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A REGULARIZED FRICTIONAL CONTACT PROBLEM WITH UNILATERAL CONSTRAINTS

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Abstract We consider a mathematical model which describes the equilibrium of an elastic body in contact with a deformable foundation. We describe the contact model and we establish the existence of a unique weak solution to the problem. Then, we prove the continuous dependence of the solution with respect to the data. Finally, we introduce a sequence of regularized problems depending on a positive parameter for which we study the convergence when the regularization parameter is very small.

Key words: Coulomb's law, deformable foundation, frictional contact, lower semicontnuity, regularized problem.

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1 Introduction

Variational inequalities have been an interesting field of study for a long time. The reason for this is the intensive development of applications of variational inequality in many areas of engineering and economics field such as solid and fluid mechanics and equilibrium problems. References in this subject are [1, 2, 6, 7, 9, 15, 21, 22, 24], for instance. Numerical analysis of variational inequalities, including solution algorithms, error estimates and numerical simulations, can be found in [4, 8, 10, 23, 25]. Some results on optimal control of variational inequalities can be found in [5, 11, 14, 18, 19, 20].

In this paper we consider a mathematical model which describes the contact between an elastic body and an obstacle. We assume that the foundation is made of a rigid material covered by a deformable layer of thickness k and yield limit β. The body is acted upon by body forces of density $\boldsymbol{\varphi}_0$ and by tractions of density $\boldsymbol{\varphi}_2$, which act on a part of its boundary. The variational formulation of the model is in a form of an elliptic variational inequality with unilateral constraints in which the data are the density of applied forces $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_2)$, the thickness k, the yield limit β and the friction bound ξ. The aim of this paper is to study the continuous dependence of the solution with respect to the data. Also, we introduce a regularized variational problem depending on a positive parameter ρ, whose solution converges to the weak solution of our contact problem when ρ tends to zero.

The paper is structured as follows. In Section 2 we describe the contact model and prove its unique weak solvability, Theorem 2.1. In Section 3 we study the dependence of the solution with respect to the density of applied forces, the unilateral constraints, the yield limit of the obstacle and the friction bound, Theorem 3.1. Finally, in Section 4 we introduce and analyze a regularized problem and prove a convergence result.

2 The contact model

We consider an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, composed with three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is fixed on Γ_1 and surface tractions of density $\boldsymbol{\varphi}_2$ is acted on Γ_2. On Γ_3, the body is in frictional contact with a rigid-deformable obstacle, the so-called foundation. Denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. Then, the classical formulation of the contact problem is as follows.

Problem $\mathcal{Q}$. *Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{\varphi}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{\varphi}_2 \qquad \text{on} \quad \Gamma_2, \tag{2.4}$$

$$\left.\begin{aligned} & u_\nu \le k, \sigma_\nu + \pi \le 0, \\ & (u_\nu - k)(\sigma_\nu + \pi) = 0, \\ & 0 \le \pi \le \beta, \quad \pi = \beta\, \frac{u_\nu^+}{|u_\nu|} \quad \text{if} \quad u_\nu \ne 0 \end{aligned}\right\} \qquad \text{on} \quad \Gamma_3, \tag{2.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le \xi, \quad \boldsymbol{\sigma}_\tau = -\xi\, \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if} \quad \boldsymbol{u}_\tau \ne 0 \qquad \text{on} \quad \Gamma_3. \tag{2.6}$$

Now, we give a short description of the equations and boundary conditions of Problem $\mathcal{Q}$. First, equation (2.1) represents the elastic constitutive law of the material in which $\mathcal{A}$ is the elasticity operator. Equation (2.2) is the equation of equilibrium in which $\boldsymbol{\varphi}_0$ denotes the density of body forces. Conditions (2.3), (2.4) represent the displacement and traction boundary conditions, respectively.

Condition (2.5) is the contact condition with unilateral constraints which models the contact with a foundation made of a rigid body covered by a layer made of rigid-elastic material. Here and below, u_ν, $\boldsymbol{u}_\tau$ represent the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. Also, σ_ν and $\boldsymbol{\sigma}_\tau$ represent the normal and tangential stress on Γ, that is, $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$.

Now, we present a description of the contact condition (2.5). Here $k > 0$, is a given bound which represents the thickness of the deformable layer, β is a given function which represents the yield limit of this layer and r^+ denotes the positive part of r, i.e., $r = \max\{r, 0\}$. It can be derived in the following way. First, we assume that the penetration is limited by the bound k and, therefore, the normal displacement satisfies the following inequality

$$u_\nu \le k \quad \text{on} \quad \Gamma_3. \tag{2.7}$$

Next, we assume that σ_ν has the following additive decomposition

$$\sigma_\nu = \sigma_\nu^1 + \sigma_\nu^2 \quad \text{on} \quad \Gamma_3, \tag{2.8}$$

where σ_ν^1 describes the reaction of the deformable layer and σ_ν^2 describes the reaction of the rigid body. We assume that σ_ν^1 satisfies the condition

$$0 \le -\sigma_\nu^1 \le \beta, \quad -\sigma_\nu^1 = \begin{cases} \beta & \text{if} \quad 0 < u_\nu, \\ 0 & \text{if} \quad u_\nu < 0, \end{cases} \tag{2.9}$$

on Γ_3. Next, we assume that the part σ_ν^2 satisfies the Signorini condition in the form with the gap k, i.e.,

$$\sigma_\nu^2 \le 0, \quad \sigma_\nu^2(u_\nu - k) = 0 \quad \text{on} \qquad \Gamma_3. \tag{2.10}$$

The Signorini contact conditions were considereded, for example, in [3, 4]. We denote $-\sigma_\nu^1 = \pi$ and we use (2.8) to see that

$$\sigma_\nu^2 = \sigma_\nu + \pi \quad \text{on} \quad \Gamma_3.$$

Then, we substitute this equality in (2.10) and use (2.7), (2.9) to obtain the contact condition (2.5).

Finally, (2.6) represents the contact with Coulomb's friction law where ξ is a given friction bound. Frictional contact problems, where considered, for example in [12, 13, 16, 18].

To provide the variational analysis of Problem $\mathcal{Q}$, we need some notation and preliminaries material. Here and below, the indices i, j run from 1 to d and the summation convention over repeated indices is used. Moreover, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x} = (x_i)$. Also, $\boldsymbol{\varepsilon}$ and Div denote the deformation and the divergence operators, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

We recall that the inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{v} &= u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma}\cdot\boldsymbol{\tau} &= \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Everywhere in this paper, we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $H = L^2(\Omega)^d$, $H_2 = L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. For an element $\boldsymbol{v} \in H^1(\Omega)^d$ we sometimes write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ. In addition, we consider the following spaces

$$\begin{aligned} V &= \{\, \boldsymbol{v} \in H^1(\Omega)^d :\ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\}, \\ Q &= \{\, \boldsymbol{\sigma} = (\sigma_{ij}) :\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,\}. \end{aligned}$$

The spaces V and Q are real Hilbert spaces equipped with the inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx.$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Also, recall that the completeness of the space V follows from the assumption $meas\,(\Gamma_1)>0$ which allows the use of Korn's inequality.

We denote by $\mathbf{0}_V$ the zero element of V and, for a regular stress function $\boldsymbol{\sigma}$, the following Green's formula holds

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \quad \forall \boldsymbol{v}\in H^1(\Omega)^d. \tag{2.11}$$

We also recall that there exists $c_{tr}>0$ which depends on Ω and Γ_1 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \le c_{tr}\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v}\in V. \tag{2.12}$$

Inequality (2.12) represents a consequence of the Sobolev trace theorem.

In the study of the mechanical problem (2.1)−−(2.6), we assume that the elasticity operator $\mathcal{A}$ satisfies the following conditions

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\
\qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\
\qquad\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c) There exists } m_{\mathcal{A}}>0 \text{ such that}\\
\qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\
\qquad\quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\mathbf{0}) \text{ belongs to } Q.
\end{array}\right. \tag{2.13}$$

We also assume that the densities of body forces and tractions satisfy

$$\boldsymbol{\varphi}_0\in H, \tag{2.14}$$
$$\boldsymbol{\varphi}_2\in H_2. \tag{2.15}$$

Finally, the yield limit β, the friction bound ξ and the thickness k satisfy the following conditions

$$\beta\in L^2(\Gamma_3), \qquad \beta(\boldsymbol{x})\ge 0 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_3, \tag{2.16}$$
$$\xi\in L^2(\Gamma_3), \qquad \xi(\boldsymbol{x})\ge 0 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_3, \tag{2.17}$$
$$k>0. \tag{2.18}$$

Under these assumptions we introduce the set $U\subset V$, the operator $A:V\to V$ and the function $j:V\to\mathbb{R}$ defined by

$$U = \{ \boldsymbol{v} \in V : v_\nu \le k \ \text{ a.e. on } \ \Gamma_3 \}, \tag{2.19}$$

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{2.20}$$

$$j(\boldsymbol{v}) = (\beta, v_\nu^+)_{L^2(\Gamma_3)} + (\xi, \|\boldsymbol{u}_\tau\|)_{L^2(\Gamma_3)} \qquad \forall\, \boldsymbol{v} \in V. \tag{2.21}$$

Then, following a standard approach based on Green formula (2.11), we can derive the following variational formulation of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}_V$. *Find a displacement field* $\boldsymbol{u} \in U$ *such that*

$$\begin{aligned} &(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{v}) - j(\boldsymbol{u}) \\ &\quad \ge (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u})_{H_2} \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{2.22}$$

In the study of this problem, we have the following existence and uniqueness result.

Theorem 2.1. *Assume that* (2.13)–(2.18) *hold. Then, Problem* $\mathcal{Q}_V$ *has a unique solution* $\boldsymbol{u} \in U$.

Proof. First, we use the definition (2.19)to see that U is a nonempty, closed and convex subset of V.

Next, we use the definition (2.20)and assumption (2.13)(c) to see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \ge m_{\mathcal{A}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{2.23}$$

On the other hand, assumption (2.13)(b) implies that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \le L_{\mathcal{A}} \|\boldsymbol{u} - \boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{2.24}$$

We conclude from inequalities (2.23) and (2.24) that A is a strongly monotone Lipschitz continuous operator on the space V.

Moreover, using $(2.16) - -(2.17)$ and (2.12) it is easy to see that the functional j defined by (2.21) is a seminorm on the space V and, in addition,

$$j(\boldsymbol{v}) \le c_{tr}(\|\beta\|_{L^2(\Gamma_3)} + \|\xi\|_{L^2(\Gamma_3)})\|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V.$$

It follows from here that j is a continuous seminorm and, therefore, it is convex and lower semicontinuous.

Finally, by the Riesz representation theorem, we deduce that there exists a unique element $\boldsymbol{\varphi} \in V$ such that

$$(\boldsymbol{\varphi}, \boldsymbol{v})_V = (\boldsymbol{\varphi}_0, \boldsymbol{v})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v})_{H_2} \qquad \forall\, \boldsymbol{v} \in V.$$

Theorem 2.1 now is a direct consequence of Theorem 2.8 in [24]. □

3 Continuous dependence result

In this section, we study the dependence of the solution $\boldsymbol{u}$ of the variational inequality (2.22) with respect to the data $\boldsymbol{\varphi}_0$, $\boldsymbol{\varphi}_2$, β, ξ and k. For each $\epsilon > 0$, we consider a perturbation $\boldsymbol{\varphi}_{0\epsilon}$, $\boldsymbol{\varphi}_{2\epsilon}$, β_ϵ, ξ_ϵ and k_ϵ of $\boldsymbol{\varphi}_0$, $\boldsymbol{\varphi}_2$, β, ξ and k, respectively, which satisfy (2.14) − −(2.18). Next, we introduce the set $U_\epsilon \subset V$ and the functional $j_\epsilon : V \to \mathbb{R}$ defined by

$$U_\epsilon = \{\, \boldsymbol{v} \in V : v_\nu \leq k_\epsilon \ \text{ a.e. on } \ \Gamma_3 \,\}, \tag{3.1}$$

$$j_\epsilon(\boldsymbol{v}) = (\beta_\epsilon, v_\nu^+)_{L^2(\Gamma_3)} + (\xi_\epsilon, \|\boldsymbol{v}_\tau\|)_{L^2(\Gamma_3)} \qquad \forall\, \boldsymbol{v} \in V, \tag{3.2}$$

and, we consider the following variational problem.

Problem $\mathcal{Q}_V^\epsilon$. *Find a displacement field $\boldsymbol{u}_\epsilon \in U_\epsilon$ such that*

$$\begin{aligned}(A\boldsymbol{u}_\epsilon, \boldsymbol{v} - \boldsymbol{u}_\epsilon)_V + j_\epsilon(\boldsymbol{v}) - j_\epsilon(\boldsymbol{u}_\epsilon)\\ \geq (\boldsymbol{\varphi}_{0\epsilon}, \boldsymbol{v} - \boldsymbol{u}_\epsilon)_H + (\boldsymbol{\varphi}_{2\epsilon}, \boldsymbol{v} - \boldsymbol{u}_\epsilon)_{H_2} \quad \forall\, \boldsymbol{v} \in U_\epsilon.\end{aligned} \tag{3.3}$$

It follows from Theorem 2.1 that, for each $\epsilon > 0$, Problem $\mathcal{Q}_V^\epsilon$ has a unique solution $\boldsymbol{u}_\epsilon \in U_\epsilon$. The main result of this section is the following.

Theorem 3.1. *Assume that*

$$(\boldsymbol{\varphi}_{0\epsilon}, \boldsymbol{\varphi}_{2\epsilon}) \rightharpoonup (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_2) \quad \text{in} \quad H \times H_2 \quad \text{as} \ \ \epsilon \to 0, \tag{3.4}$$

$$(\xi_\epsilon, \beta_\epsilon) \to (\xi, \beta) \quad \text{in} \quad L^2(\Gamma_3)^2 \quad \text{as} \ \ \epsilon \to 0, \tag{3.5}$$

$$k_\epsilon \to k \quad \text{as} \ \ \epsilon \to 0. \tag{3.6}$$

Then, the solution $\boldsymbol{u}_\epsilon$ of Problem $\mathcal{Q}_V^\epsilon$ converges to the solution $\boldsymbol{u}$ of Problem $\mathcal{Q}_V$, i.e.,

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \quad \text{in} \quad V \quad \text{as} \ \ \epsilon \to 0. \tag{3.7}$$

The proof of Theorem 3.1 will be carried out in several steps. Let $\epsilon > 0$. We start by considering the intermediate problem of finding an element $\bar{\boldsymbol{u}}_\epsilon \in U$ such that the below inequality holds

$$\begin{aligned}(A\bar{\boldsymbol{u}}_\epsilon, \boldsymbol{v} - \bar{\boldsymbol{u}}_\epsilon)_V + j_\epsilon(\boldsymbol{v}) - j_\epsilon(\bar{\boldsymbol{u}}_\epsilon)\\ \geq (\boldsymbol{\varphi}_{0\epsilon}, \boldsymbol{v} - \bar{\boldsymbol{u}}_\epsilon)_H + (\boldsymbol{\varphi}_{2\epsilon}, \boldsymbol{v} - \bar{\boldsymbol{u}}_\epsilon)_{H_2} \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{3.8}$$

It follows from Theorem 2.1 that the inequality (3.8) has a unique solution $\bar{\boldsymbol{u}}_\epsilon \in U$.

The first step is provided by the following weak convergence result.

Lemma 3.1. *The sequence $\{\bar{\boldsymbol{u}}_\epsilon\}$ converges weakly in V to $\boldsymbol{u}$, that is*

$$\bar{\boldsymbol{u}}_\epsilon \rightharpoonup \boldsymbol{u} \quad \text{in} \ \ V \quad \text{as} \ \ \epsilon \to 0. \tag{3.9}$$

Proof. Let $\epsilon > 0$. We test in (3.8)with $\boldsymbol{v} = \mathbf{0}_V$ to obtain that

$$(A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon)_V + j_\epsilon(\bar{\boldsymbol{u}}_\epsilon) \leq (\boldsymbol{\varphi}_{0\epsilon}, \bar{\boldsymbol{u}}_\epsilon)_H + (\boldsymbol{\varphi}_{2\epsilon}, \bar{\boldsymbol{u}}_\epsilon)_{H_2}.$$

We now write $A\bar{\boldsymbol{u}}_\epsilon = A\bar{\boldsymbol{u}}_\epsilon - A\mathbf{0}_V + A\mathbf{0}_V$ and we use the property (2.23) and the positivity of the functional j_ϵ to see that

$$m_{\mathcal{A}}\|\bar{\boldsymbol{u}}_\epsilon\|_V \leq \|\boldsymbol{\varphi}_{0\epsilon}\|_H + \|\boldsymbol{\varphi}_{2\epsilon}\|_{H_2} + \|A\mathbf{0}_V\|_V.$$

On the other hand, we use the convergence (3.4) to deduce that the sequences $\{\boldsymbol{\varphi}_{0\epsilon}\}$, $\{\boldsymbol{\varphi}_{2\epsilon}\}$ are bounded in H and H_2, respectively. Therefore, there exists a constant $C > 0$, which does not depend on ϵ, such that

$$\|\bar{\boldsymbol{u}}_\epsilon\|_V \leq C. \tag{3.10}$$

Thus, a standard compactness argument implies that there exists an elements $\bar{\boldsymbol{u}} \in V$ such that, passing to a subsequence, again denoted $\{\bar{\boldsymbol{u}}_\epsilon\}$, we have

$$\bar{\boldsymbol{u}}_\epsilon \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } V \quad \text{as } \epsilon \to 0. \tag{3.11}$$

We shall prove the equality $\bar{\boldsymbol{u}} = \boldsymbol{u}$. To this end, we recall that U is a closed convex subset of the space V and, therefore, the property $\{\bar{\boldsymbol{u}}_\epsilon\} \subset U$ and the convergence (3.11) imply that

$$\bar{\boldsymbol{u}} \in U. \tag{3.12}$$

Let $\epsilon > 0$. We test in (3.8) with $\boldsymbol{v} = \bar{\boldsymbol{u}} \in U$ to see that

$$(A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon - \bar{\boldsymbol{u}})_V \leq (\boldsymbol{\varphi}_{0\epsilon}, \bar{\boldsymbol{u}}_\epsilon - \bar{\boldsymbol{u}})_H + (\boldsymbol{\varphi}_{2\epsilon}, \bar{\boldsymbol{u}}_\epsilon - \bar{\boldsymbol{u}})_{H_2} + j_\epsilon(\bar{\boldsymbol{u}}) - j_\epsilon(\bar{\boldsymbol{u}}_\epsilon),$$

then we pass to the upper limit as $\epsilon \to 0$ in this inequality, using the convergences (3.4) − −(3.5), (3.11) and the compactness of the trace operator, we obtain that

$$\limsup_{\epsilon\to 0}\, (A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon - \bar{\boldsymbol{u}})_V \leq 0.$$

Therefore, using (2.23) − −(2.24), the convergence (3.11) and standard arguments on pseudomonotone operators, we deduce that

$$\liminf_{\epsilon\to 0}\, (A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{v})_V \geq (A\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} - \boldsymbol{v})_V \qquad \forall\, \boldsymbol{v} \in U. \tag{3.13}$$

On the other hand, using inequality (3.8)and the convergences (3.4)− −(3.5), (3.11) and the compactness of the trace operator, we obtain that, for all $\boldsymbol{v} \in U$,

$$\liminf_{\epsilon\to 0}\, (A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{v})_V \leq (\boldsymbol{\varphi}_0, \bar{\boldsymbol{u}} - \boldsymbol{v})_H + (\boldsymbol{\varphi}_2, \bar{\boldsymbol{u}} - \boldsymbol{v})_{H_2} + j(\boldsymbol{v}) - j(\bar{\boldsymbol{u}}).$$

We combine now this inequality and (3.13) to see that, for all $\boldsymbol{v} \in U$,

$$(A\bar{\boldsymbol{u}}, \boldsymbol{v} - \bar{\boldsymbol{u}})_V + j(\boldsymbol{v}) - j(\bar{\boldsymbol{u}}) \geq (\boldsymbol{\varphi}_0, \boldsymbol{v} - \bar{\boldsymbol{u}})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \bar{\boldsymbol{u}})_{H_2}. \tag{3.14}$$

Finally, it follows from (3.12) and (3.14) that $\bar{\boldsymbol{u}}$ is a solution of inequality (2.22) and, by the uniqueness of the solution of this inequality, guaranteed by Theorem 2.1, we deduce that the equality $\bar{\boldsymbol{u}} = \boldsymbol{u}$ holds. By applying a standard compactness argument, we obtain that the whole sequence $\{\bar{\boldsymbol{u}}_\epsilon\}$ converges weakly in V to $\boldsymbol{u}$ as $\epsilon \to 0$, which concludes the proof. □

We proceed with the following strong convergence result.

Lemma 3.2. *The sequence* $\{\bar{\boldsymbol{u}}_\epsilon\}$ *converges strongly in* V *to* $\boldsymbol{u}$*, that is*

$$\bar{\boldsymbol{u}}_\epsilon \to \boldsymbol{u} \quad \text{in } V \quad \text{as } \epsilon \to 0. \tag{3.15}$$

Proof. Let $\epsilon > 0$. We write inequality (3.8) with $\boldsymbol{v} = \boldsymbol{u} \in U$ and we use the strong monotonicity of the operator A to see that

$$m_{\mathcal{A}} \|\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}\|_V^2 \leq (A\bar{\boldsymbol{u}}_\epsilon - A\boldsymbol{u}, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_V$$
$$= (A\bar{\boldsymbol{u}}_\epsilon, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_V - (A\boldsymbol{u}, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_V$$
$$\leq (\boldsymbol{\varphi}_{0\epsilon}, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_H + (\boldsymbol{\varphi}_{2\epsilon}, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_{H_2} + j_\epsilon(\boldsymbol{u}) - j_\epsilon(\bar{\boldsymbol{u}}_\epsilon) - (A\boldsymbol{u}, \bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u})_V.$$

We now pass to the limit in this inequality and use the convergences (3.4)−−(3.5), (3.9) and the compactness of the trace operator to deduce that

$$\|\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}\|_V \to 0 \quad \text{as } \epsilon \to 0,$$

which concludes the proof. □

We now have all the ingredients to provide the proof of Theorem 3.1.

Proof. Let $\epsilon > 0$. We denote by $\kappa = \dfrac{k}{k_\epsilon}$ and $\kappa_\epsilon = \dfrac{k_\epsilon}{k}$. By using the definitions (2.19)and(3.1) of the sets U and U_ϵ, respectively, we have that $\kappa\boldsymbol{u}_\epsilon \in U$ and $\kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon \in U_\epsilon$.

Now, we take $\boldsymbol{v} = \kappa\boldsymbol{u}_\epsilon \in U$ in (3.8) and multiply the resulting inequality by κ_ϵ to obtain that

$$\begin{aligned}(A\bar{\boldsymbol{u}}_\epsilon, \boldsymbol{u}_\epsilon - \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon)_V + \kappa_\epsilon j_\epsilon(\kappa\boldsymbol{u}_\epsilon) - \kappa_\epsilon j_\epsilon(\bar{\boldsymbol{u}}_\epsilon)&\\ \geq (\boldsymbol{\varphi}_{0\epsilon}, \boldsymbol{u}_\epsilon - \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon)_H + (\boldsymbol{\varphi}_{2\epsilon}, \boldsymbol{u}_\epsilon - \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon)_{H_2}&\end{aligned} \tag{3.16}$$

On the other hand, we take $\boldsymbol{v} = \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon \in U_\epsilon$ in (3.3) to find that

$$\begin{aligned}(A\boldsymbol{u}_\epsilon, \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}_\epsilon)_V + j_\epsilon(\kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon) - j_\epsilon(\boldsymbol{u}_\epsilon)&\\ \geq (\boldsymbol{\varphi}_{0\epsilon}, \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}_\epsilon)_H + (\boldsymbol{\varphi}_{2\epsilon}, \kappa_\epsilon\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}_\epsilon)_{H_2}&\end{aligned} \tag{3.17}$$

Next, adding the inequalities (3.16) − −(3.17) and using the linearity property of the integral, we obtain that

$$0 \leq (A\bar{\boldsymbol{u}}_\epsilon - A\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon - \kappa_\epsilon\,\bar{\boldsymbol{u}}_\epsilon)_V.$$

Next, we use (3.10) and the properties (2.23) − −(2.24) to obain that

$$\|\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}_\epsilon\|_V \leq \frac{L_{\mathcal{A}} C |1 - \kappa_\epsilon|}{m_{\mathcal{A}}}. \tag{3.18}$$

Therefore, by using the convergence (3.6) we deduce that

$$\|\bar{\boldsymbol{u}}_\epsilon - \boldsymbol{u}_\epsilon\|_V \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.19}$$

Finally, we use the triangle inequality, the convergences (3.15)and(3.19) to obtain that the convergence (3.7) holds, which concludes the proof of Theorem 3.1. □

In addition to the mathematical interest in the convergence result in Theorem 3.1, it is important from mechanical point of view, since it states that the weak solution of Problem $\mathcal{Q}$ depends continuously on the data.

4 A regularized problem

In this section, we introduce a regularized problem by replacing the functional j, given by (2.21), with a sequence of functions more regular. We investigate the unique solvability of the regularized problem and we establish the convergence of the sequence of the solutions when the regularization parameter ρ tends to zero.

Let $\rho > 0$. We define the functional $j_\rho : V \to \mathbb{R}$ as follows

$$j_\rho(\boldsymbol{v}) = \int_{\Gamma_3} \beta\phi_{1\rho}(v_\nu)\, da + \int_{\Gamma_3} \xi\phi_{2\rho}(\|\boldsymbol{v}_\tau\|)\, da \qquad \forall\ \boldsymbol{v} \in V, \tag{4.1}$$

where $\phi_{1\rho} : \mathbb{R} \to \mathbb{R}_+$ and $\phi_{2\rho} : \mathbb{R} \to \mathbb{R}_+$ are the differentiable functions defined by

$$\phi_{1\rho}(x) = \begin{cases} \sqrt{x^2+\rho^2} - \rho & \text{if } \ x > 0, \\ 0 & \text{if } \ x \le 0, \end{cases} \tag{4.2}$$

and

$$\phi_{2\rho}(x) = \sqrt{x^2+\rho^2} - \rho \qquad \forall x \in \mathbb{R}, \tag{4.3}$$

and, we state the following variational problem.

Problem $\mathcal{Q}_V^\rho$. *Find a displacement field $\boldsymbol{u}_\rho \in U$ such that*

$$\begin{aligned}&(A\boldsymbol{u}_\rho, \boldsymbol{v} - \boldsymbol{u}_\rho)_V + j_\rho(\boldsymbol{v}) - j_\rho(\boldsymbol{u}_\rho) \\ &\qquad \ge (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u}_\rho)_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u}_\rho)_{H_2} \qquad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{4.4}$$

We have the following existence and uniqueness result.

Theorem 4.1. *Assume that* (2.13)–(2.18) *hold. Then, Problem $\mathcal{Q}_V^\rho$ has a unique solution $\boldsymbol{u}_\rho \in U$.*

Proof. Let $\rho > 0$. Note that for all $\boldsymbol{v} \in V$, we have

$$|\phi_{1\rho}(v_\nu)| \le |v_\nu|, \tag{4.5}$$

and

$$|\phi_{2\rho}(\|\boldsymbol{v}_\tau\|)| \le \|\boldsymbol{v}_\tau\|. \tag{4.6}$$

Indeed, consider $v_\nu \le 0$, it follows from definition (4.2) that inequality (4.5) is always satisfied. Next, if $v_\nu > 0$, we note that

$$\phi_{1\rho}(v_\nu) = \sqrt{v_\nu^2+\rho^2} - \rho = \frac{v_\nu^2}{\sqrt{v_\nu^2+\rho^2}+\rho} \quad \text{on} \quad \Gamma_3.$$

By using the following inequality

$$\frac{v_\nu}{\sqrt{v_\nu^2+\rho^2}+\rho} \le 1 \quad \text{on} \quad \Gamma_3,$$

we deduce that inequality (4.5) holds. Using similar arguments, we have that

$$\phi_{2\rho}(\|\boldsymbol{v}_\tau\|) = \sqrt{\|\boldsymbol{v}_\tau\|^2+\rho^2} - \rho = \frac{\|\boldsymbol{v}_\tau\|^2}{\sqrt{\|\boldsymbol{v}_\tau\|^2+\rho^2}+\rho} \quad \text{on} \quad \Gamma_3.$$

Next, by using the inequality

$$\frac{\|\boldsymbol{v}_\tau\|}{\sqrt{\|\boldsymbol{v}_\tau\|^2+\rho^2}+\rho} \le 1 \quad \text{on} \quad \Gamma_3,$$

we deduce that inequality (4.6) holds, too.

Considering now the functional j_ρ defined by (4.1) and using(2.12), (2.16) − −(2.17) and (4.5) − −(4.6), we see that j_ρ is a seminorm on V and, in addition, it satisfies

$$\begin{aligned} j_\rho(\boldsymbol{v}) &= \int_{\Gamma_3} \beta\phi_{1\rho}(v_\nu)\, da + \int_{\Gamma_3} \xi\phi_{2\rho}(\|\boldsymbol{v}_\tau\|)\, da \\ &\le \int_{\Gamma_3} \beta|v_\nu| da + \int_{\Gamma_3} \xi\|\boldsymbol{v}_\tau\|\, da \\ &\le c_{tr}(\|\beta\|_{L^2(\Gamma_3)} + \|\xi\|_{L^2(\Gamma_3)})\|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v}\in V. \end{aligned}$$

It follows that j_ρ is a continuous seminorm and, therefore, it is a convex and lower semicontinuous function. Then, with the same arguments used in the proof of Theorem 2.1, Problem $\mathcal{Q}_V^\rho$ has a unique solution $\boldsymbol{u}_\rho \in U$. □

In the next part of this section, we deliver the following convergence result.

Theorem 4.2. *The solution of Problem $\mathcal{Q}_V^\rho$ converges to the solution of Problem $\mathcal{Q}_V$, that is*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad V \qquad \text{as} \quad \rho \to 0. \tag{4.7}$$

Proof. Let $\rho > 0$. We take $\boldsymbol{v} = \boldsymbol{u}_\rho \in U$ in (2.22) and $\boldsymbol{v} = \boldsymbol{u} \in U$ in (4.4), then we add the resulting inequalities and we use the strong monotonicity of the operator A to find that

$$\begin{aligned} & m_{\mathcal{A}}\, \|\boldsymbol{u}_\rho - \boldsymbol{u}\|_V^2 \le (A\boldsymbol{u}_\rho - A\boldsymbol{u}, \boldsymbol{u}_\rho - \boldsymbol{u})_V \\ & \le j(\boldsymbol{u}_\rho) - j(\boldsymbol{u}) + j_\rho(\boldsymbol{u}) - j_\rho(\boldsymbol{u}_\rho) \\ & \le |j_\rho(\boldsymbol{u}_\rho) - j(\boldsymbol{u}_\rho)| + |j_\rho(\boldsymbol{u}) - j(\boldsymbol{u})| \\ & \le |\int_{\Gamma_3} \beta\left(\phi_{1\rho}(u_{\rho\nu}) - u_{\rho\nu}^+\right)\, da| + |\int_{\Gamma_3} \xi\left(\phi_{2\rho}(\|\boldsymbol{u}_{\rho\tau}\|) - \|\boldsymbol{u}_{\rho\tau}\|\right)\, da| \\ & + |\int_{\Gamma_3} \beta\left(\phi_{1\rho}(u_\nu) - u_\nu^+\right)\, da| + |\int_{\Gamma_3} \xi\left(\phi_{2\rho}(\|\boldsymbol{u}_\tau\|) - \|\boldsymbol{u}_\tau\|\right)\, da|. \end{aligned} \tag{4.8}$$

Note that

$$|\phi_{1\rho}(u_\nu) - u_\nu^+| = \begin{cases} |\sqrt{u_\nu^2+\rho^2} - \rho - u_\nu| = \rho + u_\nu - \sqrt{u_\nu^2+\rho^2} & \text{if} \quad u_\nu > 0, \\ 0 & \text{if} \quad u_\nu \le 0, \end{cases}$$

on Γ_3. It is easy to see that

$$\rho + u_\nu - \sqrt{u_\nu^2+\rho^2} \le \rho \quad \text{on} \quad \Gamma_3,$$

which implies that

$$|\phi_{1\rho}(u_\nu) - u_\nu^+| \le \rho \quad \text{on} \quad \Gamma_3. \tag{4.9}$$

Similar arguments show that

$$|\phi_{1\rho}(u_{\rho\nu}) - u_{\rho\nu}^+| \le \rho \quad \text{on} \quad \Gamma_3. \tag{4.10}$$

Moreover, note that

$$\begin{aligned} |\phi_{2\rho}(\|\boldsymbol{u}_\tau\|) - \|\boldsymbol{u}_\tau\|| &= \rho + \|\boldsymbol{u}_\tau\| - \sqrt{\|\boldsymbol{u}_\tau\|^2 + \rho^2} \\ &\le \rho \quad \text{on} \quad \Gamma_3, \end{aligned} \tag{4.11}$$

and,

$$\begin{aligned} |\phi_{2\rho}(\|\boldsymbol{u}_{\rho\tau}\|) - \|\boldsymbol{u}_{\rho\tau}\|| &= \rho + \|\boldsymbol{u}_{\rho\tau}\| - \sqrt{\|\boldsymbol{u}_{\rho\tau}\|^2 + \rho^2} \\ &\le \rho \quad \text{on} \quad \Gamma_3. \end{aligned} \tag{4.12}$$

Finally, we combine (4.8) − −(4.12) to obtain that

$$\begin{aligned} \|\boldsymbol{u}_\rho - \boldsymbol{u}\|_V^2 &\le \frac{2\rho}{m_{\mathcal{A}}} \left(\int_{\Gamma_3} \beta \, da + \int_{\Gamma_3} \xi \, da \right) \\ &\le \frac{2\rho}{m_{\mathcal{A}}} \left(\|\beta\|_{L^2(\Gamma_3)} + \|\xi\|_{L^2(\Gamma_3)} \right) \sqrt{\text{meas}(\Gamma_3)}. \end{aligned}$$

It follows from this inequality that the convergence (4.7) holds, when $\rho \to 0$, which concludes the proof. □

The convergence (4.7) is important from the mechanical point of view, since it shows that the weak solution of the elastic frictional contact problem with unilateral constraints may be approached as closely as one wishes by the solution of an elastic frictional contact problem with unilateral constraints and regularized friction, with a sufficiently small regularization parameter.

References

[1] Baiocchi C. and Capelo A., *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley, Chichester, 1984.

[2] Capatina A., *Variational Inequalities and Frictional Contact Problems*, Advances in Mechanics and Mathematics, 31, Springer, New York, 2014.

[3] Cocu M., *Existence of solutions of Signorini problems with friction*, Int. J. Engng. Sci. 22 (1984), 567-581.

[4] Cocu M., Pratt E. and Raous M., *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng. Sci., 34 (1996), 783-798.

[5] Denkowski Z., Migorski S. and Ochal A., *A Class of Optimal Control Problems for Piezoelectric Frictional Contact Models*, Nonlinear Analysis: Real World Applications, 12 (2011) 1883-1895.

[6] Duvaut G. and Lions J.-L., *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[7] Eck C., Jarušek J. and Krbeč M., *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics 270, Chapman/CRC Press, New York, 2005.

[8] Glowinski R., *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[9] Hlaváček I., Haslinger J., Nečas J. and Lovíšek J., *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[10] Ionescu I. R. and Sofonea M., *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1993.

[11] Jiang C. J. and Zeng B., *Continuous dependence and optimal control for a class of variational-hemivariational inequalities*. Appl. Math. Optim., 82 (2020), 637-656 .

[12] Kaki L. A. and Denche M., *Variational analysis for some frictional contact problems*, Boletim da Sociedade Paranaense de Matemática, 38(7) (2020), 21-36.

[13] Klarbring A., Mikelič A. and Shillor M., *Frictional contact problems with normal compliance*, Int. J. Engng. Sci., 26 (1988), 811-832.

[14] Kimmerle S. J. and Moritz R., *Optimal control of an elastic tyre-damper system with road contact*, ZAMM 14 (2014), 875-876, Special Issue: 85th annual meeting of the international association of Applied Mathematics and Mechanics (GAMM), Erlangen 2014.

[15] Kinderlehrer D. and Stampacchia G., *An Introduction to Variational Inequalities and their Applications*, Classics in Applied Mathematics 31, SIAM, Philadelphia, 2000.

[16] Lerguet Z., Shillor M. and Sofonea M., *A frictional contact problem for an electro-viscoelastic body*, Electron. J. Differ. Eqn., 170 (2007), 1-16.

[17] Martins J. A. C. and Oden J. T., *Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws*, Nonlinear Analysis : Theory, Methods and Applications, 11 (1987), 407-428.

[18] Matei A. and Micu S., *Boundary optimal control for nonlinear antiplane problems*, Nonlinear Analysis: Theory, Methods and Applications, 74 (2011), 1641-1652.

[19] Mignot R. , Puel J.-P., *Optimal control in some variational inequalities*, SIAM J. Control Optim., 22 (1984) 466-476.

[20] Neittaanmaki P., Sprekels J. and Tiba D. *Optimization of elliptic systems: Theory and applications*, Springer monographs in mathematics New York, Springer, 2006.

[21] Panagiotopoulos P. D., *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[22] Shillor M., Sofonea M. and Telega J. J., *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. 655, Springer, Berlin, 2004.

[23] Sofonea M., Chau O. and Han W., *Analysis and approximation of a viscoelastic contact problem with slip dependent friction*, Dynam. Cont. Discr. Impuls. Syst., Series B, 8 (2001), 153-174.

[24] Sofonea M. and Matei A. , *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series 398, Cambridge University Press, 2012.

[25] Wriggers P., *Computational contact mechanics.* Chichester: John Wiley and Sons, 2002.

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