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# EXPLICIT FAST AND STABLE METHOD FOR SOLUTION OF SOME COEFFICIENT INVERSE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

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**Abstract** A new approach for solution of inverse coefficient problem for some types of partial differential equations with variable coefficient is considered. This approach is based on proposed by author General Ray Principle and leads to the new GR-Method, which consists in reduction of the partial differential equation to assemblage of ordinary differential equations using local traces for considered functions and operators. New method presents the solution of considered problems by explicit analytical formulas that use the direct and inverse Radon transform. In the case of noised input data the regularization with Recursive Spline Smoothing is used. Proposed method is realized by fast and stable algorithms and MATLAB software, which quality is demonstrated by numerical experiments. Applications to electric tomography and mathematical simulation in creation of nano-composite materials with special heat-conductive properties are considered.

**Key words:** inverse problems, PDE coefficients, electric tomography

**AMS Mathematics Subject Classification:** 35R30, 81U40

## 1 Introduction

Mathematical and computer simulation is very important for modern investigations in many applied areas. This simulation leads to solving direct and inverse problems [1], [2] . Basic mathematical models that describe processes include traditional differentials equations. In this paper we will consider two problems. The first problem is described by the two dimensional Laplace type equation

$$\nabla \cdot (\varepsilon(x,y)\nabla u(x,y)) = 0,\ (x,y) \in \Omega, \tag{1}$$

where $\Omega$ - some limited open region on a plane.

The second problem corresponds to the parabolic equation:

$$u_t(x,t) = (\varepsilon(x,t)u_x(x,t))_x,\ -1 < x < 1, t > t_0. \tag{2}$$

Solution of such inverse problems is very important in mathematical simulation of electrical tomography [3] and investigation of heat-conductive properties of materials [4]. But all mathematical statements and known methods for solution of considering inverse problems are non-linear [2], [3] and require a lot of time and memory in its computer realization, that is not appropriated in modern investigations in mentioned applied areas. We propose here another approach for the mathematical modelling of the distribution of the external physical field and measurement of the external data. Proposed modelling is based on General Ray Principle (GRP) [5] and use the classic Radon transformation [6]. GRP leads to the new explicit General Ray (GR) Method and a fast linear algorithm for numerical solution of inverse problems.

## 2 Coefficient Inverse Problem for the Laplace Type Equation

Let us consider a coefficient inverse problem for the Laplace type equation (1) with respect to the function $\varepsilon(x,y)$. In traditional statement of inverse problems [2], [3] it is supposed also that some family of functions $J_n(x,y)$, $u^0(x,y)$ are known on the boundary curve $\Gamma$, and the next boundary conditions are satisfied:

$$\varepsilon(x,y)\frac{\partial u(x,y)}{\partial n} = J_n(x,y), (x,y) \in \Gamma, \tag{3}$$

$$u(x,y) = u^0(x,y)), (x,y) \in \Gamma, \tag{4}$$

where $\partial/\partial n$ is the normal derivative in the points of the boundary curve $\Gamma$. Mentioned families of functions in the boundary conditions, as the rule, correspond to some scanning scheme [3].

### 2.1 General Ray Method for Laplace Type Equation

In [5] we formulated GR Principle, which for the problems under investigation means to construct an analogue of equations (1), (2) describing the distribution of the function $u(x,y)$ along of "General Local Rays", which are presented by some straight line $l$ with the Radon parametrization [5] due a parameter:$x = pcos(\varphi) - \tau sin(\varphi)$, $y = psin(\varphi) + \tau cos(\varphi)$ . Here $|p|$ is a length of the perpendicular from the center of coordinates to the line $l$, $\varphi \in [0,\pi]$ is the angle between the axis x and this perpendicular. Using this parametrization, we shall define traces of functions $u(x,y)$, $\varepsilon(x,y)$, and function $f(x,y)$(that describes some boundary conditions), at $(x,y) \in l$ for fixed p, as functions $u(\tau), \varepsilon(\tau), f(\tau)$, of variable $\tau$. We suppose that the domain $\Gamma$ is a convex one. Let us define for every fixed $p$ and $\varphi$ the functions $u_0(p,\varphi) = u(\tau_0)$, $u_1(p,\varphi) = u(\tau_1)$ for parameters $\tau_0, \tau_1$, which correspond to the points of the intersection of the line $l$ and boundary of the domain. Hence, the GR Principle leads to the assemblage (depending of $p, \varphi$ ) of ordinary differential equations:

$$(\varepsilon(\tau)u'_\tau(\tau))'_\tau = 0, \tau \in [\tau_0, \tau_1], \tag{5}$$

Equation (5) for fixed $p$ and $\varphi$ represents the local analogy of the equation (1). Boundary conditions lead to the corresponding local boundary conditions for $u(\tau)$ at points $\tau_0, \tau_1$.

We consider for fixed $p$ and $\varphi$ following boundary conditions

$$\varepsilon(\tau_0)u'_\tau(\tau_0) = J(p,\varphi), \tag{6}$$

$$u(\tau_1) - u(\tau_0) = v(p,\varphi), \tag{7}$$

for given functions $v(p, \varphi)$ and $J(p, \varphi)$ . Equations (5), (6), (7) constitute the basic mathematical model for the inverse problem of reconstructing the coefficient $\varepsilon(x, y)$. Boundary condition (7) can be obtained from the data in the boundary condition (4). But boundary condition (6) has one principle difference with (3), because (6) includes derivative by $\tau$ that corresponds to direction on straight line $l$, not to normal direction for boundary curve $\Gamma$ . It is possible to calculate function $J(p, \varphi)$ using function of boundary condition (3), if we know $\varepsilon(x, y)$ on the boundary and angle between the normal vector and straight line $l$ in each boundary point.

The family of equations (5) - (7) we consider as the basic mathematical model in application of GRP for considering type of inverse problems. Supposing that different components in the considered structure have the smooth distribution, such that the functions $\varepsilon(\tau)u_\tau^{'}(\tau)$ and $u_\tau^{'}(\tau)$ are continuous, and integrating twice equation (5) with respect to $\tau$, we obtain for $\varepsilon(x, y)$ the following formula

$$\varepsilon(x, y) = 1/R^{-1}[\frac{v(p, \varphi)}{J(p, \varphi)}], \tag{8}$$

where $R^{-1}$ is the inverse Radon transform operator. Formula (8) represents the General Ray method for the inverse problem. This formula can be generalized and applied also for structures with piecewise constant characteristics.

## 2.2 Application to the Electric Tomography

Computer Tomography consists in the image reconstruction of an interior of a body using the measurements on its surface of characteristics of some external field. It can be state mathematically as a coefficient inverse problem for a differential equation describing the distribution of the field in considered region. Coefficients are the functions of the space variables and characterize properties of a media.

Electrical Impedance Tomography (EIT) is the most developed approach for electric tomography that includes the electric resistance (ERT) or capacitance tomography (ECT) schemes [3].

In ERT the unction $J_n(x, y)$ is given, function $u^0(x, y))$ is measured. In ECT the function $u^0(x, y))$ is given, the value of the normal component of electric induction $J_n(x, y)$ is related with measured mutual capacitances [3]. In both ERT and ECT schemes the electric field is produced by the same electrodes that serve as measuring elements i.e. the electrodes are active. May be this activity of electrodes, which provokes its mutual influence, is the cause of impossibility to use a great number of electrodes and obtain the sufficiently large number of measurements.

We propose here another variant of the Electrical Tomography, when the external electromagnetic field $\vec{V}(l)$ is produced by active electrodes, located outside of the $\Omega$, initiates some distribution of the electric potential inside the domain $\Omega$. At that, we propose that measurements of necessary values would be realized on the boundary curve $\Gamma$ with another, no active electrodes. The corresponding scheme is presented at Fig. 1, where active electrodes are marked with arcs on the external circle $A$ of radius $R$ and inactive electrodes are marked as arcs on the internal circle $B$, which is the board of the domain.

In proposing new scheme we can calculate potential $u_0(\tau)$ and induction function $J(p, \varphi)$ on direction $l$, so we need to make measurements only of values of the potential $u(\tau_1)$.

It is very important that electrodes on the boundary $\Gamma$ do not produce the external electric field (are not active) and serve only for measuring data. Therefore, the proposed approach gives in principal the possibility to use a large number of electrodes and measurements of the input values of functions $u(\tau_1)$ and reconstruct the desired image more perfectly.

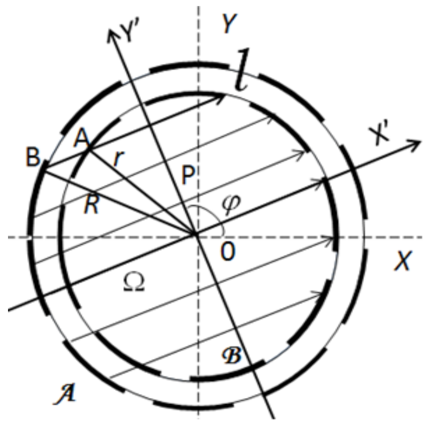


Figure 1: The measurement scheme of the external data.

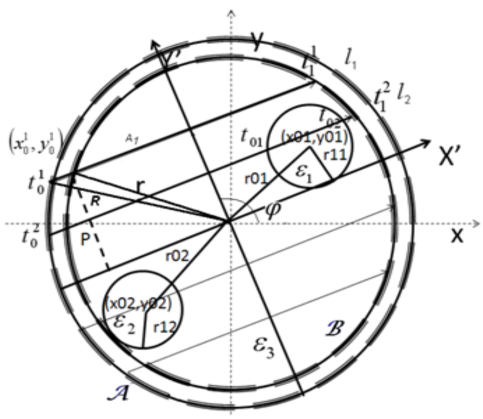


Figure 2: Illustration for synthetic example.

## 3 Coefficient Inverse Problem for the Parabolic Equation

We consider the boundary value problem for the parabolic equation in the form:

$$u_t(x,t) = (\varepsilon(x,t)u_x(x,t))_x, (x,t) \in \Omega; \tag{9}$$

$$u(x,t_0) = f^0(x),$$

$$u(x_0(t),t) = f_0(t),$$

$$u(x_1(t),t) = f_1(t),$$

$$\Omega = [x_0(t), x_1(t)] \times [t_0, t_1],$$

where unction $f^0(x)$ corresponds to initial condition, functions $f_0(t), f_1(t)$, correspond to boundary conditions. Inverse problem consists in identification of unknown function $\varepsilon(x,t)$ under additional supposition that the function $f^1(x)$ is given, such that for some moment $t_1$: $u(x,t_1) = f^1(x)$.

## 3.1 General Ray Method for Parabolic Equation

Let us use the Radon parametrization [6] of the line $l$ in domain $\Omega$ with a parameter $\tau$ for variables $x = x(\tau) = pcos(\varphi) - \tau sin(\varphi)$, $t = t(\tau) = psin(\varphi) + \tau cos(\varphi)$, and substitute these into equation (9). So, we obtain as the local analogy of equation (9) the next assemblage of ODE:

$$K(p,\varphi)u_\tau^{'}(\tau) = (\varepsilon(\tau)u_\tau^{'}(\tau))_\tau^{'}, \tau \in [\tau_0, \tau_1]; \tag{10}$$

$$K(p,\varphi) = \frac{cos(\varphi)}{sin^2(\varphi)}, \varphi \neq 0, \pi, \tag{11}$$

where $u(\tau) = u(x(\tau), t(\tau))$,$\varepsilon(\tau) = \varepsilon(x(\tau), t(\tau))$, parameters $\tau_0, \tau_1$ also correspond to the points of the intersection of the line $l$ and boundary of the domain. We suppose that the next boundary conditions are known

$$u(\tau_1) - u(\tau_0) = v(p,\varphi), \tag{12}$$

$$\varepsilon(\tau_0)u_\tau^{'}(\tau_0) = J(p,\varphi) \tag{13}$$

for given functions $v(p,\varphi)$ and $J(p,\varphi)$. Integrating equation (10) with respect to $\tau$ and using conditions (12), (13), we obtain for the following formula

$$\varepsilon(x,t) = 1/R^{-1}[\frac{1}{K(p,\varphi)} \ln(\frac{v(p,\varphi)}{J(p,\varphi)} K(p,\varphi) + 1)], \tag{14}$$

which present GR-method for solution of inverse problem for considering case.

## 3.2 Possible Application for mathematical simulation of nano-composite materials

Investigation of heat-conductive properties of materials is very important for many applied areas. Mathematical and computer simulation is one of the important steps in this investigation. This simulation leads to solving direct and inverse heat transfer problems. Modern investigations of nano-composite materials are characterized by a penetration with more substantially detailed in the structure of investigated objects and phenomena. Basic mathematical models that describe the heat-conductive processes include traditional differentials equations, nevertheless frequently with specific elements. This requires elaboration of the new analytical and numerical methods of its study, adapted to the modern requirements. One of the most important of these requirements is the possibility to obtain a sufficient increase of the exactitude at the solution of the problems in the real time allowed, or, that is equivalent, to resolve the problem with appropriate exactitude by fastest manna. The mathematical models and the known numerical methods often do not satisfy to these requirements at their computer realization, particularly for solution of desired heat transfer problems. Constructed GR-method can be recommended for mathematical simulation in creation of nano-composite materials because of its fast computer realization.

## 4 Regularization of the GR Method

Analysis of formulas for inverse Radon transformation shows that its instability for discrete noised data is equivalent to the instability of the problem of the numerical differentiation of the noised function $\overline{v}(p,\varphi)$ with respect to the variable $p$. The regularization of the inversion of Radon transform was constructed by author in [7] on the base of the Recursive Smoothing by splines (RSS), which was used in [8] for post-processing to improve the electrical capacitance tomography image reconstruction.

RSS uses the explicit formulas for two-dimensional spline on the regular uniform grid $\{p_i,\varphi_j\}$,$p_i=-1+h(i-1)$,$i=-2,...,n+2$; $\varphi_j=-1+h_\varphi(j-1)$, $i=-2,...,n+2$; $h=2/(n-1), h_\varphi=\pi/(n-1)$. Let $s_j(w)$ be a local basic cubic spline, constructed on the units $w_{i-2},...,w_{i+2}; i=0,...,n+1$; where $w$ is $p$ or $w$ is $\varphi$. Mentioned formulas are the next ones:

$$S_k(p,\varphi)=\sum_{i=1}^{n}\sum_{j=1}^{n}S_{k-1}(p_i,\varphi_j)s_i(p)s_j(\varphi), \tag{15}$$

$$k=1,2,...,\overline{K}, S_0(p_i,\varphi_j)=\overline{v}(p_i,\varphi_j). \tag{16}$$

The number of smoothes $\overline{K}$ is the regularization parameter, which can be chosen here in accordance with residual (discrepancy) principle, using the discrete estimation $\delta$ of the errors. It means, if the values of the exact function $v(p,\varphi)$ and the noised function $\overline{v}(p,\varphi)$ satisfy to the conditions

$$|v(p_i,\varphi_j)-\overline{v}(p_i,\varphi_j)|\le\delta, i,j=1,...,n, \tag{17}$$

then $\overline{K}$ is chosen as maximum among all $k$ , for which the inequality is fulfilled:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|S_k(p_i,\varphi_j)-\overline{v}(p_i,\varphi_j)|^2\le c\delta^2n^2, \tag{18}$$

where $c=const>1$. Theoretical and numerical justifications of the regularization properties of this type of smoothing are presented in [9], [10].

If for structures with piecewise constant characteristics the set $\hat{\varepsilon}=\{\varepsilon_0,\varepsilon_1,\varepsilon_2\}$ of the known values $\varepsilon_i$ of the function $\varepsilon(x,y)$ is given, then the algorithm includes also the projection of the pre-reconstructed data to the set $\hat{\varepsilon}$ with respect to the absolute or relative criterions [7].

We underline that presented algorithms are based on fast numerical realization of the inverse Radon transform and on explicit approximation formulas (15) - (16) that do not require solving any equations. It guarantees the fast property of proposed algorithms in a hole.

## 5 Numerical Experiments

We have constructed the numerical realization of formula (7) that we call "GR-algorithm". This algorithm does not require solving any equation, because the Radon transform can be inversed by fast algorithm using discrete FFT algorithm.

We tested scanning GR-algorithm on mathematically simulated model examples. The first and second presented experiments correspond to piecewise constant structure. We considered inside the unit circle $\Omega$ two different internal elements $\Omega_1, \Omega_2$ of different permittivity, as it is shown at Fig. 2. Simulation consists in the next steps:

1) calculation by electrostatic formulas values of functions $J(p, \varphi)$ and $u(\tau_0)$ for every fixed angle $\varphi$ and parameter $p$ on the boundary and analytic solution the direct Cauchy problem for equation (5);

2) calculation the value $u(\tau_1)$ and $v(p, \varphi)$;

3) numerical realization of formula (8).

The deduction of the corresponding formulas at steps 1) and 2) are presented in details at [11]. These formulas give us the important result of this simulation: the relation of functions $v(p, \varphi)/J(p, \varphi)$ dos not depend on $J(p, \varphi)$ and the value of the potential at the border, so on radius of the external circle $A$, that confirms the validity of proposed method and constructed algorithm. The step 3) was realized on discrete simulated data with n nodes for every variable.

In the first example we use exact values in n=31 discrete points for the case $\varepsilon_0(x, y) = 1$, $\varepsilon_1(x, y) = 2$, $\varepsilon_2(x, y) = 70$. It is difficult case for the reconstruction, because it corresponds to the greater scale of values $\hat{\varepsilon} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, when the post-processing (projection) is required even for the pre-reconstruction that used exact data. In Fig. 3 there are presented reconstructions of the structure image by GR-algorithm: graph (a) - exact distribution; graph (b) - reconstruction without post-processing, graph (c) - reconstruction with post-processing using the absolute criterion projection; (d) - reconstructions with post-processing using the relative criterion projection.

The second presented numerical experiment corresponds to the reconstruction of the structure for the case $\varepsilon_0 = 1, \varepsilon_1 = 2, \varepsilon_2 = 3$, using simulated noised input data, i.e. values of a function $\overline{v}(p, \varphi) = v(p, \varphi)(1 + \delta(p, \varphi))$, where $\delta(p, \varphi)$ is the randomized function with estimation: $\|\delta(p, \varphi)\|_{C[\Omega]} \leq \delta$. Results of the regularized reconstruction for n=31,$\delta = 0.05$ are presented at Fig. 4: graph (a) - exact $\varepsilon(x, y)$; graph (b) - reconstruction with noised $\overline{v}(p, \varphi)$ without regularization; graph (c) - reconstruction with noised $\overline{v}(p, \varphi)$ by regularized GR-algorithm with RSS only, without post-processing projections; graph (d) - reconstruction with noised $\overline{v}(p, \varphi)$ by regularized GR-algorithm with RSS and the post-processing, using absolute criterion projection of pre-reconstructed image.

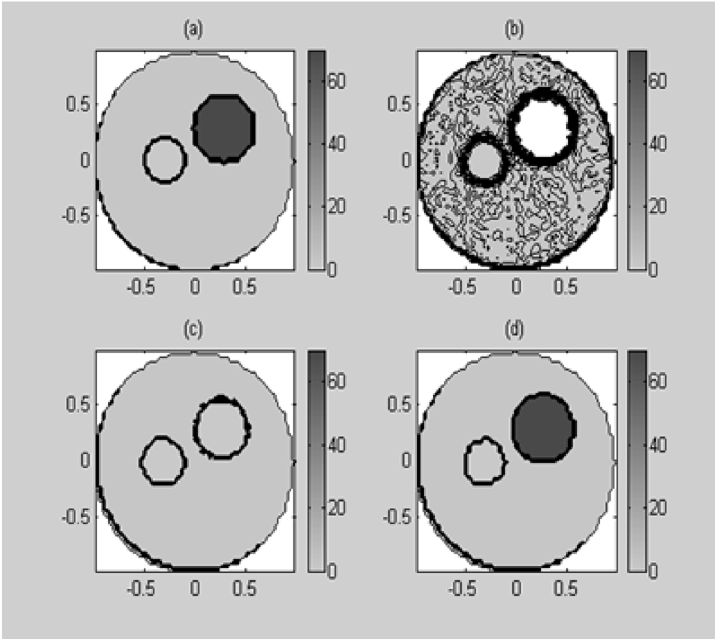


Figure 3: The 1-st experiment.

The third experiment corresponds to continue distribution of the characteristic $\varepsilon(x, y) = 1/cos(x + y)$ in the domain $\Omega$ as the circle of the radius r=0.5. The external stationary field produces in $\Omega$ the potential $u(x, y) = sin(x + y)$, such as $J(p, \varphi) = cos(\varphi) - sin(\varphi)$, exact function $v(p, \varphi) = 2cos(p(sin(\varphi) + cos(\varphi))(sin(\tau_0(sin(\varphi) + cos(\varphi))), \tau_0 = -\sqrt{0.25 - p^2}$. A noised function $\overline{v}(p, \varphi)$ we constructed in the same form as for the second example. Results of the numerical experiments for n=51, estimation $\delta = 0.03$ are presented at Fig. 5: graph (a) - exact $\varepsilon(x, y)$; graph (b) - reconstruction with exact $v(p, \varphi)$; graph (c) - reconstruction with noised $\overline{v}(p, \varphi)$ without regularization; graph (d) - reconstruction with noised $\overline{v}(p, \varphi)$ by regularized GR-algorithm with RSS.

Algorithms and the MATLAB program package to realise formula (14) are under construction.

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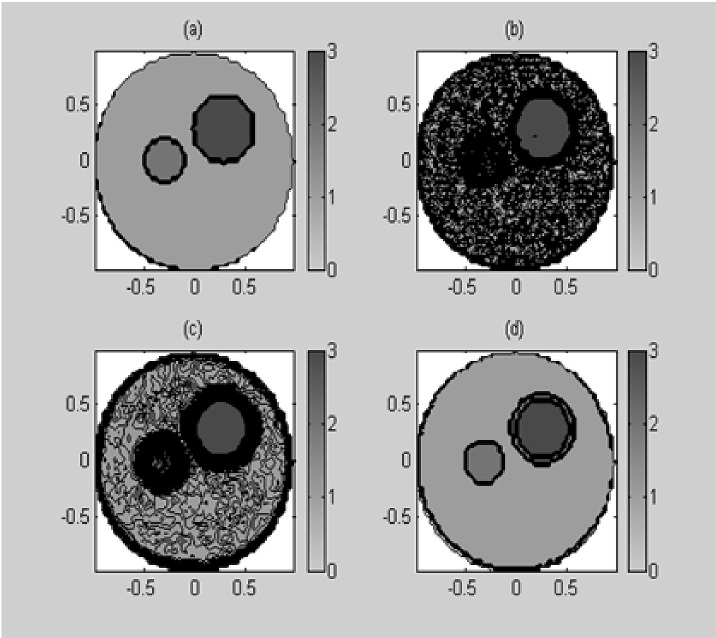


Figure 4: The 2-nd experiment.

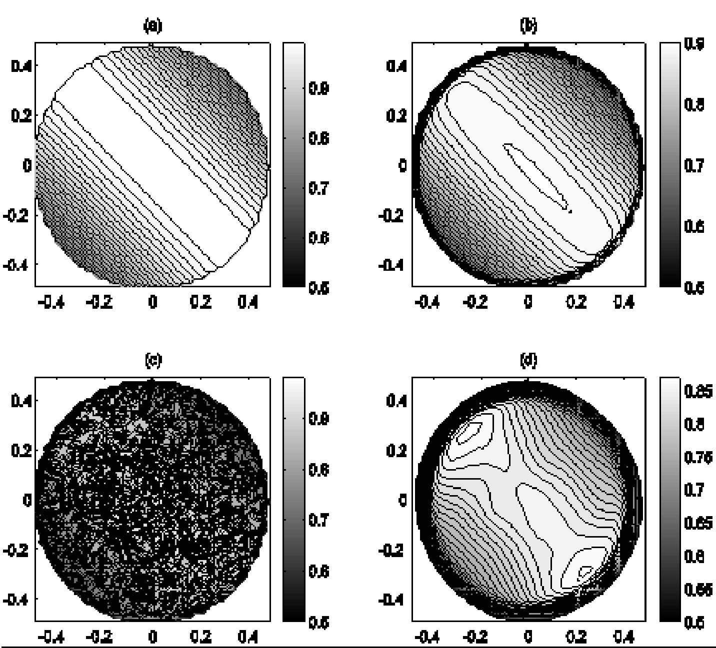


Figure 5: The 3-d experiment.

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