all test cases. And we show the error distribution by time in Figures 7 and 8. From the results, we can see that the error of the proposed algorithm is very small. Throughout the entire process, the error is less than 2%. At beginning of the process, we observe large errors because of the influence of the initial condition. Further, over time, the error decreases, which indicates the convergence of the proposed algorithm. We can conclude that the solution of the gas filtration problem by numerical homogenization allows us to significantly reduce the dimensionality of the problem, while the solution error is very small compared to the solution on a fine grid. The solution error for both linear and nonlinear equations with homogeneous and heterogeneous mediums is less than 2%.

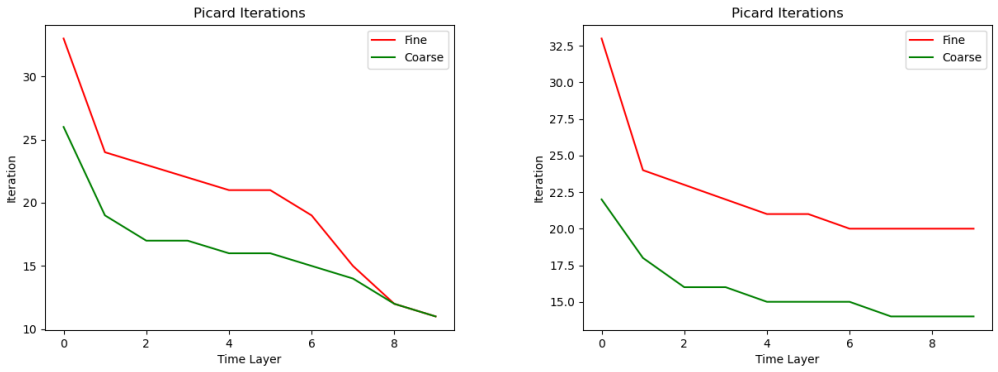
The number of Picard iterations performed to solve the problems is presented in the form of Tables 2-3. The total number of iterations and the average number per time layer are presented for several cases: *Case 3* (Table 2), *Case 4* (Table 3). In the tables, the data in the first row shows iterations in fine grid calculations with a large number of unknowns of the system, while the data at the bottom line shows the iteration count on a coarse grid using the numerical homogenization method.

Grid	Total iterations	Average iterations
Fine	201	20.1
Coarse	163	16.3

Table 2: Number of Picard iterations for the nonlinear homogeneous equation (*Case 3*).

Grid	Total iterations	Average iterations
Fine	224	22.4
Coarse	159	15.9

Table 3: Number of Picard iterations for the nonlinear heterogeneous equation (*Case 4*).



(a) (b)

Figure 9: Number of Picard iterations per time layer: (a) *Case 3* (b) *Case 4*.

From the above results, we can conclude that using the numerical homogenization method greatly reduces computational costs by solving with less iteration. For a nonlinear equation with homogeneous medium (*Case 3*), it makes $\simeq$18.9% difference, and for a nonlinear equation with heterogeneous medium (*Case 4*), it's a $\simeq$29% difference.

5 Conclusion

In this paper, the numerical homogenization method for the non-stationary gas flow equation has been investigated. A comparison has been made between the numerical homogenization method for solving the problem on a coarse mesh, containing 121 nodes, and the finite element method on a fine mesh, containing 80844 nodes. The results obtained illustrate the small error in modeling using the homogenization method for both homogeneous and heterogeneous regions. The numerical experiment showed

that the numerical homogenization method has an advantage in nonlinear problems using the Picard iteration method. Numerical homogenization required fewer Picard iterations than the fine grid solution. An algorithm based on the numerical homogenization method for the gas filtration problem allows us to obtain a solution with high accuracy on a very coarse mesh.

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References

[1] Zvezdin V.G., *Oilfield Geology: Textbook*, Perm State University [In Russian], 2007.

[2] Preziosi L., Farina A., *On Darcy's law for growing porous media*, International Journal of Non-Linear Mechanics, Vol. 37 (2002), 485–491.

[3] Engquist, Björn, Souganidis, Panagiotis E, *Asymptotic and numerical homogenization*, Acta Numerica, Vol. 17 (2008), 147–190.

[4] Huang J., Chen L., Rui H., *Multigrid methods for a mixed finite element method of the Darcy–Forchheimer model*, Journal of scientific computing, Vol. 74 (2018), 396–411.

[5] Berinde V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Analysis Forum, Vol. 9 (2004), 43–54.

[6] Spiridonov D., Huang J, Vasilyeva M., Huang Y.,Chung E. T, *Mixed generalized multiscale finite element method for Darcy-Forchheimer model*, Mathematics, Vol. 7 (2019), 1212.

[7] Tyrylgin A., Spiridonov D., Vasilyeva M., *Numerical homogenization for poroelasticity problem in heterogeneous media*, Journal of Physics: Conference Series, Vol. 1158 (2019), 042030.

[8] Smith G. D., *Numerical solution of partial differential equations: finite difference methods*, Oxford university press, 1985.

[9] Bathe, Klaus-Jürgen, *Finite element method*, Wiley encyclopedia of computer science and engineering, 2007, 1–12.

[10] Whitaker S., *Flow in porous media I: A theoretical derivation of Darcy's law*, Transport in porous media, Vol. 1 (1986), 3–25.

[11] Kononov V.M., *Oilfield Geology. Textbook for universities*, LitRes [In Russian], 2021.

[12] Bakhvalov N.S., Panasenko G., *Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials*, Springer Science & Business Media, Vol. 36 (2012).

[13] Talonov A., Vasilyeva M., *On numerical homogenization of shale gas transport*, Journal of Computational and Applied Mathematics, Vol. 301 (2016), 44–52.

[14] Hales J.D., Tonks M.R., Chockalingam K., Perez D.M., Novascone S.R., Spencer B.W., Williamson R.L., *Asymptotic expansion homogenization for multiscale nuclear fuel analysis*, Computational Materials Science, Vol. 99 (2015), 290–297.

[15] Durlofsky L. J., *Numerical calculation of equivalent grid block permeability tensors for heterogeneous porous media*, Water resources research, Vol. 27 (1991), 699–708.

[16] Efendiev Ya., Hou T. Y., *Multiscale finite element methods: theory and applications*, Springer Science & Business Media, 2009.

[17] Vabishchevich P.N., *Computational methods of mathematical physics. Non-stationary problems*, Moscow: University Book [In Russian], 2009.

[18] Pavlova M.F., Timerbaev M.R., *Sobolev spaces (embedding theorems)*, Kazan: Kazan State University [In Russian], 2010.

[19] Wu, Xiao-Hui, Efendiev Ya., Hou T.Y, *Analysis of upscaling absolute permeability*, Discrete and Continuous Dynamical Systems Series B, Vol. 2 (2002), 185–204.

[20] Logg A., Mardal Kent-Andre, Wells Garth, *Automated solution of differential equations by the finite element method: The FEniCS book*, Springer Science & Business Media, Vol. 84 (2012).

[21] Geuzaine Ch., Remacle Jean-François, *Gmsh: A 3-D finite element mesh generator with built-in pre-and post-processing facilities*, International journal for numerical methods in engineering, Vol. 79 (2009), 1309–1331.

[22] Ayachit U., *The paraview guide: a parallel visualization application*, Kitware, Inc., 2015.

[23] Hunter J. D., *Matplotlib: A 2D graphics environment*, Computing in science & engineering, Vol. 9 (2007), 90–95.

[24] Van Rossum G., Drake Jr., Fred L., *Python tutorial*, Centrum voor Wiskunde en Informatica Amsterdam, The Netherlands, Vol. 620 (1995).

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