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# ON THE DYNAMIC FRICTIONLESS CONTACT PROBLEM WITH NORMAL COMPLIANCE IN THERMO-PIEZOELECTRICITY

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**Abstract** In this work, we study, from a variational and numerical point of view, a dynamic contact problem between a thermo-electro-viscoelastic body and an electrically and thermally conductive foundation. The normal compliance condition and a nonlinear regularized electrical and thermal conditions are employed to model the contact. We provide existence and uniqueness results of a weak solution to the model using adequate auxiliary problems, Gronwall's lemma, and Banach fixed point theorem. Finally, we introduce a fully discrete approximation using the finite element method and an Euler scheme. Then we derive some error estimates, leading to the linear convergence result under suitable additional regularity conditions.

**Key words:** Dynamic process; Thermo-piezoelectric; Normal compliance; Weak solvability; Banach fixed point theorem; Finite element method; Error estimate.

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## 1 Introduction

The study of contact problems involving thermopiezoelectric materials represents a multifaceted research area in the field of materials science and engineering. In these problems, we study the interactions that occur when two or more bodies come into contact, with one or more of these bodies composed of materials that exhibit both thermal, electrical, and mechanical aspects. This intersection of diverse material behaviors and physical forces presents both scientific challenges and promising opportunities for technological advancements. In this context, we delve into the complexities of contact problems for thermopiezoelectric materials, aiming to gain a deeper understanding of their behavior and harness their capabilities for innovative applications in various domains, including sensors, actuators, and energy harvesting systems. General characteristics of thermo-piezoelectricity can be found in [15, 18, 21].

In the literature, there are some mathematical results concerning the variational analysis for this kind of problems; see, for instance, [1, 3, 4] for static models that consider the effects of mechanical, electrical, and thermal interactions in frictional contacts. The mathematical models that describe quasistatic frictional contact with thermo-piezoelectric effects are already addressed in [5, 6, 7, 11, 12], and more recently in [8]. Numerical schemes and their error estimates for the aforementioned models were also discussed for both static and quasistatic cases in [1, 5, 6, 8].

The behavior of such materials when subjected to dynamic conditions can exhibit notable differences from their behavior under static and quasistatic conditions, and can lead to complex and challenging problems in engineering applications. Then, there is significant interest in extending the contact problem for thermo-piezoelectric materials to dynamic cases.

In this paper, we deal with a general model for the dynamic process of frictionless contact between a deformable body and an electrically and thermally conductive foundation. The material obeys a thermo-electro-viscoelastic constitutive law. Moreover, a normal compliance condition and a regularized electrical and thermal conditions are used to describe the contact. We derive a variational formulation of the problem which is in the form of a system coupling a nonlinear hyperbolic variational equation for the displacement field, a nonlinear parabolic variational equation for the temperature field, and a nonlinear variational equation for the electric potential field. After this, we establish the existence of a unique weak solution to the problem. Finally, we introduce the numerical analysis of the problem and derive error estimates for numerical approximations. Let us remark that, up to date, there is no work dealing with the numerical analysis of the dynamic contact problem arising in thermo-electro-viscoelasticity, and that represents the main novelty of this work.

The paper is organized as follows. In Section 2, we state the mechanical problem. In Section 3, we list the assumptions on the data and derive a variational formulation of the model. In Section 4, we state and prove an existence and uniqueness result based on the Banach fixed point theorem. Lastly, in Section 5, we analyze a fully discrete scheme for the problem. We use the finite element method to discretize the domain and a forward Euler scheme to discretize the time derivative. Additionally, error estimates related to the scheme are derived.

## 2 Setting of the problem

In this section, we give a classic formulation of the problem of a thermo-piezoelectric body in a dynamic process contact with a rigid conductive foundation and which moves in such a way that frictional heating occurs.

Let us consider a thermo-piezoelectric body which initially occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\Gamma = \partial\Omega$. Let $[0, T]$ be time interval of interest, where $T > 0$.

For the sake of simplifying notation, we denote by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively. We will not explicitly denote the dependence of various functions on $x$. Throughout this paper the indices $i, j, k, l$ run from 1 to $d$. The summation convention over repeated indices is used. The comma following an index denotes a partial derivative with respect to the corresponding component of the spatial variable, and a dot above a variable represents the time derivative.

Additionally, we use Div and div to represent the divergence operators for tensor and vector fields, respectively, that is

$$\mathrm{Div}\sigma = (\sigma_{ij,j}), \quad \text{and} \quad \mathrm{div}D = D_{i,i}.$$

We denote the space of second order symmetric tensors on $\mathbb{R}^d$ by $\mathbb{S}^d$. In addition, the symbols " $\cdot$ " and $|\cdot|$ are used to represent the canonical inner product and its associated norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ that is

$$u \cdot v = u_i v_i, \quad |v| = \sqrt{v \cdot v}, \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = \sqrt{\tau \cdot \tau}, \quad \forall \sigma, \tau \in \mathbb{S}^d.$$

We denote by $\nu$ the unit outward normal on boundary $\Gamma$ and we shall adopt the usual notation for normal and tangential components of vectors and tensors

$$u = u_\nu \nu + u_\tau, \quad u_\nu = u \cdot \nu \quad \text{and} \quad \sigma = \sigma_\nu \nu + \sigma_\tau, \quad \sigma_\nu = (\sigma\nu) \cdot \nu.$$

In the end to present our problem, we denote by $u : \Omega \times (0,T) \to \mathbb{R}^d$ the displacement field, $\theta : \Omega \times (0,T) \to \mathbb{R}$ the temperature field, $\varphi : \Omega \times (0,T) \to \mathbb{R}$ the electric potential, $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$ the stress tensor, $D : \Omega \times (0,T) \to \mathbb{R}^d$ the electric displacement field and $q : \Omega \times (0,T) \to \mathbb{R}^d$ the heat flux vector. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

We assume that the body is acted upon by a volume forces of density $f_0$, a volume electric charges of density $q_0$ and a heat source of constant strength $q_c$. As the process is assumed to be dynamic, we have the following equations of stress equilibrium, quasi-stationary electric field and heat conduction

$$\rho\ddot{u} - \text{Div}\sigma = f_0 \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$
$$\text{div}D = q_0 \quad \text{in} \quad \Omega \times (0,T), \tag{2}$$
$$\dot{\theta} + \text{div}q = \mathcal{R}(\dot{u}, \varphi) + q_c \quad \text{in} \quad \Omega \times (0,T). \tag{3}$$

Here a non negative function $\rho$ represents the mass density and the nonlinear function $\mathcal{R}$ describe the influence of the velocity field and the electric potential on the temperature. In [16] the following function was used

$$\mathcal{R}(\dot{u}, \varphi) = \mu |\nabla\varphi|^2 - \mathcal{M}_{ij}\theta_{ref}\frac{\partial \dot{u}_i}{\partial x_j},$$

where $\mu$ and $\mathcal{M}$ are the electrical conductivity coefficient and the thermal expansion tensor, respectively. $\theta$ is measured with respect to a reference absolute temperature $\theta_{ref}$.

We assume that the material is thermo-piezoelectric and obeys the following constitutive laws

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{F}\varepsilon(u) - \mathcal{E}^* E(\varphi) - \theta\mathcal{M} \quad \text{in} \quad \Omega \times (0,T), \tag{4}$$
$$D = \mathcal{E}\varepsilon(u) + \mathcal{B}E(\varphi) - \theta\mathcal{P} \quad \text{in} \quad \Omega \times (0,T), \tag{5}$$

where $\mathcal{A} = (\mathcal{A}_{ijkl})$, $\mathcal{F} = (\mathcal{F}_{ijkl})$, $\mathcal{E} = (\mathcal{E}_{ijk})$, $\mathcal{B} = (\mathcal{B}_{ij})$ and $\mathcal{P} = (\mathcal{P}_i)$ are respectively the viscosity tensor, the elasticity tensor, the piezoelectric tensor, the electric permittivity tensor and the pyroelectric tensor. Indeed, $E(\varphi) = -\nabla\varphi$ is the electric intensity vector. Notice also that $\mathcal{E}^*$ is the transpose of the tensor $\mathcal{E}$ given by

$$\mathcal{E}^* = (\mathcal{E}^*_{ijk}), \quad \text{where} \quad \mathcal{E}^*_{ijk} = \mathcal{E}_{kij}.$$

We express the heat flux using the Fourier law of heat conduction given by

$$q = -\mathcal{K}\nabla\theta \qquad \text{in} \quad \Omega \times (0,T). \tag{6}$$

where $\mathcal{K} = (\mathcal{K}_{ij})$ is the thermal conductivity tensor.

In the end to prescribe the mechanical and temperature boundary conditions, we divide $\Gamma$ into tree measurable disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ such that $meas(\Gamma_D) > 0$. We assume that the body is fixed on $\Gamma_D$ and a surfaces traction of density $f_N$ act on $\Gamma_N$, that is

$$u = 0 \qquad \text{on} \quad \Gamma_D \times (0,T), \tag{7}$$

$$\sigma\nu = f_N \qquad \text{on} \quad \Gamma_N \times (0,T). \tag{8}$$

We also assume that the temperature vanishes on $\Gamma_D \cup \Gamma_N$, that is

$$\theta = 0 \qquad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0,T). \tag{9}$$

On the other hand, for formulate the electrical boundary conditions we consider a partition of $\Gamma_D \cup \Gamma_N$ in to two measurable disjoint parts $\Gamma_a$ and $\Gamma_b$ such that $meas(\Gamma_a) > 0$. We assume that the electrical potential vanishes on $\Gamma_a$ and a surface electric charge of density $q_b$ is prescribed on $\Gamma_b$, that is

$$\varphi = 0 \qquad \text{on} \quad \Gamma_a \times (0,T), \tag{10}$$

$$D \cdot \nu = q_b \qquad \text{on} \quad \Gamma_b \times (0,T). \tag{11}$$

In the reference configuration, the body is in contact with a thermally and electrically conductive foundation along $\Gamma_C$, then the thermoelectric contact is described with the following regularized conditions(see [4, 17])

$$D \cdot \nu = \psi_e(u_\nu - g)\phi_L(\varphi - \varphi_F) \qquad \text{on} \quad \Gamma_C \times (0,T), \tag{12}$$

$$q \cdot \nu = \psi_c(u_\nu - g)\phi_L(\theta - \theta_F) \qquad \text{on} \quad \Gamma_C \times (0,T). \tag{13}$$

Here, $g$ denotes the gap function that characterizes the separation between the body and the foundation at the contact surface. $\theta_F$ and $\varphi_F$ are respectively the temperature and the potential of the foundation. $\psi_c$ and $\psi_e$ are the thermal conductance and the surface electrical conductivity functions, respectively. $\phi_L$ is the truncate function defined by

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad |s| \le L, \\ L & \text{if} \quad s > L. \end{cases}$$

Where $L$ is a large positive number.

We assume that the contact between the body and the foundation is frictionless, i.e. the tangential component of stress is zero on the contact surface.

$$\sigma_\tau = 0 \qquad \text{on} \quad \Gamma_C \times (0,T). \tag{14}$$

We employ the following normal compliance contact condition

$$-\sigma_\nu = p(u_\nu - g) \qquad \text{on} \quad \Gamma_C \times (0,T). \tag{15}$$

Where $p$ is a prescribed non negative function which vanishes when its argument is negative, i.e. $p(r) = 0$ for $r \leq 0$. General forms of the normal compliance contact condition can be found in [14, 19] and in the references therein. For instance, one possible formulation is

$$p(r) = \frac{1}{\epsilon} r^+,$$

where $\epsilon$ is a positive constant and $r^+ = \max\{0; r\}$.

Finally, we prescribe the following initials conditions

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0 \qquad \text{in} \quad \Omega. \tag{16}$$

where $u_0$, $v_0$ and $\theta_0$ are given functions.

Thus, given the aforementioned assumptions, the dynamic frictionless contact problem for thermo-piezoelectric materials can be formulated in the following classical manner.

**Problem (P)**. Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times (0,T) \to \mathbb{R}$ and a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$ such that: (1)-(16).

# 3 Variational Formulation

In this section, we derive a weak formulation of **Problem (P)**. To this end we need to introduce some notations and preliminaries.

We use standard notation for the Lebesgue and the Sobolev spaces associated with $\Omega$ and $\Gamma$. We use the notations $H$, $\mathbf{H}^1$ and $\mathcal{H}$ for the following spaces

$$H = [L^2(\Omega)]^d = \{v = (v_i) | v_i \in L^2(\Omega)\}, \quad \mathbf{H}^1 = [H^1(\Omega)]^d,$$
$$\mathcal{H} = \{\sigma \in \mathbb{S}^d | \sigma = (\sigma_{ij}); \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}.$$

The spaces $H$, $\mathbf{H}^1$ and $\mathcal{H}$ are real Hilbert spaces endowed with the inner products given by

$$(u,v)_H = \int_\Omega u_i v_i dx, \ \forall u, v \in H,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} dx, \ \forall \sigma, \tau \in \mathcal{H},$$

$$(u,v)_{\mathbf{H}^1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \ \forall u, v \in \mathbf{H}^1.$$

The associated norms in $H$, $\mathcal{H}$ and $\mathbf{H}^1$ are denoted by $|.|_H$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathbf{H}^1}$, respectively.

Under the space $H$ we define also a modified inner product given by

$$((u,v))_H = \int_\Omega \rho u_i v_i dx, \qquad \forall u, v \in H,$$

and the associated norm is denoted by $\|.\|_H$.

Keeping in mind (7), we introduce the closed subspace of $\mathbf{H}^1$

$$V = \{w \in \mathbf{H}^1 \mid w = 0 \ \text{ on } \Gamma_D\}.$$

Since $meas(\Gamma_D) > 0$, the following Korn's inequality holds: There exists $c_k > 0$ depending only on $\Omega$ and $\Gamma_D$ such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_k \|v\|_{\mathbf{H}^1}, \quad \forall v \in V. \tag{17}$$

We define over the space $V$ the inner product

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \forall u, v \in V.$$

and its associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ is equivalent on $V$ to the usual norm $\|.\|_{\mathbf{H}^1}$, therefore $(V, \|.\|_V)$ is a real Hilbert space.

For simplicity, for an element $\omega \in \mathbf{H}^1$, we still denote by $\omega$ its trace $\gamma(\omega)$ on $\Gamma$. By trace theorem, there exists a constant $c_0 > 0$ such that

$$\|w\|_{[L^2(\Gamma_C)]^d} \leq c_0 \|w\|_V, \quad \forall w \in V. \tag{18}$$

Next, for the temperature field and the electric potential we introduce the closed functions subspaces of $H^1(\Omega)$

$$Q = \{\eta \in H^1(\Omega) \mid \eta = 0 \quad \text{on} \quad \Gamma_D \cup \Gamma_N\},$$
$$W = \{\xi \in H^1(\Omega) \mid \xi = 0 \quad \text{on} \quad \Gamma_a\}.$$

Over $Q$ and $W$, we consider the following inner products

$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \qquad (\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H, \tag{19}$$

for all $\theta, \eta \in Q$ and $\varphi, \xi \in W$, and the associated norms

$$\|\eta\|_Q = |\nabla\eta|_H, \qquad \|\xi\|_W = |\nabla\xi|_H. \tag{20}$$

Since $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$, Friedrichs-Poincaré inequality holds, therefore, there exists a constants $c_{p1} > 0$ and $c_{p2} > 0$ such that

$$|\nabla\xi|_H \geq c_{p1} \|\xi\|_{H^1(\Omega)}, \quad \forall \xi \in W. \tag{21}$$

$$|\nabla\eta|_H \geq c_{p1} \|\eta\|_{H^1(\Omega)}, \quad \forall \eta \in Q, \tag{22}$$

It follows from (19)-(22) that $\|.\|_W$ and $\|.\|_Q$ are equivalents on $W$ and $Q$, respectively with $\|.\|_{H^1(\Omega)}$ and then $(W, \|.\|_W)$ and $(Q, \|.\|_Q)$ are real Hilbert spaces. Moreover, by trace theorem, there exists a constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|\xi\|_{L^2(\Gamma_C)} \leq c_1 \|\xi\|_W, \qquad \forall \xi \in W, \tag{23}$$

$$\|\eta\|_{L^2(\Gamma_C)} \leq c_2 \|\eta\|_Q, \qquad \forall \eta \in Q. \tag{24}$$

Finally, for a real Banach space $(\mathcal{Z}, \|.\|_{\mathcal{Z}})$, we denote by $\mathcal{Z}'$ the dual space of $\mathcal{Z}$ and we use the notation $(\cdot,\cdot)_{\mathcal{Z}'\times\mathcal{Z}}$ to represent the duality pairing between $\mathcal{Z}'$ and $\mathcal{Z}$. For $1 \leq p \leq \infty$ and $m = 1, 2, \ldots$ we use the usual notation for the spaces $L^p(0,T;\mathcal{Z})$ and $W^{m,p}(0,T;\mathcal{Z})$. We denote by $C(0,T;\mathcal{Z})$ and $C^1(0,T;\mathcal{Z})$ the space of continuous an continuously differentiable functions from $[0,T]$ to $\mathcal{Z}$ respectively.

For the study of the mechanical **Problem (P)** we list these assumptions on its data.

**(H1)** The viscosity and the elasticity tensors $\mathcal{A}, \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, The electric permittivity and the thermal conductivity tensors $\mathcal{B}, \mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy

1. The usual property of symmetry and boundedness

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij} = \mathcal{A}_{ijlk} \in L^\infty(\Omega),$$
$$\mathcal{F}_{ijkl} = \mathcal{F}_{jikl} = \mathcal{F}_{klij} = \mathcal{F}_{ijlk} \in L^\infty(\Omega),$$
$$\mathcal{B}_{ij} = \mathcal{B}_{ji} \in L^\infty(\Omega), \quad \mathcal{K}_{ij} = \mathcal{K}_{ji} \in L^\infty(\Omega).$$

2. There exists $m_\mathcal{A} > 0$, $m_\mathcal{F} > 0$, $m_\mathcal{B} > 0$ and $m_\mathcal{K} > 0$ such that

$$\mathcal{A}_{ijkl}\sigma_{ij}\sigma_{kl} \geq m_\mathcal{A} |\sigma|^2, \quad \mathcal{F}_{ijkl}\sigma_{ij}\sigma_{kl} \geq m_\mathcal{F} |\sigma|^2,$$
$$\mathcal{B}_{ij}\zeta_i\zeta_j \geq m_\mathcal{B} |\zeta|^2, \quad \mathcal{K}_{ij}\zeta_i\zeta_j \geq m_\mathcal{K} |\zeta|^2.$$

for all $\sigma \in \mathbb{S}^d$ and $\zeta \in \mathbb{R}^d$.

3. There exists $M_\mathcal{A} > 0$, $M_\mathcal{F} > 0$, $M_\mathcal{B} > 0$ and $M_\mathcal{K} > 0$ such that

$$M_\mathcal{A} = \sup_{ijkl} \|\mathcal{A}_{ijkl}\|_{L^\infty(\Omega)}, \quad M_\mathcal{F} = \sup_{ijkl} \|\mathcal{F}_{ijkl}\|_{L^\infty(\Omega)},$$

$$M_\mathcal{B} = \sup_{ij} \|\mathcal{B}_{ij}\|_{L^\infty(\Omega)}, \quad M_\mathcal{K} = \sup_{ij} \|\mathcal{K}_{ij}\|_{L^\infty(\Omega)}.$$

**(H2)** The thermal expansion tensor $\mathcal{M} : \Omega \times \mathbb{R} \to \mathbb{S}^d$, the pyroelectric tensor $\mathcal{P} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ and the piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfy

1. The usual property of symmetry and boundedness

$$\mathcal{M}_{ij} = \mathcal{M}_{ji} \in L^\infty(\Omega), \quad \mathcal{P}_i \in L^\infty(\Omega), \quad \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega).$$

2. There exists $M_{\mathcal{M}} > 0$, $M_{\mathcal{P}} > 0$ and $M_{\mathcal{E}} > 0$ such that

$$M_{\mathcal{M}} = \sup_{ij} \|\mathcal{M}_{ij}\|_{L^\infty(\Omega)}, \quad M_{\mathcal{P}} = \sup_{i} \|\mathcal{P}_i\|_{L^\infty(\Omega)}, \quad M_{\mathcal{E}} = \sup_{ijk} \|\mathcal{E}_{ijk}\|_{L^\infty(\Omega)}.$$

**(H3)** The function $\mathcal{R} : V \times W \to L^2(\Omega)$ satisfies that there exists $M_{\mathcal{R}} > 0$ such that

$$\|\mathcal{R}(\zeta_1, \xi_1) - \mathcal{R}(\zeta_2, \xi_2)\|_{L^2(\Omega)} \leq M_{\mathcal{R}} \left(\|\zeta_1 - \zeta_2\|_V + \|\xi_1 - \xi_2\|_W\right),$$

for all $\zeta_1, \zeta_2 \in V$ and $\xi_1, \xi_2 \in W$.

**(H4)** The normal compliance function $p : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies

1. There exists $L_p > 0$ such that

$$|p(x, \varsigma_1) - p(x, \varsigma_2)| \leq L_p |\varsigma_1 - \varsigma_2|, \quad \forall \varsigma_1, \varsigma_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C.$$

2. $x \mapsto p(x, \varsigma)$ is measurable on $\Gamma_C$ for all $\varsigma \in \mathbb{R}$.

**(H5)** The thermal conductance function $\psi_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ and the surface electrical conductivity function $\psi_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfy

1. For $\psi = \psi_c, \psi_e$,

$$\psi(x, \varsigma) = 0, \forall \varsigma \leq 0, \text{a.e.} x \in \Gamma_C.$$

2. For $\psi = \psi_c, \psi_e$, there exists $L_\psi > 0$ such that

$$|\psi(x, \varsigma_1) - \psi(x, \varsigma_2)| \leq L_\psi |\varsigma_1 - \varsigma_2|, \ \forall \varsigma_1, \varsigma_2 \in \mathbb{R} \text{and} x \in \Gamma_C.$$

3. For $\psi = \psi_c, \psi_e$, the function $x \mapsto \psi(x, \varsigma)$ is measurable on $\Gamma_C$ for all $\varsigma \in \mathbb{R}$.

**(H6)** The forces, the traction, the volume, the surfaces charge densities and the strength of the heat source satisfy

$$\begin{aligned} &f_0 \in L^2(0, T; H), \quad f_N \in L^2(0, T; [L^2(\Gamma_N)]^d), \\ &q_0 \in L^2(0, T; L^2(\Omega)), \quad q_b \in L^2(0, T; L^2(\Gamma_b)), \\ &q_c \in L^2(0, T; L^2(\Omega)). \end{aligned}$$

**(H7)** The initial conditions, the gap, the potential and the temperature of the foundation and the mass density functions satisfy

$$\begin{aligned} u_0 \in V, \quad v_0 \in H, \quad \theta_0 \in L^2(\Omega), \qquad g \in L^2(\Gamma_C), \\ \varphi_F \in L^2(\Gamma_C), \qquad \theta_F \in L^2(\Gamma_C), \qquad \rho \in L^\infty(\Omega). \end{aligned}$$

Next, we define the elements $f(t) \in V'$, $q_e(t) \in W'$ and $q_{th}(t) \in Q'$ by

$$(f(t), v)_{V' \times V} = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_N} f_N(t) \cdot v da, \tag{25}$$

$$(q_e(t), \xi)_{W' \times W} = \int_\Omega q_0(t) \xi dx - \int_{\Gamma_b} q_b(t) \xi da, \tag{26}$$

$$(q_{th}(t), \eta)_{Q' \times Q} = \int_\Omega q_c(t) \eta dx, \tag{27}$$

for all $w \in V$, $\xi \in W$, $\eta \in Q$ and $t \in [0, T]$.

We define the mappings $j_d : V \times V \to \mathbb{R}$, $j_e : V \times W \times W \to \mathbb{R}$ and $j_{th} : V \times Q \times Q \to \mathbb{R}$ by

$$j_d(u, w) = \int_{\Gamma_C} p(u_\nu - g) w_\nu da, \tag{28}$$

$$j_e(u, \varphi, \xi) = \int_{\Gamma_C} \psi_e(u_\nu - g) \phi_L(\varphi - \varphi_F) \xi da, \tag{29}$$

$$j_{th}(u, \theta, \eta) = \int_{\Gamma_C} \psi_c(u_\nu - g) \phi_L(\theta - \theta_F) \eta da. \tag{30}$$

Finally, we apply the Banach fixed point theorem to deduce that there exists a unique element $\varphi_0 \in W$ such that for all $\xi \in W$

$$(\mathcal{B} \nabla \varphi_0, \nabla \xi)_H + (\theta_0 \mathcal{P}, \nabla \xi)_H - (\mathcal{E} \varepsilon(u_0), \nabla \xi)_H + j_e(u_0, \varphi_0, \xi) = (q_e(0), \xi)_W. \tag{31}$$

The equation (31) ensure the compatibility between the initial conditions.

Now, by utilizing Green's formula, we obtain the following variational formulation of **Problem (P)** expressed in terms of displacement field, electric potential and temperature field.

**Problem (PV)**. Find a displacement field $u : (0, T) \to V$, an electric potential $\varphi : (0, T) \to W$ and a temperature field $\theta : (0, T) \to Q$ a.e. $t \in [0, T]$ such that for all $w \in V$, $\xi \in W$ and $\eta \in Q$

$$\begin{aligned} &((\ddot{u}(t), w))_H + (\mathcal{A} \varepsilon(\dot{u}(t)), \varepsilon(w))_{\mathcal{H}} + (\mathcal{F} \varepsilon(u(t)), \varepsilon(w))_{\mathcal{H}} + (\mathcal{E}^* \nabla \varphi(t), \varepsilon(w))_{\mathcal{H}} \\ &- (\theta(t) \mathcal{M}, \varepsilon(w))_{\mathcal{H}} + j_d(u(t), w) = (f(t), w)_{V' \times V}, \end{aligned} \tag{32}$$

$$\begin{aligned} &(\mathcal{B} \nabla \varphi(t), \nabla \xi)_H + (\theta(t) \mathcal{P}, \nabla \xi)_H - (\mathcal{E} \varepsilon(u(t)), \nabla \xi)_H + j_e(u(t), \varphi(t), \xi) \\ &= (q_e(t), \xi)_{W' \times W}, \end{aligned} \tag{33}$$

$$\begin{aligned} &(\dot{\theta}(t), \eta)_{L^2(\Omega)} + (\mathcal{K} \nabla \theta(t), \nabla \eta)_H - (\mathcal{R}(\dot{u}(t), \varphi(t)), \eta)_{L^2(\Omega)} + j_{th}(u(t), \theta(t), \eta) \\ &= (q_{th}(t), \eta)_{Q' \times Q}, \end{aligned} \tag{34}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0. \tag{35}$$

# 4 Existence and uniqueness result

Our main existence and uniqueness result of the solution of **Problem (PV)** is the following.

**Theorem 4.1.** *Assume that (H1)-(H7) hold, then there exists a unique solution* $(u, \varphi, \theta)$ *of* ***Problem (PV)****, Moreover the solution satisfies*

$$u \in W^{1,2}(0,T;V) \cap C^1(0,T;H), \quad \ddot{u} \in L^2(0,T;V'), \tag{36}$$

$$\varphi \in L^2(0,T;W), \tag{37}$$

$$\theta \in L^2(0,T;L^2(\Omega)). \tag{38}$$

The proof of Theorem (4.1) will be carried out in several steps, and it is based on results of evolutionary variational equalities and Banach fixed point theorem.

Let $\zeta \in L^2(0,T;V')$. In the first step, we consider the following intermediate problem of displacement field

**Problem** $(PV_\zeta^{dp})$. Find $u_\zeta(t) \in V$ for a.e. $t \in (0,T)$ such that

$$((\ddot{u}_\zeta(t), w))_H + (\mathcal{A}\varepsilon(\dot{u}_\zeta(t)), \varepsilon(w))_{\mathcal{H}} + (\zeta(t), w)_{V'\times V} = (f(t), w)_{V'\times V}, \quad \forall w \in V, \tag{39}$$

$$u_\zeta(0) = u_0 \quad \dot{u}_\zeta(0) = v_0. \tag{40}$$

**Lemma 1.** *For all* $w \in V$ *and for a.e.* $t \in [0,T]$, *the Problem* $(PV_\zeta^{dp})$ *has a unique solution* $u_\zeta \in W^{1,2}(0,T;V) \cap C^1(0,T;H)$ *and* $\ddot{u}_\zeta \in L^2(0,T;V')$. *Moreover, if* $u_{\zeta_1}$ *and* $u_{\zeta_2}$ *are the solutions of* $(PV_{\zeta_1}^{dp})$ *and* $(PV_{\zeta_2}^{dp})$, *respectively, then there exists a constant* $C > 0$, *such that*

$$\|u_{\zeta_1}(t) - u_{\zeta_2}(t)\|_V^2 \le C \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{V'}^2 \, ds, \quad \forall t \in [0,T]. \tag{41}$$

*Proof.* By using the Riesz's representation theorem we define the element $f_\zeta(t) \in V'$ such that for all $t \in [0,T]$

$$(f_\zeta(t), w)_{V'\times V} = (f(t) - \zeta(t), w)_{V'\times V}.$$

and the operator $A : V \to V'$ defined by

$$(Au, w)_{V'\times V} = (\mathcal{A}\varepsilon(u), \varepsilon(w))_{\mathcal{H}}.$$

Recalling the Gelfand triple $V \subset H \subset V'$, let us denote

$$(u, v)_{V'\times V} = ((u, v))_H, \quad \text{for all } u \in H, v \in V.$$

Then the equation (39) can be rewritten as follows

$$\ddot{u}_\zeta(t) + A\dot{u}_\zeta(t) = f_\zeta(t), \quad \forall t \in [0,T]. \tag{42}$$

It is easy to see that, the operator $A$ is bounded, hemicontinuous, monotone and coercive. We recall that $f_\zeta \in L^2(0,T;V')$ and $v_0 \in H$, then by using a standard result on evolution equations with monotone operators (see [20, p.48]) we can prove the existence and uniqueness of $v_\zeta$ such that

$$v_\zeta \in L^2(0,T;V) \cap C(0,T;H); \quad \dot{v}_\zeta \in L^2(0,T;V'), \tag{43}$$

$$\dot{v}_\zeta(t) + Av_\zeta(t) = f_\zeta(t), \quad a.e.\ t \in [0,T], \tag{44}$$

$$v_\zeta(0) = v_0. \tag{45}$$

Let $u_\zeta : [0,T] \to V$ the function defined by

$$u_\zeta(t) = u_0 + \int_0^t v_\zeta(s)ds, \quad \forall t \in [0,T].$$

Then from (44) and (45), we conclude that $u_\zeta$ is a unique solution of the equation (42) combining with the initial conditions (40).

Now, we turn to prove the inequality (41). Let $\zeta_1$, $\zeta_2$ be two elements of $\in L^2(0,T;V')$ and let $u_{\zeta_1}$and $u_{\zeta_2}$ be the corresponding solutions of $(PV^{dp}_{\zeta_1})$ and $(PV^{dp}_{\zeta_2})$, respectively. We write the variational equality (39) successively for $u_{\zeta_1}$ and $u_{\zeta_2}$ for $w = \dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)$ , and subtract the resulting equalities we obtain that

$$\begin{aligned}
&((\ddot{u}_{\zeta_1}(s) - \ddot{u}_{\zeta_2}(s), \dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s))_H \\
&+ (\mathcal{A}\varepsilon(\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)), \varepsilon(\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)))_{\mathcal{H}} \\
&+ (\zeta_1(s) - \zeta_2(s), \dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s))_{V'\times V} = 0.
\end{aligned} \tag{46}$$

By using the coercivity of $\mathcal{A}$ and cauchy-schwartz inequality we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_H^2 + m_{\mathcal{A}}\left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V^2 \\
&\leq \left\|\zeta_1(s) - \zeta_2(s)\right\|_{V'}\left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V .
\end{aligned} \tag{47}$$

Integrating this inequality over the interval time variable $(0,t)$, and using the inequality

$$ab \leq \alpha a^2 + \frac{1}{4\alpha}b^2, \quad \forall a,b \in \mathbb{R}, \alpha > 0. \tag{48}$$

We obtain that

$$\begin{aligned}
&\frac{1}{2}\left\|\dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right\|_H^2 + m_{\mathcal{A}}\int_0^t \left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V^2 ds \\
&\leq \frac{1}{2m_{\mathcal{A}}}\int_0^t \left\|\zeta_1(s) - \zeta_2(s)\right\|_{V'}^2 ds + \frac{m_{\mathcal{A}}}{2}\int_0^t \left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V^2 ds.
\end{aligned} \tag{49}$$

Then

$$\int_0^t \left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V^2 ds \leq \frac{1}{m_{\mathcal{A}}^2}\int_0^t \left\|\zeta_1(s) - \zeta_2(s)\right\|_{V'}^2 ds. \tag{50}$$

Remembering that for all $t \in [0,T]$, we have

$$\left\|u_{\zeta_1}(t) - u_{\zeta_2}(t)\right\|_V^2 \leq T\int_0^t \left\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\right\|_V^2 ds, \quad \forall t \in [0,T]. \tag{51}$$

Finally, we combine (50) with (51) to deduce (41). □

In the second step, let $\chi \in L^2(0,T;W)$, we utilize the displacement field $u_\zeta$ obtained in Lemma 1 in the construction of the following problem in term of temperature field.

**Problem** $(PV^{th}_{\zeta,\chi})$**.**

Find a temperature field $\theta_{\zeta,\chi} : (0,T) \to Q$ such that

$$\begin{aligned}&(\dot{\theta}_{\zeta,\chi}(t), \eta)_{L^2(\Omega)} + (\mathcal{K}\nabla\theta_{\zeta,\chi}(t), \nabla\eta)_H - (\mathcal{R}(\dot{u}_\zeta(t), \chi(t)), \eta)_{L^2(\Omega)} \\ &+ j_{th}(u_\zeta(t), \theta_{\zeta,\chi}(t), \eta) = (q_{th}(t), \eta)_{Q'\times Q}, \quad \forall \eta \in Q,\end{aligned} \tag{52}$$

$$\theta_{\zeta,\chi}(0) = \theta_0. \tag{53}$$

**Lemma 2.** *For all $\eta \in Q$ and a.e. $t \in [0,T]$, the **Problem** $(PV^{th}_{\zeta,\chi})$ has a unique solution $\theta_{\zeta,\chi} \in L^2(0,T;L^2(\Omega))$. Moreover, if $\theta_{\zeta_1,\chi_1}$ and $\theta_{\zeta_2,\chi_2}$ are solutions of $(PV^{th}_{\zeta_1,\chi_1})$ and $(PV^{th}_{\zeta_2,\chi_2})$, respectively, then there exists $C > 0$ such that for all $t \in [0,T]$,*

$$\|\theta_{\zeta_1,\chi_1}(t) - \theta_{\zeta_2,\chi_2}(t)\|^2_{L^2(\Omega)} \le C \int_0^t \|(\zeta_1,\chi_1)(s) - (\zeta_2,\chi_2)(s)\|^2_{V'\times W}\, ds. \tag{54}$$

*Proof.* The existence and uniqueness result was established by Essoufi et al. in [12] by using the Faedo-Galerkin method.

On the other hand, for $(\zeta_1,\chi_1), (\zeta_2,\chi_2) \in L^2(0,T;V'\times W)$ we denote by $\theta_{\zeta_1,\chi_1}$ and $\theta_{\zeta_2,\chi_2}$ the corresponding solutions of $(PV^{th}_{\zeta_1,\chi_1})$ and $(PV^{th}_{\zeta_2,\chi_2})$, respectively. We write (52) for $(\zeta_1,\chi_1)$ and $(\zeta_2,\chi_2)$, respectively, with taking $\eta = \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)$, and subtracting the resulting equalities we obtain that

$$\begin{aligned}&(\dot{\theta}_{\zeta_1,\chi_1}(s) - \dot{\theta}_{\zeta_2,\chi_2}(s), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s))_{L^2(\Omega)} \\ &+(\mathcal{K}\nabla(\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)), \nabla(\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)))_H \\ &= (\mathcal{R}(\dot{u}_{\zeta_1}(s), \chi_1(s)) - \mathcal{R}(\dot{u}_{\zeta_2}(s), \chi_2(s)), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s))_{L^2(\Omega)} \\ &+j_{th}(u_{\zeta_2}(s), \theta_{\zeta_2,\chi_2}(s), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)) \\ &-j_{th}(u_{\zeta_1}(s), \theta_{\zeta_1,\chi_1}(s), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)).\end{aligned} \tag{55}$$

Notice that

$$\begin{aligned}&|j_{th}(u_{\zeta_2}(s), \theta_{\zeta_2,\chi_2}(s), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)) \\ &-j_{th}(u_{\zeta_1}(s), \theta_{\zeta_1,\chi_1}(s), \theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s))| \\ &\le L L_{\psi_c} c_0 c_2 \|u_{\zeta_1}(s) - u_{\zeta_2}(s)\|_V \|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|_Q.\end{aligned} \tag{56}$$

Then, using the coercivity of $\mathcal{K}$, the continuity of $\mathcal{R}$, the Cauchy-Schwartz inequality and the previous inequality, we deduce from (55) that there exists a constant $C > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|^2_{L^2(\Omega)} + m_{\mathcal{K}}\|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|^2_Q \\ &\le C\Big(\|\dot{u}_{\zeta_1}(s) - \dot{u}_{\zeta_2}(s)\|_V \|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|_{L^2(\Omega)} \\ &+ \|\chi_1(s) - \chi_2(s)\|_W \|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|_{L^2(\Omega)} \\ &+ \|u_{\zeta_1}(s) - u_{\zeta_2}(s)\|_V \|\theta_{\zeta_1,\chi_1}(s) - \theta_{\zeta_2,\chi_2}(s)\|_Q\Big).\end{aligned} \tag{57}$$

Integrating this inequality over $(0,t)$ and by using the inequality (48) several times, we obtain that there exists $C>0$ such that

$$\|\theta_{\zeta_1,\chi_1}(t)-\theta_{\zeta_2,\chi_2}(t)\|^2_{L^2(\Omega)} \leq C\bigg(\int_0^t \|\theta_{\zeta_1,\chi_1}(s)-\theta_{\zeta_2,\chi_2}(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|u_{\zeta_1}(s)-u_{\zeta_2}(s)\|^2_V\,ds + \int_0^t \|\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\|^2_V\,ds + \int_0^t \|\chi_1(s)-\chi_2(s)\|^2_W\,ds\bigg). \tag{58}$$

By Gronwall inequality, we find that there exists $C>0$ such that

$$\|\theta_{\zeta_1,\chi_1}(t)-\theta_{\zeta_2,\chi_2}(t)\|^2_{L^2(\Omega)} \leq C\bigg(\int_0^t \|u_{\zeta_1}(s)-u_{\zeta_2}(s)\|^2_V\,ds + \int_0^t \|\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\|^2_V\,ds + \int_0^t \|\chi_1(s)-\chi_2(s)\|^2_W\,ds\bigg). \tag{59}$$

Keeping in mind the inequalities (41) and (50), the inequality (54) holds. □

In the third step, we use the displacement field $u_\zeta$ obtained in Lemma 1 and the temperature field $\theta_{\zeta,\chi}$ obtained in Lemma 2 in the construction of the following problem in term of electric potential.

**Problem**$(PV^{el}_{\zeta,\chi})$.
Find an electric potential $\varphi_{\zeta,\chi}:(0,T)\to W$ such that

$$(\mathcal{B}\nabla\varphi_{\zeta,\chi}(t),\nabla\xi)_H + (\theta_{\zeta,\chi}(t)\mathcal{P},\nabla\xi)_H - (\mathcal{E}\varepsilon(u_\zeta(t)),\nabla\xi)_H + j_e(u_\zeta(t),\varphi_{\zeta,\chi}(t),\xi) = (q_e(t),\xi)_{W'\times W}, \quad \forall\xi\in W. \tag{60}$$

**Lemma 3.** *For all $\xi\in W$ and a.e. $t\in[0,T]$, the **Problem** $(PV^{el}_{\zeta,\chi})$ has a unique solution $\varphi_{\zeta,\chi}\in L^2(0,T;W)$. Moreover, if $\varphi_{\zeta_1,\chi_1}$ and $\varphi_{\zeta_2,\chi_2}$ are solutions of $(PV^{el}_{\zeta_1,\chi_1})$ and $(PV^{el}_{\zeta_2,\chi_2})$, respectively, then there exists $C>0$, such that for all $t\in[0,T]$*

$$\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\|^2_W \leq C\int_0^t \|(\zeta_1,\chi_1)(s)-(\zeta_2,\chi_2)(s)\|^2_{V'\times W}\,ds. \tag{61}$$

*Proof.* The existence and uniqueness result was established by Essoufi et al. in [12]. For $(\zeta_1,\chi_1),(\zeta_2,\chi_2)\in L^2(0,T;V'\times W)$, let $\varphi_{\zeta_1,\chi_1}$ and $\varphi_{\zeta_2,\chi_2}$ be the corresponding solutions of $(PV^{el}_{\zeta_1,\chi_1})$ and $(PV^{el}_{\zeta_2,\chi_2})$ , respectively.

By taking the substitution $(\zeta,\chi)=(\zeta_i,\chi_i)$ in (60) and choosing $\xi=\varphi_{\zeta_1,\chi_1}-\varphi_{\zeta_2,\chi_2}$ for $i=1,2$, we find that

$$\begin{aligned}&(\mathcal{B}\nabla(\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)),\nabla(\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)))_H\\&+((\theta_{\zeta_1,\chi_1}(t)-\theta_{\zeta_2,\chi_2}(t))\mathcal{P},\nabla(\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)))_H\\&=(\mathcal{E}\varepsilon(u_{\zeta_1}(t)-u_{\zeta_2}(t)),\nabla(\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)))_H\\&+j_e(u_{\zeta_2}(t),\varphi_{\zeta_2,\chi_2}(t),\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t))\\&-j_e(u_{\zeta_1}(t),\varphi_{\zeta_1,\chi_1}(t),\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)).\end{aligned} \tag{62}$$

It is easy to see that

$$
\begin{aligned}
&|j_e(u_{\zeta_2}(t),\varphi_{\zeta_2,\chi_2}(t),\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t))\\
&-j_e(u_{\zeta_1}(t),\varphi_{\zeta_1,\chi_1}(t),\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t))|\\
&\leq LL_{\psi_e}c_0c_1\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V\left\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\right\|_W.
\end{aligned}
\tag{63}
$$

Then, from the previous inequality, the coercivity of $\mathcal{B}$ and the continuity of $\mathcal{P}$ and $\mathcal{E}$ we deduce from (62) that

$$
\begin{aligned}
&m_{\mathcal{B}}\left\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\right\|_W\leq M_{\mathcal{P}}\left\|\theta_{\zeta_1,\chi_1}(t)-\theta_{\zeta_2,\chi_2}(t)\right\|_{L^2(\Omega)}\\
&+(M_{\mathcal{E}}+LL_{\psi_e}c_0c_1)\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V.
\end{aligned}
\tag{64}
$$

The inequality $(a+b)^2\leq 2a^2+2b^2$ leads to

$$
\begin{aligned}
&\left\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\right\|_W^2\leq 2\frac{M_{\mathcal{P}}^2}{m_{\mathcal{B}}^2}\left\|\theta_{\zeta_1,\chi_1}(t)-\theta_{\zeta_2,\chi_2}(t)\right\|_{L^2(\Omega)}^2\\
&+2\frac{(M_{\mathcal{E}}+LL_{\psi_e}c_0c_1)^2}{m_{\mathcal{B}}^2}\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V^2.
\end{aligned}
\tag{65}
$$

Keeping in mind inequalities (41) and (54) the inequality (61) holds. □

In the last step of the proof, for $(\zeta,\chi)\in L^2(0,T;V'\times W)$ we denote by $u_\zeta$, $\theta_{\zeta,\chi}$ and $\varphi_{\zeta,\chi}$ the solutions of problems $(PV_\zeta^{dp})$, $(PV_{\zeta,\chi}^{th})$ and $(PV_{\zeta,\chi}^{el})$, respectively, and we consider the operator $\Lambda: L^2(0,T;V'\times W)\to L^2(0,T;V'\times W)$ defined by

$$
\Lambda(\zeta,\chi)=(\Lambda 1(\zeta,\chi),\Lambda 2(\zeta,\chi)),
\tag{66}
$$

where

$$
\begin{aligned}
&(\Lambda 1(\zeta,\chi),w)_V=(\mathcal{F}\varepsilon(u_\zeta(t)),\varepsilon(w))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi_{\zeta,\chi}(t),\varepsilon(w))_{\mathcal{H}}\\
&-(\theta_{\zeta,\chi}(t)\mathcal{M},\varepsilon(w))_{\mathcal{H}}+j_d(u_\zeta(t),w),
\end{aligned}
\tag{67}
$$

$$
\Lambda 2(\zeta,\chi)(t)=\varphi_{\zeta,\chi}(t),
\tag{68}
$$

for all $w\in V$ and a.e. $t\in[0,T]$.

We have the following result.

**Lemma 4.** *There exists a unique $(\zeta^*,\chi^*)\in L^2(0,T;V'\times W)$ such that $\Lambda(\zeta^*,\chi^*)=(\zeta^*,\chi^*)$.*

*Proof.* Let $(\zeta_1,\chi_1),(\zeta_2,\chi_2)\in L^2(0,T;V'\times W)$ and for $i=1,2$, let $u_{\zeta_i}$, $\theta_{\zeta_i,\chi_i}$ and $\varphi_{\zeta_i,\chi_i}$ the solutions of $(PV_{\zeta_i}^{dp})$, $(PV_{\zeta_i,\chi_i}^{th})$ and $(PV_{\zeta_i,\chi_i}^{el})$ respectively.
By (67)-(68) and after some algebra, we obtain

$$
\begin{aligned}
&\left\|\Lambda 1(\zeta_1,\chi_1)(t)-\Lambda 1(\zeta_2,\chi_2)(t)\right\|_{V'}^2\leq(2M_{\mathcal{F}}^2+2c_0^2L_P^2)\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V^2\\
&+2M_{\mathcal{E}}^2\left\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\right\|_W^2+2M_{\mathcal{M}}^2\left\|\theta_{\zeta_1,\chi_1}-\theta_{\zeta_2,\chi_2}(t)\right\|_{L^2(\Omega)}^2,
\end{aligned}
\tag{69}
$$

$$
\left\|\Lambda 2(\zeta_1,\chi_1)(t)-\Lambda 2(\zeta_2,\chi_2)(t)\right\|_W^2\leq\left\|\varphi_{\zeta_1,\chi_1}(t)-\varphi_{\zeta_2,\chi_2}(t)\right\|_W^2.
\tag{70}
$$

Combining (69)-(70) with the estimates (41), (54) and (61), we conclude that there exists a constant $C > 0$ such that

$$\begin{aligned} &\|\Lambda(\zeta_1,\chi_1)(t)-\Lambda(\zeta_2,\chi_2)(t)\|^2_{V'\times W} \\ &\leq C\int_0^t \|(\zeta_1,\chi_1)(s)-(\zeta_2,\chi_2)(s)\|^2_{V'\times W}\,ds. \end{aligned} \tag{71}$$

Here and below, we denote by $\Lambda^k$ the power of the operator $\Lambda$. Reiterating the inequality (71) $k$ times and after some algebra we get

$$\begin{aligned} &\left\|\Lambda^k(\zeta_1,\chi_1)(t)-\Lambda^k(\zeta_2,\chi_2)(t)\right\|^2_{V'\times W} \\ &\leq \frac{C^k t^{k-1}}{(k-1)!}\int_0^t \|(\zeta_1,\chi_1)(s)-(\zeta_2,\chi_2)(s)\|^2_{V'\times W}\,ds, \end{aligned} \tag{72}$$

for all $t\in[0,T]$, that leads to obtain the formula

$$\begin{aligned} &\left\|\Lambda^k(\zeta_1,\chi_1)-\Lambda^k(\zeta_2,\chi_2)\right\|^2_{L^2(0,T;V'\times W)} \\ &\leq \frac{C^k T^k}{k!}\int_0^T \|(\zeta_1,\chi_1)(s)-(\zeta_2,\chi_2)(s)\|^2_{V'\times W}\,ds. \end{aligned} \tag{73}$$

Finally, we get

$$\begin{aligned} &\left\|\Lambda^k(\zeta_1,\chi_1)-\Lambda^k(\zeta_2,\chi_2)\right\|_{L^2(0,T;V'\times W)} \\ &\leq \sqrt{\frac{C^k T^k}{k!}}\,\|(\zeta_1,\chi_1)-(\zeta_2,\chi_2)\|_{L^2(0,T;V'\times W)}\,. \end{aligned} \tag{74}$$

This inequality shows that for a sufficiently large $k$ the operator $\Lambda^k$ is a contraction on the Banach space $L^2(0,T;V'\times W)$, therefore, Banach fixed point theorem shows that $\Lambda$ admits a unique fixed point $(\zeta^*,\chi^*)\in L^2(0,T;V'\times W)$. □

We are now ready to prove Theorem (4.1).

*Proof of Theorem* (4.1). Let $(\zeta^*,\chi^*)\in L^2(0,T;V'\times W)$ be the unique fixed point of the operator $\Lambda$, we denote by $u_{\zeta^*}$ , $\theta_{\zeta^*,\chi^*}$and $\varphi_{\zeta^*,\chi^*}$ the solutions of problems $(PV^{dp}_{\zeta^*})$, $(PV^{th}_{\zeta^*,\chi^*})$ and $(PV^{el}_{\zeta^*,\chi^*})$, respectively. Therefore, we conclude that $(u_{\zeta^*},\varphi_{\zeta^*,\chi^*},\theta_{\zeta^*,\chi^*})$ is the solution of **Problem (PV)**.
The uniqueness of the solution of **Problem (PV)** is a consequence of the uniqueness of the fixed point of $\Lambda$. □

# 5 Fully discrete approximation and error estimates

In this section, we introduce a fully discrete approximation for the **Problem(PV)** and we establish an error bound for the resulting approximate solution.

Let $T^h=\{T_r\}_r$ a finite element triangulation of $\overline{\Omega}$ compatible with the boundary partitions where $h$ represents the spacial discretization parameter and let $\mathbb{P}^1(T_r)$ the space of polynomials of global degree less or equal to 1 in $T_r$, We then define the following finite-dimensional spaces

$$V^h = \{w^h \in [C(\bar{\Omega})]^d, w^h_{|T_r} \in [\mathbb{P}^1(T_r)]^d, w^h = 0 \text{ on } \Gamma_D\} \subset V,$$
$$W^h = \{\xi^h \in C(\bar{\Omega}), \xi^h_{|T_r} \in \mathbb{P}^1(T_r), \xi^h = 0 \text{ on } \Gamma_a\} \subset W,$$
$$Q^h = \{\eta^h \in C(\bar{\Omega}), \eta^h_{|T_r} \in \mathbb{P}^1(T_r), \eta^h = 0 \text{ on } \Gamma_D \cup \Gamma_N\} \subset Q,$$

approximating the spaces $V$, $W$ and $Q$, respectively. For the time interval, we consider a uniform partition $t_0 = 0 < t_1 < ... < t_N = T$ of $[0, T]$. We denote by $k$ the time step size given by $k = \frac{T}{N}$. Moreover, for a continuous function $g$ we denote $g(t_n) = g_n$, and for a sequence $\{y_n\}_{n=0}^N$ we denote $\delta y_n = \frac{y_n - y_{n-1}}{k}$.

Let $u_0^h$, $v_0^h$, $\theta_0^h$ and $\varphi_0^h$ be the appropriate approximations of the initial conditions $u_0$, $v_0$, $\theta_0$ and $\varphi_0$, respectively.

To simplify again the notations we introduce the velocity field $v$ such that

$$u(t) = u_0 + \int_0^t v(s)ds, \quad \forall t \in [0, T]. \tag{75}$$

The fully discrete approximation of the **Problem (PV)** can be expressed using the backward Euler scheme as follows.

***Problem*** $(PV^{hk})$**:** Find a discrete displacement field $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$, a discrete electric potential $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h$ and a discrete temperature field $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ such that fora all $w^h \in V^h$, $\xi^h \in W^h$ and $\eta^h \in Q^h$

$$\begin{aligned}&((\delta v_n^{hk}, w^h))_H + (\mathcal{A}\varepsilon(v_n^{hk}), \varepsilon(w^h))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_{n-1}^{hk}), \varepsilon(w^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_{n-1}^{hk}, \varepsilon(w^h))_{\mathcal{H}}\\ &- (\theta_{n-1}^{hk}\mathcal{M}, \varepsilon(w^h))_{\mathcal{H}} + j_d(u_{n-1}^{hk}, w^h) = (f_n, w^h)_{V'\times V},\end{aligned} \tag{76}$$

$$\begin{aligned}&(\mathcal{B}\nabla\varphi_n^{hk}, \nabla\xi^h)_H + (\theta_n^{hk}\mathcal{P}, \nabla\xi^h)_H - (\mathcal{E}\varepsilon(u_n^{hk}), \nabla\xi^h)_H + j_e(u_n^{hk}, \varphi_{n-1}^{hk}, \xi^h)\\ &= (q_{e_n}, \xi^h)_{W'\times W},\end{aligned} \tag{77}$$

$$\begin{aligned}&(\delta\theta_n^{hk}, \eta^h)_{L^2(\Omega)} + (\mathcal{K}\nabla\theta_n^{hk}, \nabla\eta^h)_H - (\mathcal{R}(v_n^{hk}, \varphi_{n-1}^{hk}), \eta^h)_{L^2(\Omega)} + j_{th}(u_n^{hk}, \theta_{n-1}^{hk}, \eta^h)\\ &= (q_{th_n}, \eta^h)_{Q'\times Q},\end{aligned} \tag{78}$$

$$u^{hk}(0) = u_0^h, \quad v^{hk}(0) = v_0^h, \quad \theta^{hk}(0) = \theta_0^h, \quad \varphi^{hk}(0) = \varphi_0^h. \tag{79}$$

Here the discrete displacement field and the velocity field $v^{hk} = \{v_n^{hk}\}$ are related by

$$u_n^{hk} = \delta v_n^{hk} \quad \text{and} \quad u_n^{hk} = u_0^h + k\sum_{i=0}^n v_i^{hk}, \quad n = 1, ..., N.$$

Using the same arguments presented in the previous section, it can be shown that Problem $(PV^{hk})$ has a unique solution $(u^{hk}, \varphi^{hk}, \theta^{hk}) \subset V^h \times W^h \times Q^h$. Our goal here is to estimate the following numerical errors $\|u_n - u_n^{hk}\|_V$, $\|v_n - v_n^{hk}\|_H$, $\|\varphi_n - \varphi_n^{hk}\|_W$, $\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}$ and $\|\theta_n - \theta_n^{hk}\|_Q$.

**Theorem 5.1.** *Let the assumptions of Theorem* (4.1) *hold and* $\varphi_0$ *is the solution of* (31). *Let* $(u, \varphi, \theta)$ *and* $(u^{hk}, \varphi^{hk}, \theta^{hk})$ *denote the solutions of problems* ***(PV)*** *and* $(PV^{hk})$*, respectively. Under the following regularity conditions*

$$v \in C(0,T;H^2(\Omega)^d) \cap H^1(0,T;V) \cap H^2(0,T;H),\ v_{|\Gamma_c} \in C(0,T;H^2(\Gamma_C)^d), \tag{80}$$

$$\varphi \in C(0,T;H^2(\Omega)), \tag{81}$$

$$\theta \in C(0,T;H^2(\Omega)) \cap H^1(0,T;Q) \cap H^2(0,T;L^2(\Omega)),\ \dot{\theta} \in L^2(0,T;H^1(\Omega)). \tag{82}$$

*There exists* $c > 0$ *independent of* $h$ *and* $k$ *such that*

$$\begin{aligned} &\max_{1\le n\le N} \left\{ \left\|u_n - u_n^{hk}\right\|_V + \left\|v_n - v_n^{hk}\right\|_H + \left\|\varphi_n - \varphi_n^{hk}\right\|_W \right. \\ &\left. + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)} + k\left\|\theta_n - \theta_n^{hk}\right\|_Q \right\} \le c(h+k). \end{aligned} \tag{83}$$

*Proof.* everywhere below, we denote by $c$ various positive constants which are depends on the problem data, but they are independent of the discretization parameters $h$ and $k$ and their value may vary from line to line.

Let us first obtain an error estimate on the velocity field. We take (32) at time $t = t_n$ for $w = w^h \in V^h$ and subtracting it from (76) to obtain that for all $w^h \in V^h$, we have

$$\begin{aligned} &((\dot{v}_n - \delta v_n^{hk}, w^h))_H + (\mathcal{A}\varepsilon(v_n - v_n^{hk}), \varepsilon(w^h))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n - u_{n-1}^{hk}), \varepsilon(w^h))_{\mathcal{H}} \\ &+ (\mathcal{E}^*\nabla(\varphi_n - \varphi_{n-1}^{hk}), \varepsilon(w^h))_{\mathcal{H}} - ((\theta_n - \theta_{n-1}^{hk})\mathcal{M}, \varepsilon(w^h))_{\mathcal{H}} \\ &+ j_d(u_n, w^h) - j_d(u_{n-1}^{hk}, w^h) = 0. \end{aligned} \tag{84}$$

We write the previous relation for $w^h - v_n^{hk}$ to obtain that for all $w^h \in V^h$

$$\begin{aligned} &((\delta(v_n - v_n^{hk}), v_n - v_n^{hk}))_H + (\mathcal{A}\varepsilon(v_n - v_n^{hk}), \varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} \\ &= ((\delta(v_n - v_n^{hk}), v_n - w^h))_H + ((\delta v_n - \dot{v}_n, w^h - v_n^{hk}))_H \\ &+ (\mathcal{A}\varepsilon(v_n - v_n^{hk}), \varepsilon(v_n - w^h))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n - u_{n-1}^{hk}), \varepsilon(v_n^{hk} - w^h))_{\mathcal{H}} \\ &+ (\mathcal{E}^*\nabla(\varphi_n - \varphi_{n-1}^{hk}), \varepsilon(v_n^{hk} - w^h))_{\mathcal{H}} + ((\theta_n - \theta_{n-1}^{hk})\mathcal{M}, \varepsilon(w^h - v_n^{hk}))_{\mathcal{H}} \\ &+ j_d(u_{n-1}^{hk}, w^h - v_n^{hk}) - j_d(u_n, w^h - v_n^{hk}). \end{aligned} \tag{85}$$

From properties of the function $p$, we can easily see that

$$\left|j_d(u_{n-1}^{hk}, w^h - v_n^{hk}) - j_d(u_n, w^h - v_n^{hk})\right| \le L_p c_0^2 \left\|u_n - u_{n-1}^{hk}\right\|_V \left\|w^h - v_n^{hk}\right\|_V. \tag{86}$$

Keeping in mind that

$$((\delta(v_n - v_n^{hk}), v_n - v_n^{hk}))_H \ge \frac{1}{2k}\left(\left\|v_n - v_n^{hk}\right\|_H^2 - \left\|v_{n-1} - v_{n-1}^{hk}\right\|_H^2\right), \tag{87}$$

the coercivity of $\mathcal{A}$, the continuity of $\mathcal{A}$, $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{M}$ , Cauchy-Schwartz inequality and the inequality (48) we can deduce from (85) that there exists a constant $c > 0$

such that

$$\begin{aligned}&\left\|v_n-v_n^{hk}\right\|_H^2-\left\|v_{n-1}-v_{n-1}^{hk}\right\|_H^2+k\left\|v_n-v_n^{hk}\right\|_V^2\leq ck\left\{\|\dot{v}_n-\delta v_n\|_H^2\right.\\&+\left\|v_n-w^h\right\|_H^2+\left\|v_n-w^h\right\|_V^2+\left\|u_{n-1}-u_{n-1}^{hk}\right\|_V^2+\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\\&\left.+\left\|\varphi_{n-1}-\varphi_{n-1}^{hk}\right\|_W^2+\|u_n-u_{n-1}\|_V^2+\|\theta_n-\theta_{n-1}\|_{L^2(\Omega)}^2+\|\varphi_n-\varphi_{n-1}\|_W^2\right\}\\&+2k((\delta(v_n-v_n^{hk}),v_n-w^h))_H.\end{aligned}\tag{88}$$

Secondly, we proceed to estimate the numerical error on the temperature field, we take (34) at time $t=t_n$ for $\eta=\eta^h\in Q^h$ and we subtract it to (78) to obtain that for all $\eta^h\in Q^h$ we have

$$\begin{aligned}&(\dot{\theta}_n-\delta\theta_n^{hk},\eta^h)_{L^2(\Omega)}+(\mathcal{K}\nabla(\theta_n-\theta_n^{hk}),\nabla\eta^h)_{H*}\\&=(\mathcal{R}(v_n,\varphi_n)-\mathcal{R}(v_n^{hk},\varphi_{n-1}^{hk}),\eta^h)_{L^2(\Omega)}\\&+j_{th}(u_n^{hk},\theta_{n-1}^{hk},\eta^h)-j_{th}(u_n,\theta_n,\eta^h).\end{aligned}\tag{89}$$

Thus, we substitute $\eta^h$ by $\eta^h-\theta_n^{hk}$ to get

$$\begin{aligned}&(\delta(\theta_n-\theta_n^{hk}),\theta_n-\theta_n^{hk})_{L^2(\Omega)}+(\mathcal{K}\nabla(\theta_n-\theta_n^{hk}),\nabla(\theta_n-\theta_n^{hk}))_H\\&=(\delta\theta_n-\dot{\theta}_n,\eta^h-\theta_n^{hk})_{L^2(\Omega)}+(\delta(\theta_n-\theta_n^{hk}),\theta_n-\eta^h)_{L^2(\Omega)}\\&+(\mathcal{K}\nabla(\theta_n-\theta_n^{hk}),\nabla(\theta_n-\eta^h))_H+(\mathcal{R}(v_n,\varphi_n)-\mathcal{R}(v_n^{hk},\varphi_{n-1}^{hk}),\eta^h-\theta_n^{hk})_{L^2(\Omega)}\\&+j_{th}(u_n^{hk},\theta_{n-1}^{hk},\eta^h-\theta_n^{hk})-j_{th}(u_n,\theta_n,\eta^h-\theta_n^{hk}).\end{aligned}\tag{90}$$

From the properties of the functions $\psi_c$ and $\phi_L$, there exists $C_1>0$ such that

$$\begin{aligned}&\left|j_{th}(u_n^{hk},\theta_{n-1}^{hk},\eta^h-\theta_n^{hk})-j_{th}(u_n,\theta_n,\eta^h-\theta_n^{hk})\right|\\&\leq LL_{\psi_c}c_0c_2\left\|u_n-u_n^{hk}\right\|_Q\left\|\eta^h-\theta_n^{hk}\right\|_Q.\end{aligned}\tag{91}$$

Using an analogue idea to (87), we have

$$\begin{aligned}&(\delta(\theta_n-\theta_n^{hk}),\theta_n-\theta_n^{hk})_{L^2(\Omega)}\\&\geq\frac{1}{2k}\left(\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\right).\end{aligned}\tag{92}$$

By using the coercivity of $\mathcal{K}$, the continuity of $\mathcal{K}$ and $\mathcal{R}$, the Cauchy-Shwartz inequality and Keeping in mind (91) and (92), we can deduce from (90) that for all $\eta^h\in Q^h$ we have

$$\begin{aligned}&\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2+k\left\|\theta_n-\theta_n^{hk}\right\|_Q^2\\&\leq ck\left\{\left\|\dot{\theta}_n-\delta\theta_n\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta^h\right\|_Q^2+\left\|v_n-v_n^{hk}\right\|_V^2+\left\|\varphi_{n-1}-\varphi_{n-1}^{hk}\right\|_W^2\right.\\&\left.+\|\varphi_n-\varphi_{n-1}\|_W^2+\left\|u_n-u_n^{hk}\right\|_V^2\right\}+2k(\delta(\theta_n-\theta_n^{hk}),\theta_n-\eta^h)_{L^2(\Omega)}.\end{aligned}\tag{93}$$

Thirdly, we turn to an estimation of the error estimate on the electric potential. We take (33) at time $t = t_n$ for $\xi = \xi^h \in W^h$ and we subtract it from (77) to obtain that for all $\xi^h \in W^h$ we have

$$
\begin{aligned}
&(\mathcal{B}\nabla(\varphi_n - \varphi_n^{hk}), \nabla\xi^h)_H + ((\theta_n - \theta_n^{hk})\mathcal{P}, \nabla\xi^h)_H - (\mathcal{E}\varepsilon(u_n - u_n^{hk}), \nabla\xi^h)_H \\
&+ j_e(u_n, \varphi_n, \xi^h) - j_e(u_n^{hk}, \varphi_{n-1}^{hk}, \xi^h) = 0.
\end{aligned}
\tag{94}
$$

Thus, if we substitute $\xi^h$ by $\xi^h - \varphi_n^{hk}$ in the previous equality, we get

$$
\begin{aligned}
&(\mathcal{B}\nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \varphi_n^{hk}))_H = (\mathcal{B}\nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \xi^h))_H \\
&- ((\theta_n - \theta_n^{hk})\mathcal{P}, \nabla(\xi^h - \varphi_n^{hk}))_H + (\mathcal{E}^*\varepsilon(u_n - u_n^{hk}), \nabla(\xi^h - \varphi_n^{hk}))_H \\
&+ j_e(u_n^{hk}, \varphi_{n-1}^{hk}, \xi^h - \varphi_n^{hk}) - j_e(u_n, \varphi_n, \xi^h - \varphi_n^{hk}).
\end{aligned}
\tag{95}
$$

We also have that

$$
\begin{aligned}
&\left| j_e(u_n^{hk}, \varphi_{n-1}^{hk}, \xi^h - \varphi_n^{hk}) - j_e(u_n, \varphi_n, \xi^h - \varphi_n^{hk}) \right| \\
&\leq L L_{\psi_e} c_0 c_1 \left\| u_n - u_n^{hk} \right\|_V \left\| \xi^h - \varphi_n^{hk} \right\|_W .
\end{aligned}
\tag{96}
$$

Keeping in mind (96), the coercivity of $\mathcal{B}$, the continuity of $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{E}$, Cauchy-Shwartz inequality and the inequality (48) we deduce from (95) that there exists $c > 0$ such that for all $\xi^h \in W^h$

$$
\left\| \varphi_n - \varphi_n^{hk} \right\|_W^2 \leq c \left\{ \left\| u_n - u_n^{hk} \right\|_V^2 + \left\| \theta_n - \theta_n^{hk} \right\|_{L^2(\Omega)}^2 + \left\| \xi^h - \varphi_n \right\|_W^2 \right\}. \tag{97}
$$

Now, we combine (88), (93) and (97) to obtain that there exists a constant $c > 0$ such that for all $\{w_i^h\}_{i=1}^n \subset V^h$, $\{\xi_i^h\}_{i=1}^n \subset W^h$ and $\{\eta_i^h\}_{i=1}^n \subset Q^h$

$$
\begin{aligned}
&\left\| v_n - v_n^{hk} \right\|_H^2 + \left\| \theta_n - \theta_n^{hk} \right\|_{L^2(\Omega)}^2 + \left\| \varphi_n - \varphi_n^{hk} \right\|_W^2 + k \sum_{i=1}^n \Big( \left\| v_i - v_i^{hk} \right\|_V^2 \\
&+ \left\| \theta_n - \theta_n^{hk} \right\|_Q^2 \Big) \leq \left\| v_0 - v_0^h \right\|_H^2 + \left\| \theta_0 - \theta_0^h \right\|_{L^2(\Omega)}^2 + \left\| \varphi_0 - \varphi_0^h \right\|_W^2 \\
&+ c \Bigg[ k \sum_{i=1}^n \Big( \| \dot{v}_n - \delta v_n \|_H^2 + \left\| \dot{\theta}_n - \delta\theta_n \right\|_{L^2(\Omega)}^2 + \left\| v_n - w_i^h \right\|_H^2 \\
&+ \left\| v_n - w_i^h \right\|_V^2 + \left\| \theta_i - \eta_i^h \right\|_Q^2 + \left\| \varphi_i - \xi_i^h \right\|_W^2 \Big) \\
&+ k \sum_{i=1}^n \Big( \left\| u_i - u_i^{hk} \right\|_V^2 + \left\| \theta_i - \theta_i^{hk} \right\|_{L^2(\Omega)}^2 + \left\| \varphi_i - \varphi_i^{hk} \right\|_W^2 \Big) \\
&+ k \sum_{i=1}^n \Big( \| u_i - u_{i-1} \|_V^2 + \| \theta_i - \theta_{i-1} \|_{L^2(\Omega)}^2 + \| \theta_i - \theta_{i-1} \|_Q^2 + \| \varphi_i - \varphi_{i-1} \|_W^2 \Big) \Bigg] \\
&+ 2k \sum_{i=1}^n ((\delta(v_i - v_i^{hk}), v_i - w_i^h))_H + 2k \sum_{i=1}^n (\delta(\theta_i - \theta_i^{hk}), \theta_i - \eta_i^h)_{L^2(\Omega)}.
\end{aligned}
\tag{98}
$$

Recalling that

$$2k\sum_{i=1}^{n}((\delta(v_i - v_i^{hk}), v_i - w_i^h))_H \leq \gamma_1 \left\| v_n - v_n^{hk} \right\|_H^2 + c\left\{ \left\| v_n - w_n^h \right\|_H^2 \right.$$
$$\left. + \left\| v_0 - v_0^h \right\|_H^2 + \left\| v_1 - w_1^h \right\|_H^2 \right\} + k\sum_{i=1}^{n-1} \left\| v_i - v_i^{hk} \right\|_H^2 \tag{99}$$
$$+ \frac{1}{k}\sum_{i=1}^{n-1} \left\| v_i - w_i^h - (v_{i+1} - w_{i+1}^h) \right\|_H^2 .$$

and

$$2k\sum_{i=1}^{n}(\delta(\theta_i - \theta_i^{hk}), \theta_i - \eta_i^h)_{L^2(\Omega)} \leq \gamma_2 \left\| \theta_n - \theta_n^{hk} \right\|_{L^2(\Omega)}^2 + c\left\{ \left\| \theta_n - \eta_n^h \right\|_{L^2(\Omega)}^2 \right.$$
$$\left. + \left\| \theta_0 - \theta_0^h \right\|_{L^2(\Omega)}^2 + \left\| \theta_1 - \eta_1^h \right\|_{L^2(\Omega)}^2 \right\} + k\sum_{i=1}^{n-1} \left\| \theta_i - \theta_i^{hk} \right\|_{L^2(\Omega)}^2 \tag{100}$$
$$+ \frac{1}{k}\sum_{i=1}^{n-1} \left\| \theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h) \right\|_{L^2(\Omega)}^2 .$$

where $\gamma_1 > 0$ and $\gamma_2$ are two parameters chosen to be small enough.

Also, we have( see [2] for more details)

$$\left\| u_i - u_i^{hk} \right\|_v^2 \leq ck\sum_{j=1}^{i} \left\| v_j - v_j^{hk} \right\|_V^2 + c(h^2 + k^2), \quad i = 1, ..., N. \tag{101}$$

Combining (98),(99), (100) and (101), using a discrete version of Gronwall's inequality, we obtain the following error estimates for all $\{w_i^h\}_{i=1}^N \subset W^h$, $\{\eta_i^h\}_{i=1}^N \subset Q^h$ and

$\{\xi_i^h\}_{i=1}^N \subset W^h$

$$\max_{1\le n\le N}\left\{\left\|v_n-v_n^{hk}\right\|_H^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\varphi_n-\varphi_n^{hk}\right\|_W^2\right\}+k\sum_{i=1}^N\left(\left\|v_i-v_i^{hk}\right\|_V^2\right.$$
$$\left.+\left\|\theta_i-\theta_i^{hk}\right\|_Q^2\right)\le c\left[\left\|v_0-v_0^h\right\|_H^2+\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}^2+\left\|\varphi_0-\varphi_0^h\right\|_W^2\right.$$
$$+\max_{1\le n\le N}\left\{\left\|v_n-w_n^h\right\|_H^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2+\left\|\varphi_n-\xi_n^h\right\|_W^2\right\}$$
$$+k\sum_{i=1}^N\left(\|\dot v_i-\delta v_i\|_H^2+\left\|\dot\theta_i-\delta\theta_i\right\|_{L^2(\Omega)}^2+\left\|v_i-w_i^h\right\|_V^2+\left\|\theta_i-\eta_i^h\right\|_Q^2\right)$$
$$+k\sum_{i=1}^N\left(\|u_i-u_{i-1}\|_V^2+\|\theta_i-\theta_{i-1}\|_{L^2(\Omega)}^2+\|\varphi_i-\varphi_{i-1}\|_W^2\right) \tag{102}$$
$$+\left\|v_1-w_1^h\right\|_H^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2+\frac{1}{k}\sum_{i=1}^{N-1}\left\|v_i-w_i^h-(v_{i+1}-w_{i+1}^h)\right\|_H^2$$
$$\left.+\frac{1}{k}\sum_{i=1}^{N-1}\left\|\theta_i-\eta_i^h-(\theta_{i+1}-\eta_{i+1}^h)\right\|_{L^2(\Omega)}^2+h^2+k^2\right].$$

Next, we denote by $\Pi_{\mathcal{Z}}^h$ the standard finite element interpolation operator over the space $\mathcal{Z}$. We choose $w_n^h=\Pi_V^h v_n$, $\xi_n^h=\Pi_W^h\varphi_n$ and $\eta_n^h=\Pi_Q^h\theta_n$ the finite element interpolants of $v_n$, $\varphi_n$ and $\theta_n$, respectively. Using standard finite element interpolation error estimates[10], we have the following approximations

$$\left\|v_n-w_n^h\right\|_V\le ch\|v\|_{C(0,T;H^2(\Omega)^d)}, \tag{103}$$
$$\left\|v_n-w_n^h\right\|_H\le ch^2\|v\|_{C(0,T;H^2(\Omega)^d)}, \tag{104}$$
$$\left\|\varphi_n-\xi_n^h\right\|_W\le ch\|\varphi\|_{C(0,T;H^2(\Omega))}, \tag{105}$$
$$\left\|\theta_n-\eta_n^h\right\|_Q\le ch\|\theta\|_{C(0,T;H^2(\Omega))}. \tag{106}$$

We assume that the discrete initial conditions $u_0^h$, $v_0^h$, $\theta_0^h$ and $\varphi_0^h$ are defined by

$$u_0^h=\Pi_V^h u_0,\quad v_0^h=\Pi_V^h v_0,\quad \theta_0^h=\Pi_Q^h\theta_0,\quad \varphi_0^h=\Pi_W^h\varphi_0. \tag{107}$$

Then (see [10, 13])

$$\left\|u_0-u_0^h\right\|_V\le ch\|u\|_{C(0,T;H^2(\Omega)^d)}, \tag{108}$$
$$\left\|v_0-v_0^h\right\|_H\le ch\|u\|_{C^1(0,T;V)}, \tag{109}$$
$$\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}\le ch\|\theta\|_{C(0,T;Q)}, \tag{110}$$
$$\left\|\varphi_0-\varphi_0^h\right\|_W\le ch\|\varphi\|_{C(0,T;H^2(\Omega))}. \tag{111}$$

Moreover, from (80) and (82) it is easy to check that

$$k\sum_{i=1}^{N}\left(\|\dot{v}_i-\delta v_i\|_H^2+\left\|\dot{\theta}_i-\delta\theta_i\right\|_{L^2(\Omega)}^2\right)\le ck^2\left(\|v\|_{H^2(0,T;H)}^2+\|\theta\|_{H^2(0,T;L^2(\Omega))}^2\right). \quad (112)$$

Proceeding as in [9] we obtain that

$$\begin{aligned}&\frac{1}{k}\sum_{i=1}^{N-1}\left(\left\|v_i-w_i^h-(v_{i+1}-w_{i+1}^h)\right\|_H^2+\left\|\theta_i-\eta_i^h-(\theta_{i+1}-\eta_{i+1}^h)\right\|_{L^2(\Omega)}\right)\\&\le ch^2\left(\|v\|_{H^1(0,T;V)}^2+\|\theta\|_{H^1(0,T;L^2(\Omega))}^2\right),\end{aligned} \quad (113)$$

and from [13] we find

$$\begin{aligned}&k\sum_{i=1}^{n}\left(\|u_i-u_{i-1}\|_V^2+\|\theta_i-\theta_{i-1}\|_{L^2(\Omega)}^2+\|\varphi_i-\varphi_{i-1}\|_W^2\right)\\&\le ck^2\left(\|u\|_{H^1(0,T;V)}^2+\|\theta\|_{H^1(0,T;L^2(\Omega))}^2+\|\varphi\|_{H^1(0,T;W)}^2\right).\end{aligned} \quad (114)$$

Now, we combine the estimates (103)-(114) with (102) to find that there exists a constant $c>0$ such that

$$\begin{aligned}&\max_{1\le n\le N}\left\{\left\|v_n-v_n^{hk}\right\|_H^2+\left\|\varphi_n-\varphi_n^{hk}\right\|_W^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2\right.\\&\left.+\sum_{i=1}^{N}k\left(\left\|v_i-v_i^{hk}\right\|_V^2+\left\|\theta_i-\theta_i^{hk}\right\|_Q^2\right)\right\}\le c(h^2+k^2).\end{aligned} \quad (115)$$

Finally, we combine (115) with (101) it leads to (83). □

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