EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 12, Issue 1 (2024) 41 – 56

INVERSE PROBLEM OF DETERMINING THE TIME-DEPENDENT BEAM STIFFNESS COEFFICIENT IN THE BEAM VIBRATION EQUATION USING THE FINITE DIFFERENCE METHOD

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Abstract In this study, we consider the inverse problem of determining the time-dependent beam stiffness coefficient $q(t)$ and transverse bending vibrations $u(x,t)$ of a homogeneous beam using the finite difference method. The problem is governed by a fourth-order partial differential equation, which is discretized in space and time using numerical techniques to obtain an accurate and stable solution. An additional condition in the form of an integral is also included to determine $q(t)$. The results obtained for different discretization grid points are compared, and a simple test example is presented to demonstrate the accuracy and agreement of the numerical solutions with analytical solutions. The paper provides a comprehensive overview of the methodology used to solve the inverse problem, and presents a detailed analysis of the results obtained.

Key words: direct problem, inverse problem, finite difference method, beam vibration equation.

AMS Mathematics Subject Classification: 35R30, 65M32.

DOI: 10.32523/2306-6172-2024-12-1-41-56

1 Introduction

In the past few years, there has been growing interest in the study of direct and inverse problems for the equation of beam vibrations [2]–[15]. The development of modern technology has facilitated the creation of programs for solving these equations, which has become a major field in engineering and science. The numerical solutions of the beam vibration equation are given in [7]–[15]. Articles [8] and [9] present approximate methods for solving direct and inverse problems described by the inhomogeneous Bernoulli-Euler equation of beam vibrations. In [15], an analytical solution of the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain was obtained for various boundary conditions.

An inverse problem is a type of problem that often arises in many branches of science, when the values of the model parameters must be obtained from the observed data. The method of proof for inverse dynamic problems of local theorems of existence and uniqueness of solutions, theorems of uniqueness and conditional stability, and numerical approaches to finding their solutions are considered in [16]-[25].The main results of [20] comprise local existence and global uniqueness theorems, as well as a stability estimate for the solution of the problem of determining the reaction coefficient in a time-fractional diffusion equation. In [24], a comparison of finite difference and

Fourier spectral numerical methods for an inverse problem of simultaneously determining an unknown coefficient in a parabolic equation with the usual initial and boundary conditions is proposed.

In this study, we derive an efficient numerical algorithm based on the finite difference method for the inverse problem of determining the time-dependent unknown coefficient in the one-dimensional beam vibration equation, together with its solution. The remainder of this paper is organized as follows. In the next section, we describe the mathematical formulation of the inverse problem. Section 3 presents the numerical setup and finite-difference discretization for the inverse problem under consideration. In Section 4, we present numerical results obtained for a test example. Finally, conclusions are presented in Section 5.

2 Mathematical formulation of the inverse problem

We consider an inverse problem of determining the unknown coefficient $q(t)$ and the transverse bending vibrations $u(x,t)$ of a homogeneous beam of length l that satisfy the fourth-order partial differential equation

$$u_{tt} + a^2 u_{xxxx} + q(t)u = f(x,t), \quad (x,t) \in (0,l) \times (0,T] =: G \tag{1}$$

with the initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in [0,l], \tag{2}$$

and the boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \quad t \in [0,T]. \tag{3}$$

In the direct problem, it is required to determine the function

$$u(x,t) \in C^{4,2}_{x,t}(G) \cap C^{2,1}_{x,t}(\bar{G}), \tag{4}$$

satisfying relations (1)–(3), given numbers a, l, T and sufficiently smooth functions $\varphi(x)$, $\psi(x)$, $q(t)$, $f(x,t)$.

Inverse problem. Find the coefficient $q(t)$ if the following additional information about the solution of the direct problem (1)–(3) is available:

$$\int_0^l u(x,t)h(x)dx = g(t), \tag{5}$$

where the functions $g(t)$ and $h(x)$ are given functions.

We assume that the data of the problem (1)–(5) satisfy the following conditions:

(A_1) $\varphi(x) \in C^5[0,l]$, $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \varphi^{(4)}(0) = \varphi^{(4)}(l) = 0$,

(A_2) $\psi(x) \in C^3[0,l]$, $\psi(0) = \psi(l) = \psi''(0) = \psi''(l) = 0$,

(A_3) $f(x,t) \in C(\overline{G}) \cap C^3_x(\overline{G})$, $f(0,t) = f(l,t) = f''_{xx}(0,t) = f''_{xx}(l,t) = 0$, $0 \le t \le T$,

(A_4) $h(x) \in C^4[0,l]$, $h(0) = h(l) = h_{xx}(0) = h_{xx}(l) = 0$,

(A_5) $g(t) \in C^2[0,T]$, $|g(t)| \ge g_0 > 0$, $g_0 = const.$

2.1 Study of the direct problem

We will look for the solution to direct problem (1)–(3) in the form

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x), \tag{6}$$

where

$$u_n(t)=\sqrt{\frac{2}{l}}\int_0^l u(x,t)\sin\mu_n x dx, \quad X_n(x)=\sqrt{\frac{2}{l}}\sin\mu_n x, \quad \mu_n=\frac{\pi n}{l}.$$

Applying the formal scheme of the Fourier method and using (1), (2), we obtain

$$u_n''(t)+a\mu_n^4u_n(t)=f_n(t)-q(t)u_n(t), \quad n=1,2,\ldots, \quad 0<t<T, \tag{7}$$

$$u_n(0)=\varphi_n, \quad u_n'(0)=\psi_n, \quad n=1,2,\ldots, \tag{8}$$

где

$$\varphi_n=\int_0^l\varphi(x)X_n(x)dx, \quad \psi_n=\int_0^l\psi(x)X_n(x)dx, \quad f_n(t)=\int_0^l f(x,t)X_n(x)dx.$$

Using the methodology of [4], we present the solution to the problem (7), (8) in the form of an integral equation

$$u_n(t)=\varphi_n\cos a\mu_n^2t+\frac{\psi_n}{a\mu_n^2}\sin a\mu_n^2t+\frac{1}{a\mu_n^2}\int_0^t f_n(s)\sin a\mu_n^2(t-s)ds-$$

$$-\frac{1}{a\mu_n^2}\int_0^t q(s)u_n(s)\sin a\mu_n^2(t-s)ds. \tag{9}$$

For each fixed n, the equation (9) is a Volterra integral equation of the second kind with respect to u_n. According to the general theory of integral equations, under appropriate conditions on the functions $\varphi(x)$, $\psi(x)$, $q(t)$, $f(x,t)$ it has a unique solution. The solution to this integral equation can be found by the method of successive approximations.

In addition, from (9) can obtain an estimate for $u_n(t)$:

$$|u_n(t)|\le\left|\varphi_n\cos a\mu_n^2t+\frac{\psi_n}{a\mu_n^2}\sin a\mu_n^2t\right|+\left|\frac{1}{a\mu_n^2}\int_0^t f_n(s)\sin a\mu_n^2(t-s)ds\right|+$$

$$+\left|\frac{1}{a\mu_n^2}\int_0^t q(s)u_n(s)\sin a\mu_n^2(t-s)ds\right|\le|\varphi_n|+\frac{1}{a\mu_n^2}|\psi_n|+\frac{1}{a\mu_n^2}\|f_n\|T+\frac{1}{a\mu_n^2}\|q\|\int_0^t u_n(s)ds,$$

where $\|f_n\| = \max\limits_{0\le t\le T} |f_n(t)|$, $\|q\| = \max\limits_{0\le t\le T} |q(t)|$.

Hence, due to Gronwall's inequality and the relations $\mu_1 < \mu_2 < \ldots$, we obtain the estimate

$$|u_n(t)| \le \left(|\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{1}{a\mu_n^2}\|f_n\|T\right) \exp\left\{\frac{1}{a\mu_1^2}\|q\|T\right\}. \tag{10}$$

Taking into account (7) and (10) the estimation results for $u_n''(t)$:

$$|u_n''(t)| \le \|f_n\| + (\|q\| + a^2\mu_n^4)\left(|\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{1}{a\mu_n^2}\|f_n\|T\right) \exp\left\{\frac{1}{a\mu_1^2}\|q\|T\right\}. \tag{11}$$

Thus, the following statement is true:

Lemma 2.1. *For any $t \in [0, T]$ the following estimates hold:*

$$|u_n(t)| \le C_1 \left(|\varphi_n| + \frac{1}{n^2}|\psi_n| + \frac{1}{n^2}\|f_n\|\right), \tag{12}$$

$$|u_n''(t)| \le C_2 \left(n^4|\varphi_n| + n^2|\psi_n| + n^2\|f_n\|\right), \tag{13}$$

where C_i, $i = 1, 2$ – positive constants depending on T, a, l and $\|q\|$.

Formally, from (6) by term-by-term differentiation we compose the series

$$u_{tt} = \sum_{n=1}^{\infty} u_n''(t) X_n(x), \tag{14}$$

$$u_{xxxx} = \sum_{n=1}^{\infty} u_n(t) X_n^{(4)}(x) = \sum_{n=1}^{\infty} \mu_n^4 u_n(t) X_n(x). \tag{15}$$

The series (6), (14) and (15) for any $(x, t) \in \overline{G}$ based on Lemma 1 are majorized by the series

$$C_3 \sum_{n=1}^{\infty} \left(n^4|\varphi_n| + n^2|\psi_n| + n^2\|f_n\|\right), \tag{16}$$

where the constant C_3 depends on T, l.

Lemma 2.2. *Under the conditions A_1, A_2, A_3, one has the relations*

$$\varphi_n = \frac{1}{\mu_n^5}\varphi_n^{(5)}, \quad \psi_n = -\frac{1}{\mu_n^3}\psi_n''', \quad f_n(t) = -\frac{1}{\mu_n^3} f_n'''(t), \tag{17}$$

where

$$\varphi_n^{(5)} = \sqrt{\frac{2}{l}} \int_0^l \varphi^{(5)}(x) \cos\mu_n x dx, \qquad \psi_n''' = \sqrt{\frac{2}{l}} \int_0^l \psi'''(x) \cos\mu_n x dx,$$

$$f_n'''(t) = \sqrt{\frac{2}{l}} \int_0^l f_{xxx}(x, t) \cos\mu_n x dx$$

with the estimates

$$\sum_{n=1}^{\infty} |\varphi_n^{(5)}|^2 \le \|\varphi^{(5)}\|_{L_2[0,l]}, \qquad \sum_{n=1}^{\infty} |\psi_n'''|^2 \le \|\psi'''\|_{L_2[0,l]}. \tag{18}$$

Integrating by parts five times in the integrals for φ_n and three times in the integrals for ψ_n and $f_n(t)$ and taking into account the assumptions of Lemma 2.1, we obtain the relation (17). Inequality (18) is the Bessel inequality for the coefficients of the Fourier expansions of the functions $\varphi_n^{(5)}$ and ψ_n''' in the cosine system $\{\sqrt{2/l}\cos\mu_n x\}$ on the interval $[0,l]$.

If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 2, then, by virtue of (17) and (18), the series (16) converges, and consequently the series (6), (14) and (15) converge absolutely and uniformly in the rectangle $\overline{G}$, thus the sum of the series (6) satisfies the relations (1)–(4).

2.2 Study of the inverse problem

Having multiplied both parts of (1) by $h(x)$ and integrated from 0 to l over x, in view of conditions (5) and A_5, we obtain

$$g''(t)+a^2\int_0^l u(x,t)h^{(4)}(x)dx+q(t)g(t)=\int_0^l f(x,t)h(x)dx.$$

By solving this equation for $q(t)$, we find that

$$q(t)=\frac{1}{g(t)}\int_0^l f(x,t)h(x)dx-\frac{g''(t)}{g(t)}-\frac{a^2}{g(t)}\int_0^l u(x,t)h^{(4)}(x)dx. \tag{19}$$

Now we substitute the expression (19) for $q(t)$ into (6) and arrive at an integral equation for $u(x,t)$:

$$u(x,t)=\Psi(x,t)-\int_0^t\int_0^l\int_0^l u(y,s)G_1(x,\xi,y,t,s)d\xi dyds-\int_0^t\int_0^l u(y,s)G_2(x,y,t,s)dyds-$$
$$-\int_0^t\int_0^l\int_0^l u(\xi,s)u(y,s)G_3(x,\xi,y,t,s)dyd\xi ds. \tag{20}$$

where

$$\Psi(x,t)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left(\varphi_n\cos\omega_n t+\frac{\psi_n}{\omega_n}\sin\omega_n t\right)\sin\mu_n x+$$
$$+\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int_0^t\int_0^l f(x,s)\sin\omega_n(t-s)\sin\mu_n xdxds,$$

$$G_1(x,\xi,y,t,s)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n g(s)}f(\xi,s)h(s)h^{(4)}(y)\sin\omega_n(t-s)\sin\mu_n(y)\sin\mu_n x,$$

$$G_2(x,y,t,s)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{g''(s)}{\omega_n g(s)}h^{(4)}(y)\sin\omega_n(t-s)\sin\mu_n(y)\sin\mu_n x,$$

$$G_3(x,\xi,y,t,s)=\frac{2a^2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n g(s)}h^{(4)}(\xi)h^{(4)}(y)\sin\omega_n(t-s)\sin\mu_n(y)\sin\mu_n x.$$

Denote the operator taking the function $u(x,t)$ to the right-hand side of (20) by A. Then, (20) is written as the operator equation

$$u=Au. \tag{21}$$

Consider the functional space of function $u(x,t)\in C(G)$ with the norm given by the relation

$$\|u\|=\max_{(x,t)\in\bar{G}}|u(x,t)|$$

For simplicity, we denote

$$\Psi_0=\max_{(x,t)\in G}|\Psi(x,t)|,\quad \lambda_1=\max_{(x,t)\in G,\ \xi,y\in[0,l],\ s\in[0,T]}\left|G_1(x,\xi,y,t,s)\right|,$$

$$\lambda_2=\max_{(x,t)\in G,\ y\in[0,l],\ s\in[0,T]}|G_2(x,y,t,s)|,\quad \lambda_3=\max_{(x,t)\in G,\ \xi,y\in[0,l],\ s\in[0,T]}|G_3(x,\xi,y,t,s)|, \tag{22}$$

Let $B(\Psi,\Psi_0)=\{u:\|u-\Psi\|\le\Psi_0\}$. Obviously, $\|u\|\le 2\Psi_0$ for $u(x,t)\in B(\Psi,\Psi_0)$. We use the Banach principle to prove the existence and uniqueness of solution to the operator equation (21).

Theorem 2.1. *Let conditions A_1, A_2, A_3, A_4, A_5 are satisfied. For all $u(x,t)\in B(\Psi,\Psi_0)$ and $T\le\dfrac{1}{2l\left(\lambda_1 l+\lambda_2+2\lambda_3\Psi_0 l\right)}$ the solution to the operator equation (21) in the class $C^{4,2}_{x,t}(G)$ exists and it is unique.*

Proof

Let us prove that for suitable T the operator A maps the ball $B(\Psi,\Psi_0)$ into itself; i.e., the condition $u\in B(\Psi,\Psi_0)$ implies that $Au\in B(\Psi,\Psi_0)$. For this, we have

$$\|Au-\Psi\|=\max_{(x,t)\in\bar{G}}|Au-\Psi|\le\left|\int_0^t\int_0^l\int_0^l u(y,s)G_1(x,\xi,y,t,s)d\xi dyds\right|+$$
$$+\left|\int_0^t\int_0^l u(y,s)G_2(x,y,t,s)dyds\right|+\left|\int_0^t\int_0^l\int_0^l u(\xi,s)u(y,s)G_3(x,\xi,y,t,s)dyd\xi ds\right|\le$$
$$\le 2\Psi_0 lT\left(\lambda_1 l+\lambda_2+2\lambda_3\Psi_0 l\right)$$

If T satisfy the condition $T\le 1/[2l\left(\lambda_1 l+\lambda_2+2\lambda_3\Psi_0 l\right)]$, then $Au\in B(\Psi,\Psi_0)$. Now we check the second condition of a fixed point argument. Let $u_1,u_2\in B(\Psi,\Psi_0)$ then we get

$$
\begin{aligned}
\|Au_1 - Au_2\| \le \max_{(x,t)\in\bar{G}} |Au_1 - Au_2| \le \\
\le \left| \int_0^t \int_0^l \int_0^l (u_1(y,s) - u_2(y,s)) G_1(x,\xi,y,t,s) d\xi dy ds \right| + \\
+ \left| \int_0^t \int_0^l (u_1(y,s) - u_2(y,s)) G_2(x,y,t,s) dy ds \right| + \\
+ \left| \int_0^t \int_0^l \int_0^l (u_1(\xi,s)u_1(y,s) - u_2(\xi,s)u_2(y,s)) G_3(x,\xi,y,t,s) dy d\xi ds \right| \le \\
\le lT(\lambda_1 l + \lambda_2 + 4\lambda_3 \Psi_0 l)\|u_1 - u_2\|
\end{aligned}
$$

Therefore, if the number T is small enough to satisfy the condition $T < \dfrac{1}{2l\left(\lambda_1 l + \lambda_2 + 2\lambda_3 \Psi_0 l\right)}$, then A is a contraction operator on $B(\Psi, \Psi_0)$. Then, by the Banach principle, equation (21) has a unique solution in $B(\Psi, \Psi_0)$. The proof of the Theorem 2.1 is complete.

3 Numerical procedure

Consider a one-dimensional domain denoted as $\Omega \in (0, l)$. We divide this domain into discrete points using a total of N_x discretization points. The spatial step size is defined as $\Delta x = \frac{l}{N_x}$. The discretized points are represented as $x_i = i\Delta x$, where $0 \le i \le N_x$ and i is a positive integer. To approximate the solution $u(x_i, t_n)$, we introduce the notation u_i^n, where $t_n = n\Delta t$ and $0 \le n \le N_t$. Here, $\Delta t = \frac{T}{N_t}$ represents the temporal step size, and N_t is the total number of time steps. The discretization scheme is depicted in Figure 1.

3.1 Fully explicit finite difference scheme

We use the explicit finite difference scheme with the central second order approximations for the temporal and spacial derivatives in Eq. (1).

$$
u_{tt} \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}
$$

and

$$
u_{xxxx} \approx \frac{u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n}{\Delta x^4} = Du_i^n, \tag{23}
$$

where D is the finite difference differentiation matrix for the fourth-order spacial derivative. Then, the fully explicit finite difference form of Eq. (1) reads as

$$
\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + a^2 Du_i^n + q^n u_i^n = f_i^n.
$$

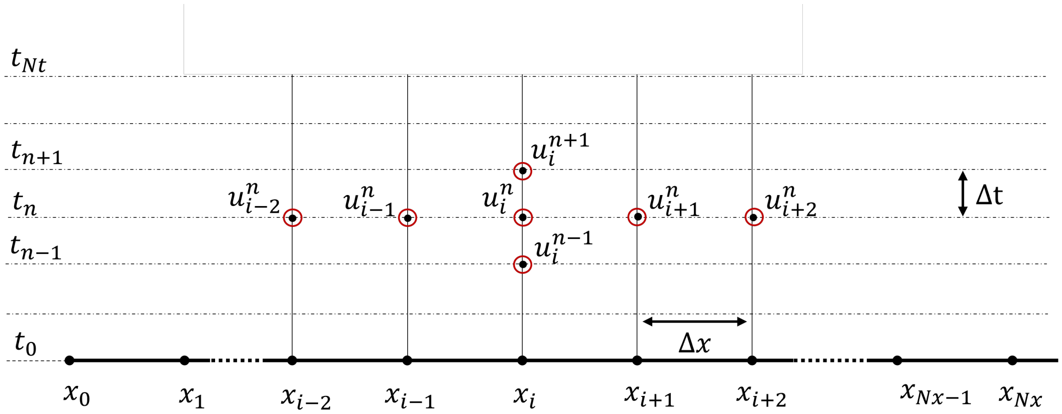


Figure 1: A schematic node ordering representation for a one-dimensional finite difference grid.

Note that Du_i^n is written here as a matrix-vector multiplication. u_i^{n+1} at grid point i for the time step $n+1$ results in

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \Delta t^2 \left(f_i^n - q^n u_i^n - a^2 D u_i^n \right) \tag{24}$$

The time loop in (24) starts from $n = 1$. Hence, u_i^0 and u_i^1 must be known, and are obtained by using the initial conditions (2), i.e. $u_i^0 = \phi_i$. We employ the forward Euler time marching for the second initial condition, $u_t|_{t=0} = \psi(x)$,

$$\frac{u_i^1 - u_i^0}{\Delta t} = \psi_i, \quad u_i^1 = u_i^0 + \Delta t \psi_i \quad \text{or} \quad u_i^1 = \phi_0 + \Delta t \psi_i.$$

3.2 Finite difference differentiation matrix

In order to find the differentiation matrix D for the fourth-order spacial derivative, we write Eq. (23) at grid points $i = 1, 2, 3, ..., N_x - 2, N_x - 1$ in ascending indices, i.e.

$$\begin{aligned}
i = 1 \rightarrow \quad & \frac{1}{\Delta x^4} \left(u_{-1}^n - 4u_0^n + 6u_1^n - 4u_2^n + u_3^n \right), \\
i = 2 \rightarrow \quad & \frac{1}{\Delta x^4} \left(u_0^n - 4u_1^n + 6u_2^n - 4u_3^n + u_4^n \right), \\
i = 3 \rightarrow \quad & \frac{1}{\Delta x^4} \left(u_1^n - 4u_2^n + 6u_3^n - 4u_4^n + u_5^n \right), \\
& \vdots \\
i = N_x - 2 \rightarrow \quad & \frac{1}{\Delta x^4} \left(u_{N_x-4}^n - 4u_{N_x-3}^n + 6u_{N_x-2}^n - 4u_{N_x-1}^n + u_{N_x}^n \right), \\
i = N_x - 1 \rightarrow \quad & \frac{1}{\Delta x^4} \left(u_{N_x-3}^n - 4u_{N_x-2}^n + 6u_{N_x-1}^n - 4u_{N_x}^n + u_{N_x+1}^n \right).
\end{aligned} \tag{25}$$

We do not consider $i = 0$ and $i = N_x$ due to the boundary conditions (3), i.e. $u_0^n = u_{N_x}^n = 0$. As it can be noticed, there are two grid points u_{-1}^n and $u_{N_x+1}^n$ out of the domain. These points are determined by applying the second order central finite difference approximations to the boundary conditions (3), i.e. $u_{xx}|_{x=0} = u_{xx}|_{x=l} = 0$,

$$\frac{u_1^n - 2u_0^n + u_{-1}^n}{\Delta t^2} = 0, \quad \frac{u_{N_x+1}^n - 2u_{N_x}^n + u_{N_x-1}^n}{\Delta t^2} = 0.$$

The above equation results in

$$u_{-1}^n = -u_1^n \quad \text{and} \quad u_{N_x+1}^n = -u_{N_x-1}^n. \tag{26}$$

By inserting (26) in (25), the expression (25) can be rewritten as a matrix-vector multiplication

$$\frac{1}{\Delta x^4}\begin{pmatrix} 5 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 5 \end{pmatrix}\begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \\ \vdots \\ u_{N_x-4}^n \\ u_{N_x-3}^n \\ u_{N_x-2}^n \\ u_{N_x-1}^n \end{pmatrix} = Du_i^n,$$

where $i \in [1, N_x - 1]$, and the matrix itself is denoted as D.

3.3 Time-dependent unknown coefficient

Now, let us construct a predicting mechanism to find the time-dependent unknown coefficient $q(t)$. First, multiplying both sides of Eq. (1) by $h(x)$, and integrating it over the domain of interest, from 0 to l, we obtain

$$\int_0^l h(x)u_{tt}(x,t)dx + a^2\int_0^l h(x)u_{xxxx}(x,t)dx + q(t)\int_0^l h(x)u(x,t)dx = \int_0^l h(x)f(x,t)dx.$$

In addition, employing the overdetermination condition (4), the above integral form is reduced to

$$g_{tt}(t) + a^2\int_0^l h(x)u_{xxxx}(x,t)dx + q(t)g(t) = \int_0^l h(x)f(x,t)dx. \tag{27}$$

The integral in the left-hand side of Eq. (27) has been rewritten in terms of $u(x,t)$ by using the integration by parts (four times) and boundary conditions (3), resulting in

$$\int_0^l h(x)u_{xxxx}(x,t)dx = \int_0^l h_{xxxx}(x)u(x,t)dx.$$

From Eq. (27), $q(t)$ is calculated as

$$q(t) = \frac{1}{g(t)}\left(\int_0^l h(x)f(x,t)dx - g_{tt}(t) - a^2\int_0^l h_{xxxx}(x)u(x,t)dx\right).$$

We further calculate $q(t)$ numerically for each time step by applying the second order finite difference approach for $g_{tt}(t)$ and trapezoidal rule to approximate the integrals, i.e. q^n,

$$q^n = \frac{1}{g^n}\left(F^n - \frac{g_i^{n+1} - 2g_i^n + g_i^{n-1}}{\Delta t^2} - a^2 H^n\right) \tag{28}$$

where

$$F^n \approx \int_0^l h(x) f(x,t) dx,$$
$$H^n \approx \int_0^l h_{xxxx}(x) u(x,t) dx.$$

The same finite difference differentiation matrix D, as in 25, has been used for the fourth derivative, i.e. $h_{xxxx}(x) \approx Dh_i$. Based on the equations (24) and (28), Algorithm 1 has been constructed for the simultaneous determination of $u(x,t)$ and the unknown coefficient $q(t)$, and implemented in open-source programming language Python.

Algorithm 1 Finite difference scheme for the evolution of $u(x,t)$ and the unknown coefficient $q(t)$

Require: $\Omega = [0,l]$, N_x, T, N_t, a, f_i^n, h_i, g^n
  $\Delta x = l/N_x$, $x_i = i\Delta x$, $N_x \leftarrow i$
  $\Delta t = T/N_t$, $t_n = n\Delta t$, $N_t \leftarrow n$
Ensure: $u_i^0 = \varphi_i$, $u_i^1 = \psi_i + \Delta t \varphi_i$, D
  $(g_{tt})^n$, Dh_i, F^0, H^0 ▷ Finite difference and trapezoidal approximations
  compute q^0 ▷ use Eq. (3.6)
  while $n \le N_t$ **do** ▷ n starts from 1
    F^n, H^n
    compute q^n ▷ use Eq. (3.6)
    $u_i^{n+1} = 2u_i^n - u_i^{n-1} + \Delta t^2 \left(f_i^n - q^n u_i^n - a^2 D u_i^n\right)$
    $u_0^{n+1} = 0$, $u_{N_x}^{n+1} = 0$ ▷ Enforce the boundary conditions
  end while

4 Results and discussions

4.1 Example

In this section, the numerical results, obtained by using Algorithm 1, are presented for the test example (30). The computational details have already been provided in Section 3. The results have been analyzed by calculating the absolute error between the exact and estimated solutions, defined as,

$$\begin{aligned} \eta(u) &= max_{1\le i\le N_x} |u_i^{numerical} - u_i^{exact}| \\ \eta(q_n) &= |q_n^{numerical} - q_n^{exact}| \end{aligned} \tag{29}$$

We solve the inverse problem (1)-(4) with the following input data:

$$\begin{aligned} &h(x) = \sin x, \quad \phi(x) = 4\sin x, \quad \psi(x) = 2\sin x, \quad g(t) = 2\pi(e^t + 1) \\ &f(x,t) = 2(e^t+1)e^t \sin x + 8(e^t+1)\sin x + 2e^t \sin x \end{aligned} \tag{30}$$

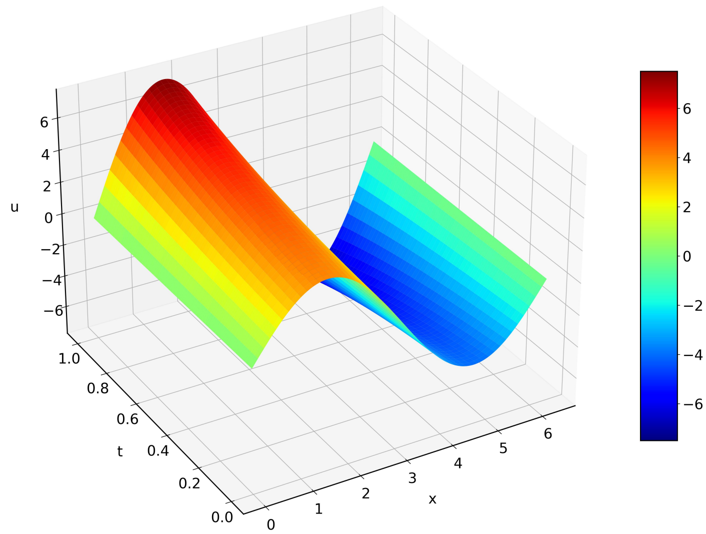


Figure 2: Evolution of the numerical solutions of $u(x,t)$ over time with N_x=300 grid points.

for $x \in (0, l = 2\pi)$ and $t \in (0, T = 1)$. The exact solution is given by

$$u(x,t) = 2(e^t + 1)\sin x \quad \text{and} \quad q(t) = e^t. \tag{31}$$

The one-dimensional domain was discretized with N_x=300 grid points with grid spacing of $\Delta x = l/N_x$ in the program. The time increment Δt between the time steps was set to 10^{-5}.

4.2 Numerical results

Figure 2 illustrates the numerical solution of $u(x,t)$ over time. As time evolves, the amplitudes of $u(x,t)$ increase, as shown in the color bar of the three-dimensional plot, Figure 2. It is analogous, looking at the analytical solution of $u(x,t)$,(31), if t gets higher the exponential e^t ascends. For clarity, the comparison of numerical and analytical solutions are presented in Figure 4, although, it is hard to distinguish numerical and analytical solutions due to the large number of grid points. Therefore, we have analyzed the relative errors (29) between the estimated and exact solutions for each time. The results are shown in Figure 3 for N_x=300 and N_x=600. The results of the time-dependent unknown coefficient $q(t)$ are represented in Figure 5 for N_x=300 and N_x=600 as well. As can be seen, although, this coefficient depends on time, it shows higher accuracy for larger number of grid points N_x. In Eq. (28), $q(t)$ contains two integrals, and we have used the same grid spacing for the integral approximation as in the spacial derivative approximation. Therefore, as grid spacing gets finer, the accuracy of the numerical results improves. Figure 6 shows that the relative error decreases linearly as the total number of grid points increases.

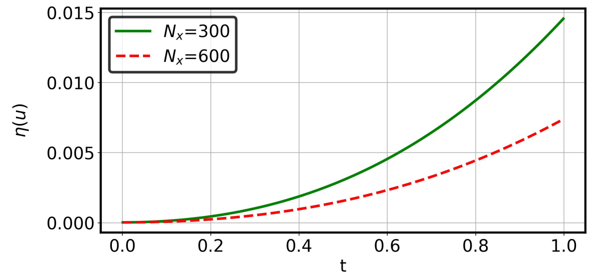


Figure 3: The relative error between the estimated and exact solutions of $u(x,t)$ for each time step. The error increases exponentially as time evolves.

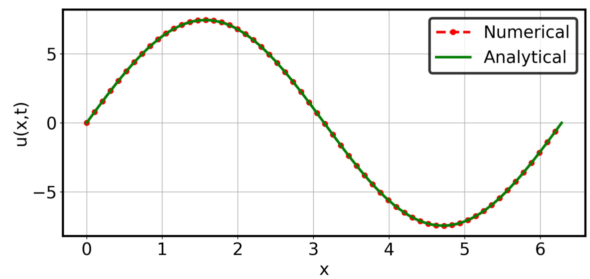


Figure 4: Comparison of the analytical and numerical solutions of $u(x, t = T)$.

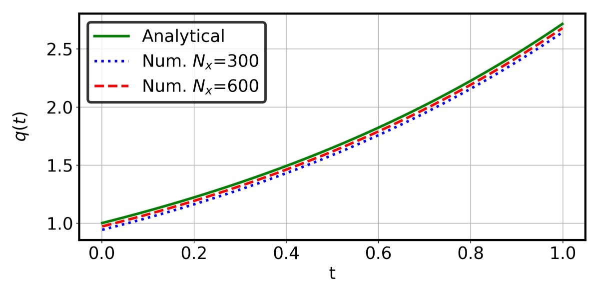


Figure 5: Comparison of the analytical and numerical solutions of $q(t)$.

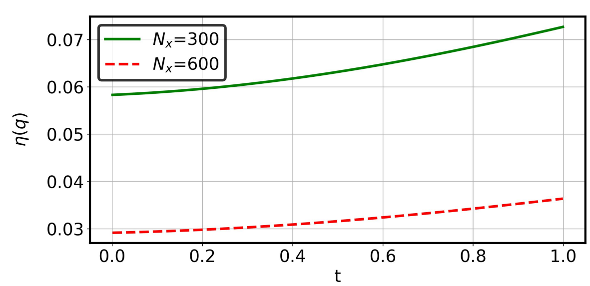


Figure 6: The absolute error between the numerical and analytical solutions of $q(t)$.

4.3 Numerical results under noise

In an inverse problem, the goal is to determine the input parameters or causes that produced a particular set of observed output data. In many cases, this is a difficult task due to the presence of measurement errors, model uncertainties, and other sources of noise. Adding noise to an inverse problem can be beneficial because it can help to regularize the problem and prevent overfitting. Without noise, the solution to an inverse problem is very sensitive to small variations in the input data, and may not generalize well to new or noisy data. By adding noise to the data, the inverse problem becomes more well-posed, meaning that it has a unique and stable solution that is less sensitive to small perturbations in the input data. Therefore, the proposed algorithm is tested by adding noise to the additional condition (4), for example:

$$\int_0^l u(x,t)h(x)dx = g(t)\left(1+\frac{P}{100}\xi\right),$$

where P - error in percantage, ξ - a random number from a uniform distribution over (-1,1).

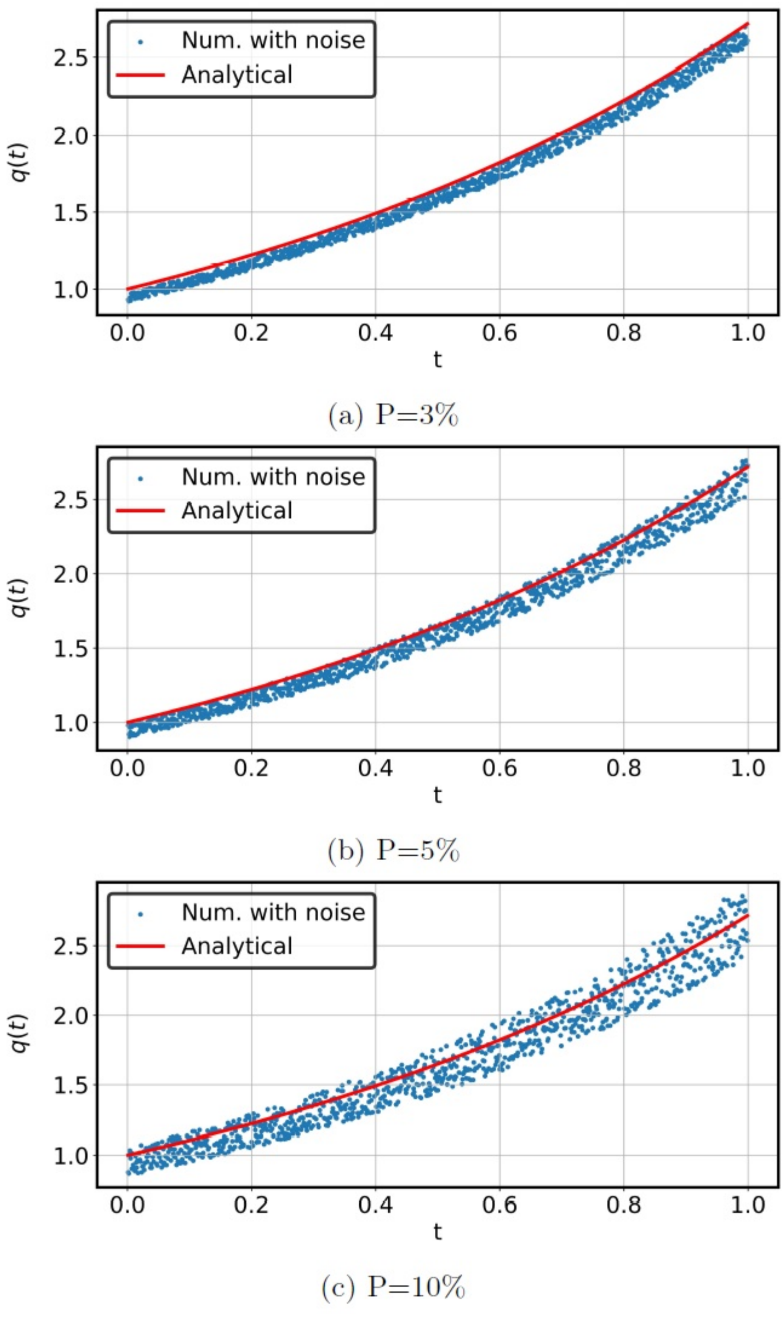
Based on the results, Figure 7 shows two plots: one representing the analytical solution to $q(t)$ and the other representing the numerical solution to $q(t)$ with added noise, where P is 3%, 5% and 10%. The two plots are overlaid on each other, with the analytical solution shown as a smooth curve and the numerical solution with noise shown as a series of fluctuating data points. It is important to note that the level of error P can have a significant impact on the accuracy and stability of the simulation results. The 3% error (Figure 7a) would represent a relatively small level of uncertainty, while the 10% error (Figure 7c) would represent a much larger level of uncertainty. In general, higher levels of error will lead to more variability in the results, and may make it more difficult to identify trends or patterns in the data. On the other hand, lower levels of error may not accurately reflect the amount of uncertainty that is present in the real-world data, and may lead to overly confident or biased conclusions.

5 Conclusion

In this study, we have presented a methodology for solving the inverse problem of determining the time-dependent beam stiffness coefficient $q(t)$ and transverse bending vibrations $u(x,t)$ of a homogeneous beam using the finite difference method. The results obtained demonstrate the accuracy and agreement of the numerical solutions with analytical solutions, even in the presence of noise. The additional condition in the form of an integral is shown to be effective in determining $q(t)$, and the effect of discretization grid points on the accuracy of the solutions is also analyzed.

The methodology presented in this study can have important practical applications in various fields of engineering, such as the design and analysis of structural systems, and the prediction of dynamic response of structures to external loads or environmental changes. The use of noise in the numerical analysis can also help to simulate measurement errors and other sources of uncertainty, which is an important consideration in real-world applications.

Overall, the methodology presented in this study provides a powerful tool for solving inverse problems in the presence of a time-dependent stiffness coefficient, and can be adapted to a wide range of problems in engineering and other fields. Further research can explore the use of other numerical methods or optimization algorithms to improve



(a) P=3%

(b) P=5%

(c) P=10%

Figure 7: Numerical solutions of $q(t)$ obtained with noise.

the accuracy and efficiency of the solution, and to address other sources of uncertainty or variability in the system.

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Received 18.09.2023, revised 12.12.2023, Accepted 15.12.2023