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OPERATOR EQUATIONS OF THE FIRST KIND AND INTEGRO-DIFFERENTIAL EQUATIONS OF DEGENERATE TYPE IN BANACH SPACES AND APPLICATIONS TO INTEGRO-DIFFERENTIAL PDE's

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Abstract We deal with linear operator integral equations (OIE) - of the first kind and first-order in time integro-differential equations of degenerate type - in a Banach space X, the related kernels being piecewise continuous with linear closed operator values in X. The curve when such operators may have a jump is $s = \alpha(t)$, where α has the properties (1.3), (1.4). For the solutions to equations (OIE), possibly endowed with initial conditions, we prove some existence and uniqueness results. Applications are given to linear integro-differential equations with kernels of "elliptic" and "parabolic" type.

Key words: linear operator integral equations, first-order in time operator integro-differential equations of degenerate type, piecewise continuous kernels with linear closed operator values in a Banach space, existence and uniqueness results, analytic semi-groups, applications to linear integro-differential equations with kernels of "elliptic" and "parabolic" type.

AMS Mathematics Subject Classification: 45A05, 45D05, 45K05, 45N05, 47D06.

1 Introduction

This paper deals with the following two classes of operator equations in a Banach space X:

$$\int_0^{\alpha(t)} K_1(t,s)u(s)\,ds + \int_{\alpha(t)}^t K_2(t,s)u(s)\,ds = f(t), \quad t \in [0,T], \tag{1.1}$$

and

$$L_0(t)u(t) + \int_0^{\alpha(t)} [K_{1,0}(t,s)u'(s) + K_{1,1}(t,s)u(s)]\,ds$$
$$+ \int_{\alpha(t)}^t [K_{2,0}(t,s)u'(s) + K_{2,1}(t,s)u(s)]\,ds = f(t), \quad t \in [0,T], \tag{1.2}$$

where $\alpha : [0,T] \to \mathbb{R}$ is any function enjoying the properties

$$\alpha \in C^1([0,T];\mathbb{R}), \quad \alpha \text{ is increasing on } [0,T], \tag{1.3}$$

$$0 \le \alpha(t) < t, \quad t \in (0,T], \quad \alpha(0) = 0, \quad 0 \le \alpha'(0) < 1. \tag{1.4}$$

Set

$$\omega_1(T) = \{(t,s) \in \mathbb{R}^2 : 0 < s < \alpha(t) < t < T\}, \tag{1.5}$$

$$\omega_2(T) = \{(t,s) \in \mathbb{R}^2 : \alpha(t) < s < t < T\}. \tag{1.6}$$

Finally, the kernels $K_j(t,s)$ and $K_{j,k}(t,s)$, $j = 1,2$ and $k = 0,1$, are linear operators with domains contained in X for all $(t,s) \in \omega_j(T)$, while $L_0(t)$ is a linear bounded operator for all $t \in [0,T]$.
We shall consider the following two cases:

(C0) let Y and X be a pair of Banach spaces with $Y \hookrightarrow X$ and let K_j and $D_t K_j$ belong to $C(\overline{\omega_j(T)}; \mathcal{L}(Y;X))$, $j = 1,2$;

(C1) let $L_0(t)$ be a family of (possible non-invertible) linear bounded operators in $\mathcal{L}(X)$ defined on the interval $[0,T]$ such that $L_0 \in C^1([0,T];\mathcal{L}(X))$. Let $K_{j,0}$, $D_t K_{j,0}$ and $K_{j,1}$, $D_t K_{j,0}$ belong, respectively, to $C(\overline{\omega_j(T)}; \mathcal{L}(X))$ and $C(\overline{\omega_j(T)}; \mathcal{L}(Y;X))$. $j = 1,2$, let $L_0(t) + K_{2,0}(t,t)$ belong to $\mathcal{L}(X)$ and be invertible in $\mathcal{L}(X)$ for all $t \in [0,T]$ and $\|[L_0(t) + K_{2,0}(t,t)]^{-1}\|_{\mathcal{L}(X)} \le \mu_1$, for some positive μ_1.

In our applications $K_2(t,t)$, and $[L_0(t) + K_{2,0}(t,t)]^{-1}[L_0'(t) + K_{2,1}(t,t)]$, corresponding, respectively, to the cases (C0) and (C1), will be uniformly invertible or uniformly parabolic for all $t \in [0,T]$.
First we notice that in case (C0) we have an actual operator equation of the first kind with a (possibly) discontinuous operator kernel. We stress that pure integral equations *of the first kind*, even in the scalar case, are *ill-posed*. Then we stress that in case (C1) the operators $K_j(t,s,D_s) = K_{2,0}(t,s)D_s - K_{j,1}(t,s)$, $j = 1,2$, are of the first-order in D_t, so that the principal operator in (1.2) is inside the latter integral, so that also in the scalar case, when dealing with pure integral equations, we have a (singular) equation *of the third kind* - which is known to be an *ill-posed problem* - with a (possibly) discontinuous operator kernel.
Moreover, even after differentiation with respect to t (cf. Section 2), operator $L_0(t)u'(t)$ needs not to be the principal part in the differentiated equation, since $L_0(t)$ may be not invertible for all $t \in [0,T]$, while $L_0(t) + K_{2,0}(t,t)$ is, according to assumption (C1).
Possible motivations for dealing with equations of this type are twofold: the first consists in a natural generalization of the results proved in [4], [12]-[17], concerning the pure integral case (C0), while the latter consists in interpreting the integro-differential case (C1), i.e. equation (1.2), as the limit case of the equation

$$\begin{aligned}
&\varepsilon A(t,D_t)u(t) + L_0(t)u(t) + \int_0^{\alpha(t)} [K_{1,0}(t,s)u'(s) - K_{1,1}(t,s)u(s)]\, ds \\
&+ \int_{\alpha(t)}^{t} [K_{2,0}(t,s)u'(s) - K_{2,1}(t,s)u(s)]\, ds = f(t), \quad t \in [0,T],
\end{aligned} \tag{1.7}$$

where $A(t,D_t)$ is a family of linear linear differential operators defined on the interval $[0,T]$ and taking their values in the space of linear closed, but discontinuous operators, while ε is a positive (small) parameter.

To our knowledge a complete study concerning the integral operator equation (1.1) of the first kind with the approximating equation (1.7), (with $A(t, D_t) = I$) is carried out in [1, Chapt. 5]. There the author deals with a general operator kernel $K(t, s)$ acting in scales of Hilbert ($\{H\}_{s \geq 0}$) or Banach ($\{X\}_{s \geq 0}$) spaces, but the aim consists in establishing the uniqueness of the solution to (1.1)(with $L_0(t) \equiv 0$) and approximating it - in suitable spaces - by the solutions of the family of equations

$$\varepsilon v_\varepsilon(t) + \int_0^t K(t, s) v_\varepsilon(s)\, ds = f(t), \quad t \in [0, T],\ \varepsilon \in \mathbb{R}_+.$$

We stress that the assumptions made in [1] to deduce the desired results are remarkably different from ours, and, of course, no existence result can be found in [1, Chapt. 5]. Apart from this book [1], we observe that there is not a widely developed theory, as far as Banach spaces are concerned. Indeed, all the other results we have found [2, 5, 6, 8, 10, 11, 17] are related to specific integral or integro-differential equations, while the monograph [3] is devoted to *scalar* Volterra integral equations of the first kind, only.
We give now the plan of the paper. Section 2 provides existence and uniqueness results for the solution to problem (1.1) under assumptions (C0). Section 3 supplies auxiliary results for problem (1.2) under assumptions (C1), while Section 4 provides existence and uniqueness results for such a problem. Finally, Section 5 deals with a boundary value problem for an elliptic integro-differential equation "of the first kind". Then an initial and boundary value problem is dealt with for a first-order in time integro-differential equation "of the third kind", or *of degenerate type*. In the latter case the existence and uniqueness results are proved via Semigroup Theory. Finally, Section 6 (Appendix) supplies the outlines of the proof of Theorem 3.4 in Section 3.

2 The first abstract integral equation in the (C0)-case

Associated with function $\alpha : [0, T] \to \mathbb{R}$ enjoying the properties (1.3) and (1.4) we consider the integral operator equation of the first kind (1.1) related to the pair of Banach spaces Y and X, where the operator-valued functions $K_j : \omega_j(T) \to \mathcal{L}(Y; X)$, $j = 1, 2$, and the right-hand side f enjoy the following properties:

$$K_j, D_t K_j \in C(\omega_j(T); \mathcal{L}(Y; X)), \quad j = 1, 2, \tag{2.1}$$

the pair $(K_j, D_t K_j)$ can be continuously extended to $\overline{\omega_j(T)}$, (cf. (1.5)), (1.6)) by a pair still denoted by $(K_j, D_t K_j)$, $j = 1, 2$;

$$K_2(t, t) \text{ is invertible for any } t \in [0, T] \text{ and } t \to K_2(t, t)^{-1} \in C([0, T]; \mathcal{L}(X; Y)); \tag{2.2}$$

$$f \in C^1([0, T]; X), \quad f(0) = 0. \tag{2.3}$$

The main result of this section is

[1] where also scalar cases can be found.

Theorem 2.1. *Under assumptions (C0), (2.1)-(2.3) and the following*

$$\alpha'(0)\|K_2(0,0)^{-1}[K_2(0,0)-K_1(0,0)]\|_{\mathcal{L}(Y;X)} < 1 \tag{2.4}$$

equation (1.1) admits a unique global solution $u \in C([0,T];\mathcal{L}(Y;X))$ continuously depending on the data $f \in C^1([0,T];X)$.

First we need the following lemma.

Lemma 2.1. *The linear mappings*

$$\begin{aligned}
\mathcal{K}_1 u(t) &= \int_0^{\alpha(t)} K_1(t,s)u(s)\,ds, \qquad t \in [0,T],\\
\mathcal{K}_2 u(t) &= \int_{\alpha(t)}^{t} K_2(t,s)u(s)\,ds, \qquad t \in [0,T],
\end{aligned}$$

map $C([0,T];Y)$ into $C([0,T];X)$ and satisfy the estimates

$$\|\mathcal{K}_1 u\|_{C([0,T];X)} \le \alpha(T)\|K_{1,0}\|_{C(\overline{\omega_1(T)};\mathcal{L}(Y,X))}\|u\|_{C([0,T];Y)}, \tag{2.5}$$

$$\|\mathcal{K}_2 u\|_{C([0,T];X)} \le \max_{t\in[0,T]}[t-\alpha(t)]\|K_{2,0}\|_{C(\overline{\omega_2(T)};\mathcal{L}(Y,X))}\|u\|_{C([0,T];Y)}. \tag{2.6}$$

Proof. For the sake of simplicity we limit ourselves to considering the case $j=2$. Let $t_1, t_2 \in (0,T]$ so that

$$0 < \alpha(t_1) < \alpha(t_2).$$

Consider then the following identity

$$\begin{aligned}
&\int_{\alpha(t_2)}^{t_2} K_2(t_2,s)u(s)\,ds - \int_{\alpha(t_1)}^{t_1} K_2(t_1,s)u(s)\,ds\\
&= \int_{t_1}^{t_2} K_2(t_2,s)u(s)\,ds + \int_{\alpha(t_2)}^{t_1}[K_{2,0}(t_2,s)-K_{2,0}(t_1,s)]u(s)ds - \int_{\alpha(t_1)}^{\alpha(t_2)} K_2(t_1,s)u(s)ds.
\end{aligned}$$

Whence we easily imply that $\mathcal{K}_2 u$ is right–continuous at t_1, since the segments (t_2,s), with $s \in [t_1,t_2]$ and $s \in [\alpha(t_2),t_1]$ are contained in the compact set $\overline{\omega_2(T)}$ where $K_{2,0}$ is *continuous.*
Interchanging the roles of t_1 and t_2 we show that $\mathcal{K}_2 u$ is also left–continuous for any $t_1 \in (0,T]$. Finally, the continuity of $\mathcal{K}_2 u$ at $t=0$ easily follows from the estimate

$$\|\mathcal{K}_2 u(t)\| \le \|K_2\|_{C(\overline{\omega_2(T)};\mathcal{L}(Y,X))}\|u\|_{C([0,T];Y)}[t-\alpha(t)], \quad t\in[0,T]. \tag{2.7}$$

From (2.7) we immediately derive the latter estimate in (2.6).
Proceeding similarly, we can prove the analogous property for $\mathcal{K}_1$. $\square$

Proof of Theorem 2.1. We now differentiate both sides of the integral equation (1.1). We obtain the following equation, which is easily seen to be equivalent to (1.1) due to the consistency conditions (2.3):

$$\begin{aligned} &K_2(t,t)u(t) - \alpha'(t)[K_2(t,\alpha(t)) - K_1(t,\alpha(t))]u(\alpha(t)) \\ &+ \int_0^{\alpha(t)} D_t K_1(t,s)u(s)\,ds + \int_{\alpha(t)}^t D_t K_2(t,s)u(s)\,ds = f'(t), \quad t \in [0,T]. \end{aligned} \tag{2.8}$$

Applying the operator $K_2(t,t)^{-1}$ (cf. (C0)) to both sides in (2.8), we easily get that u solves the integro-functional equation

$$\begin{aligned} u(t) = K_3(t)u(\alpha(t)) &+ \int_0^{\alpha(t)} K_4(t,s)u(s)\,ds \\ &+ \int_{\alpha(t)}^t K_5(t,s)u(s)\,ds + f_1(t), \quad t \in [0,T], \end{aligned} \tag{2.9}$$

where the operators $K_3 \in C([0,T];\mathcal{L}(Y))$, $K_4 \in C(\overline{\omega_1(T)};\mathcal{L}(Y;X))$, $K_5 \in C(\overline{\omega_2(T)};\mathcal{L}(Y;X))$ and the function $f_1 \in C([0,T];X)$ are defined, respectively, by

$$\begin{aligned} &K_3(t) = \alpha'(t)K_2(t,t)^{-1}[K_2(t,\alpha(t)) - K_1(t,\alpha(t))], \\ &K_{3+j}(t,s) = K_2(t,t)^{-1}D_t K_j(t,s), \quad j = 1,2, \\ &f_1(t) = K_2(t,t)^{-1}f'(t). \end{aligned}$$

We now first show that equation (2.9) can be solved *locally in time*. For this task we observe that

$$K_3(t) \to \alpha'(0)K_2(0,0)^{-1}[K_2(0,0) - K_1(0,0)] \quad \text{in } \mathcal{L}(Y;X) \text{ as } t \to 0+. \tag{2.10}$$

From assumption (2.4) we easily deduce the following estimate for any $t \in [0,T]$:

$$\begin{aligned} &\left\| K_3(t)u(\alpha(t)) + \int_0^{\alpha(t)} K_4(t,s)u(s)\,ds + \int_{\alpha(t)}^t K_5(t,s)u(s)\,ds \right\|_X \\ \le &\left[\|K_3(t)\|_{\mathcal{L}(Y,X)} + \int_0^{\alpha(t)} \|K_4(t,s)\|_{\mathcal{L}(Y,X)}\,ds + \int_{\alpha(t)}^t \|K_5(t,s)\|_{\mathcal{L}(Y,X)}\,ds \right] \\ &\times \|u\|_{C([0,T];Y)}. \end{aligned}$$

Whence, for all $\tau \in (0,T]$, we conclude that

$$\begin{aligned} &\left\| K_3 + \int_0^{\alpha(\cdot)} K_4(\cdot,s)\,ds + \int_{\alpha(\cdot)}^{\cdot} K_5(\cdot,s)\,ds \right\|_{C([0,\tau];\mathcal{L}(Y;X))} \\ \le &\|K_3\|_{C([0,\tau];\mathcal{L}(Y;X))} + \alpha(\tau)\|K_4\|_{C(\overline{\omega_1(\tau)};\mathcal{L}(Y;X))} + \tau\|K_5\|_{C(\overline{\omega_2(\tau)};\mathcal{L}(Y;X))}. \end{aligned}$$

Consequently, owing to the limit relation (2.10), equation (2.9) admits a unique solution for any small enough τ. Then, taking advantage of property (1.4), [2] the same procedure described in [4] allows to prove that equation (2.9) admits a unique *global* solution $u \in C([0,T];\mathcal{L}(Y;X))$ continuously depending on the datum $f \in C^1([0,T];X)$. □

3 The abstract integro-differential equation (1.2) in the (C1)-case: auxiliary results

Before making our assumptions on the data necessary to solve a Cauchy problem related to equation (1.2), we need to introduce some Hölder spaces.
For any pair of Banach spaces X_1 and X_2 we denote by $C^{0,\delta}(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))$, (resp. $C^\delta([0,T];X)$), $\delta \in (0,1)$, the vector space consisting of all functions $K \in C(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))$ (resp. $f \in C^\delta([0,T];X)$) such that

$$|K|_{C^{1,\delta}(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))} = \sup_{(t_1,s_1),(t_2,s_2)\in\overline{\omega_j(T)},\ s_1\neq s_2}$$

$$\big[|t_2-t_1|+|s_2-s_1|^\delta\big]^{-1}\|K(t_2,s_2)-K(t_1,s_1)\|_{\mathcal{L}(X_1;X_2)} < +\infty, \tag{3.1}$$

$$\Big(|f|_{C^\delta([0,T];X)} = \sup_{t_1,t_2\in[0,T],\ t_1\neq t_2} |t_2-t_1|^{-\delta}\|f(t_2)-f(t_1)\| < +\infty.\Big) \tag{3.2}$$

We observe that $C^{0,\delta}(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))$, (resp. $C^\delta([0,T];X)$) turns out to be a Banach space when endowed with the norm

$$\|K\|_{C^{0,\delta}(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))} = \|K\|_{C(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))} + |K|_{C^{0,\delta}(\overline{\omega_j(T)};\mathcal{L}(X_1;X_2))}. \tag{3.3}$$

$$\Big(\|f\|_{C^\delta([0,T];X)} = \|f\|_{C([0,T];X)} + |f|_{C^\delta([0,T];X)}.\Big) \tag{3.4}$$

In this section we will assume that here the pair (α, f) satisfies, the following properties stricter than (1.3), (1.4) and (2.3):

$$\alpha \in C^{1+\beta}([0,T];\mathbb{R}), \tag{3.5}$$

$$\beta \in (0,1), \quad \alpha(0)=0, \quad 0 \le \alpha'(t) < 1, \quad t\in[0,T], \tag{3.6}$$

$$f \in C^{1+\beta}([0,T];X), \quad f(0)=0. \tag{3.7}$$

Moreover, we assume that the sixtuplet $(K_{1,0},K_{2,0},K_{1,1},K_{2,1},L_0,u_0)$ satisfies, in addition to assumptions (C1), also the following ones for some $\delta \in (0,1)$:

$$K_{j,0} \in C^{1,\delta}(\overline{\omega_2(T)};\mathcal{L}(Y;X)), \quad j=1,2, \tag{3.8}$$

$$t \to K_{2,0}(t,t) \in C^\delta([0,T];\mathcal{L}(X)), \tag{3.9}$$

$$t \to K_{2,1}(t,t) \in C^\delta([0,T];\mathcal{L}(Y;X)), \tag{3.10}$$

[2]For a similar procedure see also Section 3.

$$\alpha'(0)[K_{2,0}(0,0) - K_{1,0}(0,0)] = O, \tag{3.11}$$

$$L_0 \in C^{1+\delta}([0,T];\mathcal{L}(X)), \quad u_0 \in N(L_0(0)) \cap Y, \tag{3.12}$$

$$\begin{aligned} &f'(0) + [L_0(0) + K_{2,0}(0,0)]^{-1} \\ &\times\Big\{L_0'(0) + K_{2,1}(0,0) + \alpha'(0)[K_{2,0}(0,0) - K_{1,0}(0,0)]\Big\}u_0 \\ &\in (Y;X)_{\beta,\infty}, \end{aligned} \tag{3.13}$$

$(Y;X)_{\beta,\infty}$ denoting the intermediate space between Y and X of order β and index $p = +\infty$, (cf. [7, Chapt. 2]).

We will deal with the initial problem: *look for a function* $u \in C^1([0,T];X) \cap C([0,T];Y)$ *satisfying*

$$\begin{aligned} &L_0(t)u(t) + \int_0^{\alpha(t)} [K_{1,0}(t,s)u'(s) + K_{1,1}(t,s)u(s)]\,ds \\ &+ \int_{\alpha(t)}^t [K_{2,0}(t,s)u'(s) + K_{2,1}(t,s)u(s)]\,ds = f(t), \quad t \in [0,T], \end{aligned} \tag{3.14}$$

$$u(0) = u_0 \in N(L_0(0)) \cap Y. \tag{3.15}$$

We now explain why the initial datum u_0 has to be restricted to $N(L_0(0))$. This is an immediate consequence of the fact that from equation (1.2) we deduce that the necessary condition $L_0(0)u(0) = f(0) = 0$ (cf. (3.7)) must be fulfilled by the initial datum $u(0)$.
Finally, we observe that, if $N(L_0(0)) = \{0\}$, then $u_0 = 0$.
The main result of this section is

Theorem 3.1. *Under assumptions (C1) and (3.5)-(3.13) problem (3.14), (3.15) admits a unique global solution* $u \in C^1([0,T];X) \cap C([0,T];Y)$ *continuously depending on the data* $(f,u_0) \in C^1([0,T];X) \times Y$.

Proof of Theorem 3.1. Differentiating both sides of (3.14) with respect to t, we obtain the first-order differential-functional equation

$$\begin{aligned} &[L_0(t) + K_{2,0}(t,t)]u'(t) - \alpha'(t)[K_{2,0}(t,\alpha(t)) - K_{1,0}(t,\alpha(t))]u'(\alpha(t)) \\ &+[L_0'(t) + K_{2,1}(t,t)]u(t) - \alpha'(t)[K_{2,1}(t,\alpha(t)) - K_{1,1}(t,\alpha(t))]u(\alpha(t)) \\ &+ \int_0^{\alpha(t)} [D_tK_{1,0}(t,s)u'(s) + D_tK_{1,1}(t,s)u(s)]\,ds \\ &+ \int_{\alpha(t)}^t [D_tK_{2,0}(t,s)u'(s) + D_tK_{2,1}(t,s)u(s)]\,ds = f'(t), \quad t \in [0,T]. \end{aligned} \tag{3.16}$$

Introduce then the operators $K_3, K_5 : [0,T] \to \mathcal{L}(X)$, $K_4, K_5 : [0,T] \to \mathcal{L}(Y;X)$, $K_{5+j} : \overline{\omega_j(T)} \to \mathcal{L}(X)$, $K_{7+j} : \overline{\omega_j(T)} \to \mathcal{L}(Y;X)$, $j = 1,2$, and the function $g : [0,T] \to Y$ defined by

$$K_3(t) = [L_0(t) + K_{2,0}(t,t)]^{-1}[K_{2,0}(t,\alpha(t)) - K_{1,0}(t,\alpha(t))], \tag{3.17}$$

$$K_4(t) = -[L_0(t) + K_{2,0}(t,t)]^{-1}[L_0'(t) + K_{2,1}(t,t)], \tag{3.18}$$

$$K_5(t) = [L_0(t) + K_{2,0}(t,t)]^{-1}[K_{2,1}(t,\alpha(t)) - K_{1,1}(t,\alpha(t))], \tag{3.19}$$

$$K_{5+j}(t,s) = [L_0(t) + K_{2,0}(t,t)]^{-1}D_t K_{j,0}(t,s), \ j = 1,2, \tag{3.20}$$

$$K_{7+j}(t,s) = [L_0(t) + K_{2,0}(t,t)]^{-1}D_t K_{j,1}(t,s), \ j = 1,2, \tag{3.21}$$

$$g(t) = [L_0(t) + K_{2,0}(t,t)]^{-1} f'(t). \tag{3.22}$$

Applying the operator $[L_0(t) + K_{2,0}(t,t)]^{-1}$ to both sides of (3.16), we deduce that u solves the following Cauchy problem, which is equivalent to (3.14), (3.15):

$$u'(t) - \alpha'(t)K_3(t)u'(\alpha(t)) - K_4(t)u(t) - \alpha'(t)K_5(t)u(\alpha(t))$$

$$+ \int_0^{\alpha(t)} [K_6(t,s)u'(s) + K_8(t,s)u(s)]\, ds$$

$$+ \int_{\alpha(t)}^t [K_7(t,s)u'(s) + K_9(t,s)u(s)]\, ds = g(t), \quad t \in [0,T],$$

$$u(0) = u_0 \in N(L_0(0)) \cap Y. \tag{3.23}$$

We stress that, to uniquely solve the first-order in time equation (3.23) we do need an initial condition like (3.24). Moreover, if we did not prescribe any condition to equation (3.14), (3.23) would admit infinitely many solutions depending on a vector $c \in N(L_0(0))$.

Theorem 3.2. *For any $h \in C^\beta([0,T];X)$, $\beta \in (0,\delta)$, the functional equation*

$$\zeta(t) - \alpha'(t)K_3(t)\zeta(\alpha(t)) = h(t), \quad t \in [0,T], \tag{3.24}$$

admits a unique solution $\zeta \in C^\beta([0,T];X)$, with $\zeta(0) = h(0)$, represented by

$$\zeta(t) = h(0) + \widetilde{h}(t) + (Z\widetilde{h})(\alpha(t)), \quad t \in [0,T], \tag{3.25}$$

where

$$\widetilde{h}(t) = h(t) - [I - \alpha'(t)K_3(t)]h(0), \quad t \in [0,T]. \tag{3.26}$$

Moreover, there exists $\tau_0 \in (0,T]$ such that Z satisfies the estimate

$$\|Z\|_{\mathcal{L}(C^\beta([0,\tau];\mathcal{L}(X))))} \le \frac{J_1(K_3,\tau)}{1 - J_1(K_3,\tau)}, \quad \tau \in (0,\tau_0], \tag{3.27}$$

where $J_1(K_3,\tau)$, $\tau \in (0,\tau_0]$, is defined by

$$J_1(K_3,\tau) := \max\Big\{\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}|\alpha'|^\beta_{C^\beta([0,\tau];\mathbb{R})},\ \tau^\beta|\alpha'|_{C^\beta([0,T];\mathbb{R})}\|K_3\|_{C([0,T];\mathcal{L}(X))}$$

$$+ \|\alpha'\|_{C([0,T];\mathbb{R})}|K_3|_{C^\beta([0,T];\mathcal{L}(X))}\Big\} < 1. \tag{3.28}$$

Finally, the restriction of Z to $[0,\tau_0]$ is defined by

$$(Z\widetilde{h})(t) = \sum_{n=1}^{+\infty} \Big[\prod_{j=0}^{n-1} (\alpha' \circ \alpha^j)(t) K_3(\alpha^j(t))\Big] \widetilde{h}(\alpha^{n-1}(t)), \quad t \in [0,\tau], \tag{3.29}$$

where $\circ$ denotes composition and $\alpha^j = (\alpha\circ)^{j-1}\alpha$, $j \in \mathbb{N}\backslash\{0\}$, $\alpha^0(t) = t$.

Proof. First we observe that, according to our assumption on α', there exists a $\tau_1 \in (0,T]$ satisfying the following inequalities for all $\tau \in (0,\tau_1]$ (cf. (1.4)):

$$0 \le \alpha'(\tau) \le 1, \tag{3.30}$$

$$\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} |\alpha'|^{\beta}_{C^\beta([0,\tau];\mathbb{R})} \le \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} < 1, \tag{3.31}$$

$$\tau^\beta |\alpha'|_{C^\beta([0,T];\mathbb{R})} \|K_3\|_{C([0,T];\mathcal{L}(X))} + \|\alpha'\|_{C([0,T];\mathbb{R})} |K_3|_{C^\beta([0,T];\mathcal{L}(X))} < 1. \tag{3.32}$$

First we want now to prove that the linear operator

$$M\zeta(t) = \alpha'(t) K_3(t) \zeta(\alpha(t)), \quad t \in [0,T], \tag{3.33}$$

maps $C([0,\tau];X)$ into itself and satisfies, for $\zeta \in C([0,\tau];X)$,

$$\|M\zeta(t)\|_{\mathcal{L}(X)} \le \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} \|\zeta\|_{C([0,t];X)}, \quad t \in [0,\tau_1].$$

Consequently, since $\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} < 1$ according to assumption (3.31), equation (3.25) admits a unique solution $\zeta_1 = (I-M)^{-1}h$ in $C([0,\tau];X)$ represented by (3.26). Hence, according to Neumann's theorem $I - M$ is invertible from $C([0,\tau];$ $\mathcal{L}(X))$ into itself. Moreover,

$$\begin{aligned}
&\|(Zh)(t)\|_{\mathcal{L}(X)} = \|[(I-M)^{-1} - I]h(t)\|_{\mathcal{L}(X)} = \|(I-M)^{-1}Mh(t)\|_{\mathcal{L}(X)} \\
&\le \|h\|_{C([0,t];X)} \sum_{n=1}^{+\infty} \|M\|^n_{\mathcal{L}(X)} \le \|h\|_{C([0,t];X)} \sum_{n=1}^{+\infty} \|\alpha' K_3\|^n_{C([0,\tau];\mathcal{L}(X))} \\
&= \frac{\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}}{1 - \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}} \|h\|_{C([0,t];X)}, \quad t \in [0,\tau].
\end{aligned}$$

To show that such our local solution ζ can be extended to a *global* one, introduce, as in [4], the finite sequence of positive real points defined by

$$\tau_{j+1} = \alpha^{-1}(\tau_j), \quad j = 1,\ldots,n, \quad (\tau_n \le T < \tau_{n+1}). \tag{3.34}$$

We proceed by recurrence and assume to have shown that equation (3.25) admits a solution ζ_j defined on the interval $[\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{N}$, with $\tau_0 = 0$.
Consider then the following function ζ_{j+1} defined by

$$\zeta_{j+1}(t) = \alpha'(t) K_3(t) \zeta_j(\alpha(t)) + h(t), \quad t \in [\tau_j, \tau_{j+1}].$$

So, we conclude that equation (4.30) admits a unique *global* solution $\zeta_{j+1} \in C([\tau_j, \tau_{j+1}]; Y)$ such that $\zeta_{j+1}(\tau_j) = \zeta_j(\tau_j)$. Consequently, the function

$$\zeta(t) = \zeta_j(t), \quad t \in [\tau_{j-1}, \tau_j],\ j = 1,\ldots,n, \tag{3.35}$$

solves the equation (3.25) in the interval $[0,T]$. Moreover, the functions ζ_j satisfy the recurrence estimates for $j=0,\dots,n-1$:

$$\begin{aligned}\|\zeta_{j+1}\|_{C([\tau_j,\tau_{j+1}];X)} \le \|\alpha' K_3\|_{C([\tau_j,\tau_{j+1}];\mathcal{L}(X))}\|\zeta_j\|_{C([\tau_{j-1},\tau_j];X)}\\ + \|h\|_{C([\tau_j,\tau_{j+1}];X)}.\end{aligned} \tag{3.36}$$

Since $\|\zeta_1\|_{C([0,\tau];X)} \le \big[1-\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}\big]^{-1}$, from (3.37), we deduce the estimate

$$\begin{aligned}\|\zeta_{j+1}\|_{C([\tau_j,\tau_{j+1}];X)} &\le \big[1-\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}\big]^{-1}\prod_{i=0}^{j}\|\alpha' K_3\|_{C([\tau_i,\tau_{i+1}];\mathcal{L}(X))}\\ &+\sum_{i=1}^{j}\|h\|_{C([\tau_i,\tau_{i+1}];X)}\prod_{k=i+1}^{j}\|\alpha' K_3\|_{C([\tau_k,\tau_{k+1}];\mathcal{L}(X))},\quad j=0,\dots,n-1,\end{aligned}$$

where $\prod_{k=j+1}^{j}\cdots=1$, $j=0,\dots,n-1$.
We have so proved that ζ satisfies the estimate

$$\|\zeta\|_{C([0,T];X)} \le J_2(\alpha' K_3)\|h\|_{C([0,T];X)},$$

for some positive constant $J_2(\alpha' K_3)$ depending on the norm $\|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))}$, only. Then we want to prove that M maps $C^\beta([0,T];X)$ into itself. For this task we need the following estimates for all $t_1,t_2\in[0,\tau]$, $t_1\le t_2$ and $\tau\in(0,T]$, (cf. (C1)):

$$|\alpha'(t_2)-\alpha'(t_1)| \le |\alpha'|_{C^\beta([0,t])}|t_2-t_1|^\beta, \tag{3.37}$$

$$\begin{aligned}\|\zeta\circ\alpha(t_2)-\zeta\circ\alpha(t_1)\| &\le |\zeta|_{C^\beta([0,t];X)}|\alpha(t_2)-\alpha(t_1)|^\beta\\ &\le |\zeta|_{C^\beta([0,t];X)}|\alpha'|^\beta_{C([0,t])}|t_2-t_1|^\beta,\end{aligned} \tag{3.38}$$

$$\begin{aligned}&\|K_{j,0}(t_2,\alpha(t_2))-K_{j,0}(t_1,\alpha(t_1))\|_{\mathcal{L}(X)}\\ &\le \|K_{j,0}\|_{C^{1,\delta}([0,t];\mathcal{L}(X))}\big[|t_2-t_1|+|\alpha(t_2)-\alpha(t_1)|^\delta\big]\\ &\le \|K_{j,0}\|_{C^{1,\delta}([0,t];\mathcal{L}(X))}|t_2-t_1|^\delta\big[|t_2-t_1|^{1-\delta}+\|\alpha'\|^\delta_{C([0,T])}\big]\\ &\le \|K_{j,0}\|_{C^{1,\delta}([0,t];\mathcal{L}(X))}|t_2-t_1|^\beta\tau^{\delta-\beta}\big[\tau^{1-\delta}+\|\alpha'\|^\delta_{C([0,T])}\big],\quad j=1,2,\end{aligned} \tag{3.39}$$

$$\begin{aligned}&\|[L_0(t_2)+K_{2,0}(t_2,t_2)]^{-1}-[L_0(t_1)+K_{2,0}(t_1,t_1)]^{-1}\|_{\mathcal{L}(X)}\\ &\le \|L_0(t_2)-L_0(t_1)+K_{2,0}(t_2,t_2)-K_{2,0}(t_1,t_1)\|_{\mathcal{L}(X)}\\ &\quad\times\prod_{j=1}^{2}\|[L_0(t_j)+K_{2,0}(t_j,t_j)]^{-1}\|_{\mathcal{L}(X)}\\ &\le \mu_1^{-2}|t_2-t_1|^\delta\big[\tau^{1-\delta}\|L_0\|_{C^1([0,T];\mathcal{L}(X))}+\|\widetilde{K}_{2,0}\|_{C^\delta([0,T];\mathcal{L}(X))}\big]\\ &\le \mu_1^{-2}|t_2-t_1|^\beta\tau^{\delta-\beta}\big[\tau^{1-\delta}\|L_0\|_{C^1([0,T];\mathcal{L}(X))}+\|\widetilde{K}_{2,0}\|_{C^\delta([0,T];\mathcal{L}(X))}\big],\end{aligned} \tag{3.40}$$

where $\widetilde{K}_{2,0}(t)=K_{2,0}(t,t)$, $t\in[0,T]$.

Consequently, from (3.41) and definition (3.17) we deduce the following estimate for all pair $t_1, t_2 \in [0,\tau]$:

$$\|K_3(t_2) - K_3(t_1)\|_{\mathcal{L}(X)} \leq |t_2 - t_1|^\beta \tau^{\delta-\beta} J_2(K_{1,0}, K_{2,0}, L_0, T), \tag{3.41}$$

J_2 being a functional continuous depending on T and the norms $\|K_{j,0}\|_{C^{1,\delta}(\overline{\omega_j(T)};\mathcal{L}(X))}$, $j = 1, 2$, and $\|L_0\|_{C^1([0,T];\mathcal{L}(X))}$.
Assume now $h(0) = 0$, so that $\zeta(0) = h(0) = 0$. Hence, from definition (3.34) we easily deduce the following estimates, holding for all $t \in [0,\tau]$:

$$\begin{aligned}
&|M\zeta|_{C^\beta([0,t];\mathcal{L}(X))} \leq \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} |\zeta \circ \alpha|_{C^\beta([0,t];X)} \\
&+ |\alpha' K_3|_{C^\beta([0,t];\mathcal{L}(X))} \|\zeta\|_{C([0,t];X)} \leq \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} |\alpha'|^\beta_{C([0,T];\mathbb{R})} |\zeta|_{C^\beta([0,t];X)} \\
&+ \|\zeta\|_{C([0,t];X)} \Big[|\alpha'|_{C^\beta([0,T];\mathbb{R})} \|K_3\|_{C([0,\tau];\mathcal{L}(X))} + \|\alpha'\|_{C([0,\tau];\mathbb{R})} |K_3|_{C^\beta([0,\tau];\mathcal{L}(X))} \Big] \\
&\leq \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} |\alpha'|^\beta_{C([0,T];\mathbb{R})} |\zeta|_{C^\beta([0,t];X)} + t^\beta |\zeta|_{C^\beta([0,t];X)} \\
&\quad \times \Big[|\alpha'|_{C^\beta([0,T];\mathbb{R})} \|K_3\|_{C([0,T];\mathcal{L}(X))} + \|\alpha'\|_{C([0,T];\mathbb{R})} |K_3|_{C^\beta([0,T];\mathcal{L}(X))} \Big].
\end{aligned} \tag{3.42}$$

Finally, setting $t \in [0,\tau]$, from (3.42) and (3.43) we deduce the following estimate for M:

$$\begin{aligned}
&\|M\zeta\|_{C^\beta([0,\tau];\mathcal{L}(X))} \leq \|\zeta\|_{C^\beta([0,\tau];X)} \max \Big\{ \|\alpha' K_3\|_{C([0,\tau];\mathcal{L}(X))} |\alpha'|^\beta_{C^\beta([0,\tau];\mathbb{R})}, \\
&\tau^\beta |\alpha'|_{C^\beta([0,T];\mathbb{R})} \|K_3\|_{C([0,T];\mathcal{L}(X))} + \|\alpha'\|_{C([0,T];\mathbb{R})} |K_3|_{C^\beta([0,T];\mathcal{L}(X))} \Big\} \\
&=: \|\zeta\|_{C^\beta([0,t];X)} J_1(K_3, \tau), \quad t \in [0,\tau].
\end{aligned} \tag{3.43}$$

Observe now that, according to inequalities (3.31) and (3.33), for any $\tau \in (0, \tau_0]$ we have $J_1(K_3, \tau) < 1$. Consequently, the linear operator $I - M$ maps continuously $C^\beta([0,\tau]; X)$ into itself and is invertible in $C^\beta([0,\tau]; \mathcal{L}(X))$ for all $\tau \in [0, \tau_0]$. Moreover the following estimate holds true for all $\tau \in [0, \tau_0]$:

$$\begin{aligned}
\|Z\|_{\mathcal{L}(C^\beta([0,\tau];\mathcal{L}(X))} &= \|(I - M)^{-1} - I\|_{\mathcal{L}(C^\beta([0,t];\mathcal{L}(X))} \\
&\leq J_1(K_3, \tau) \big[1 - J_1(K_3, \tau)\big]^{-1}.
\end{aligned}$$

Consequently, the linear operator $I - M$ maps continuously $C^\beta([0,\tau]; X)$ into itself and is invertible in $C^\beta([0,\tau]; \mathcal{L}(X))$ for any $\tau \in (0, \tau_0]$. Therefore $\zeta_1 \in C^\beta([0,\tau]; X)$ and satisfies the estimate

$$\|\zeta_1\|_{C^\beta([\tau_0,\tau_1];X)} \leq \big[1 - J_1(K_3, T)\big]^{-1}.$$

Reasoning as in the first part of this proof we conclude that each function ζ_j, $j = 1, \ldots, n$, defined above belongs to $C^\beta([\tau_j, \tau_{j+1}]; X)$ and satisfies the equality $\zeta_{j+1}(\tau_j) = \zeta_j(\tau_j)$ as well as the following recurrence estimates for $j = 1, \ldots, n-1$:

$$\|\zeta_{j+1}\|_{C^\beta([\tau_j,\tau_{j+1}];X)} \leq \|\alpha' K_3\|_{C^\beta([\tau_j,\tau_{j+1}];\mathcal{L}(X))} \|\zeta_j\|_{C^\beta([\tau_{j-1},\tau_j];X)}$$

$$+\|h\|_{C^\beta([\tau_j,\tau_{j+1}];X)}.$$

Since $\|\zeta_1\|_{C^\beta([0,\tau];X)} \le J_1(K_3,\tau)$, from (3.37), we deduce the estimate

$$\|\zeta_{j+1}\|_{C^\beta([\tau_j,\tau_{j+1}];X)} \le J_1(K_3,\tau)\prod_{i=0}^{j}\|\alpha' K_3\|_{C^\beta([\tau_i,\tau_{i+1}];\mathcal{L}(X))}$$
$$+\sum_{i=1}^{j}\|h\|_{C^\beta([\tau_i,\tau_{i+1}];X)}\prod_{k=i+1}^{j}\|\alpha' K_3\|_{C^\beta([\tau_k,\tau_{k+1}];\mathcal{L}(X))}, \quad j=1,\dots,n-1.$$

We have so proved, when $h(0)=0$, that ζ defined by (3.36) satisfies the estimate

$$\|\zeta\|_{C^\beta([0,T]:X)} \le J_3(\alpha',K_3,T,\tau)\|h\|_{C^\beta([0,T];X)},$$

for some positive constant $J_3(\alpha',K_3,T,\tau)$ depending on (T,τ) and the norms $\|\alpha'\|_{C^\beta([0,T])}$, $\|K_3\|_{C^\beta([0,T];\mathcal{L}(X))}$, only.
We now consider the case $h(0)\neq 0$. Introducing the function

$$z(t)=\zeta(t)-h(0).$$

It immediate to check that z solves the problem

$$z(t)-\alpha'(t)K_3(t)z(\alpha(t))=\widetilde{h}(t), \quad t\in[0,T],$$

where $\widetilde{h}$ is defined by (3.27).
Since $\widetilde{h}(0)=0$ (cf. (3.11)), we conclude that z belongs to $C^\beta([0,T]:X)$ and satisfies the estimate

$$\|z\|_{C^\beta([0,T]:X)} \le J_3(\alpha',K_3,T,\tau)\|\widetilde{h}\|_{C^\beta([0,T];X)},$$

Moreover, z admits the representation

$$z(t)=(I-M)^{-1}\widetilde{h}(t)=\widetilde{h}(t)+Z(\widetilde{h})(t), \quad t\in[0,T].$$

This concludes the proof. □

4 Solving the abstract integro-differential Cauchy problem (3.23), (3.24)

First we set

$$h(t)=g(t)+K_4(t)u(t)+L_1u(\alpha(t))-L_2u(t)-L_3u(t), \quad t\in[0,T], \tag{4.1}$$

where, for all $t\in[0,T]$, we have set

$$L_1(t)=\alpha'(t)K_5(t), \tag{4.2}$$

$$L_2u(t)=\int_0^{\alpha(t)}\big[K_6(t,s)u'(s)+K_8(t,s)u(s)\big]\,ds, \tag{4.3}$$

$$L_3u(t) = \int_{\alpha(t)}^{t} \big[K_7(t,s)u'(s) + K_9(t,s)u(s)\big]\, ds. \tag{4.4}$$

We can now rewrite problem (3.23), (3.24) in the form

$$u'(t) - \alpha'(t)K_3(t)u'(\alpha(t)) = h(t), \quad t \in [0,T], \tag{4.5}$$

$$u(0) = u_0 \in N(L_0(0)) \cap Y, \tag{4.6}$$

Observe now that

$$h(0) = g(0) + K_4(0)u_0 + \alpha'(0)K_5(0)u_0, \tag{4.7}$$

Define then

$$\begin{aligned} \widetilde{h}(t) &= \alpha'(t)K_3(t)h(0) + h(t) - h(0) = \alpha'(t)K_3(t)h(0) + g(t) - g(0) \\ &\quad + K_4(t)[u(t) - u(0)] + L_1[u(\alpha(t)) - u(0)] - L_2u(t) - L_3u(t), \end{aligned} \tag{4.8}$$

for any $t \in [0,T]$. From (4.5), (4.7), (4.8) we deduce that u solves the Cauchy problem

$$\begin{aligned} u'(t) &= h(0) + (I-M)^{-1}\widetilde{h}(t) = h(0) + (I-M)^{-1}(\alpha' K_3 h(0))(t) \\ &\quad + (I-M)^{-1}(g - g(0))(t) + (I-M)^{-1}(K_4(u - u(0))(t) \\ &\quad + (I-M)^{-1}(L_1((u\circ\alpha) - u(0)))(t)) - (I-M)^{-1}(L_2u)(t) \\ &\quad - (I-M)^{-1}(L_3u)(t), \quad t \in [0,T], \end{aligned} \tag{4.9}$$

$$u(0) = u_0 \in N(L_0(0)) \cap Y. \tag{4.10}$$

Setting $t = 0$ in equation (4.5) and using the latter condition in (3.11), we easily compute

$$u'(0) = h(0). \tag{4.11}$$

Introduce now the new unknown defined by

$$v(t) = u(t) - u_0, \quad t \in [0,T] \quad \Longrightarrow \quad v(0) = 0,\ v'(0) = h(0). \tag{4.12}$$

Therefore, owing to Theorem 3.2 and the representation formula (3.26), via the identity $(I-M)^{-1}(K_4v)(t) = (K_4v)(t) + Z(K_4v)(\alpha(t))$, problem (4.5), (4.6) is equivalent to the following

$$\begin{aligned} v'(t) - K_4(t)v(t) &= \widetilde{g}(t) + Z(K_4v)(\alpha(t)) + (I-M)^{-1}(L_1(v\circ\alpha))(t) \\ &\quad - (I-M)^{-1}(L_2v)(t) - (I-M)^{-1}(L_3v)(t), \quad t \in [0,T], \end{aligned} \tag{4.13}$$

$$v(0) = 0. \tag{4.14}$$

The family of operators $\{K_4(t)\}_{t\in[0,T]}$ is assumed to satisfy the following assumptions:

(H1) the domain of $K_4(t)$ is independent of $t \in [0,T]$, its spectrum contains the set $\Sigma_\phi \cup \{0\}$, where

$$\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg \lambda| \le \phi\}, \quad \phi \in (\pi/2, \pi); \tag{4.15}$$

(H2) the following inequalities holds true:

$$\|[\lambda - K_4(t)]^{-1}\|_{\mathcal{L}(X)} \le c_0 |\lambda|^{-1}, \quad \lambda \in \Sigma_\phi,\ t \in [0,T], \tag{4.16}$$

$$\|K_4(t)K_4(0)^{-1} - K_4(t)K_4(0)^{-1}\|_{\mathcal{L}(X)} \le c_2 |t_2 - t_1|^\beta, \quad 0 \le t_1 \le t_2 \le T. \tag{4.17}$$

for some positive constants c_0 and c_2,

In particular, from (4.17) it follows that $K_4(t)K_4(0)^{-1}$ is uniformly bounded in $\mathcal{L}(X)$ with respect to $t \in [0,T]$ as well as

$$\|\big[K_4(t_2) - K_4(t_1)\big]K_4(t_1)^{-1}\|_{\mathcal{L}(X)} \le c_2 |t_2 - t_1|^\beta, \quad 0 \le t_1 \le t_2 \le T,$$

for some positive constants c_2 (cf. formula (5.5) in [8]).
Then the following estimates hold true:

$$\|e^{tK_4(s)}\|_{\mathcal{L}(X)} \le c_1 e^{t\lambda_0} \le c_1, \quad \|e^{tK_4(s)}\|_{\mathcal{L}(X)} \le c_2 t^{-1}, \quad s,t \in [0,T].$$

for some constants c_0 and c_1.
We have set here

$$\begin{aligned}\widetilde{g}(t) &= (I-M)^{-1}(\alpha' K_3 h(0))(t) + (I-M)^{-1}(g - g(0))(t)\\ &\quad - (I-M)^{-1}(L_4(u_0, h(0))(t) - (I-M)^{-1}(L_5(u_0, h(0))(t), \quad t \in [0,T],\end{aligned} \tag{4.18}$$

$$L_4(u_0, h(0))(t) = \int_0^{\alpha(t)} \big[K_6(t,s)h(0) + K_8(t,s)u_0\big]\, ds, \quad t \in [0,T], \tag{4.19}$$

$$L_5(u_0, h(0))(t) = \int_{\alpha(t)}^{t} \big[K_7(t,s)h(0) + K_9(t,s)u_0\big]\, ds, \quad t \in [0,T]. \tag{4.20}$$

Observe now that from our latter assumption in (3.11), the formula $[(I-M)^{-1}l](0) = 0$, if $l(0) = 0$, and definitions (4.18)-(4.20), we get

$$\widetilde{g}(0) = 0.$$

4.1 Showing that $\widetilde{g} \in C^\beta([0,T];X)$

Then we need to estimate $L_j u$, $j = 1, \ldots, 5$. For this task we need the following lemma.

Lemma 4.1. *Let $H_j \in C^{\beta,0}(\overline{\omega_j(T)}; \mathcal{L}(X_1; X_2))$, $j = 1,2$, X_1 and X_2 being two Banach spaces. Then the linear operators*

$$L_6 f(t) = \int_0^{\alpha(t)} H_1(t,s) f(s)\, ds, \quad L_7 f(t) = \int_{\alpha(t)}^{t} H_2(t,s) f(s)\, ds, \quad t \in [0,T], \tag{4.21}$$

map $C([0,T];X_1)$ continuously into $C^\beta([0,T];X_2)$ and satisfy the estimates

$$\|L_{5+j}f\|_{C^\beta([0,\tau];X_2)} \le J_{3+j}(H_j,\tau,X_1;X_2)\|f\|_{C([0,\tau];X_1)}, \quad \tau\in(0,T],\ j=1,2,$$

where

$$J_4(H_1,\tau,X_1;X_2) = \big(\tau^{1-\beta}+\tau\big)\|\alpha'\|_{C([0,T];\mathbb{R})}\|H_1\|_{C^{\beta,0}(\overline{\omega_j(\tau)};\mathcal{L}(X_1;X_2))},$$

$$J_5(H_2,\tau,X_1;X_2) = \big\{\tau^{1-\beta}\big(1+\|\alpha'\|_{C([0,T];\mathbb{R})}\big) + \sup_{t\in[0,\tau]}\big[t-\alpha(t)\big]\big\} \times \|H_2\|_{C^{\beta,0}(\overline{\omega_j(\tau)};\mathcal{L}(X_1;X_2))}.$$

Proof. We limit ourselves with dealing with operator L_7, since the result for L_6 can be derived analogously.
Let (t_1,t_2) be a pair such that $0\le t_1\le t_2\le T$. It suffices to consider the two cases $\alpha(t_2)\le t_1$ and $t_1<\alpha(t_2)$ and to notice that the following formulae hold

$$L_7f(t_2)-L_7f(t_1) = \int_{t_1}^{t_2} H_2(t_2,s)f(s)\,ds + \int_{\alpha(t_2)}^{t_1}[H_2(t_2,s)-H_2(t_1,s)]f(s)\,ds - \int_{\alpha(t_1)}^{\alpha(t_2)} H_2(t_1,s)f(s)\,ds, \quad \text{if } \alpha(t_2)\le t_1(\le t_2), \tag{4.22}$$

$$L_7f(t_2)-L_7f(t_1) = \int_{t_1}^{t_2} H_2(t_2,s)f(s)\,ds + \int_{\alpha(t_1)}^{t_1}[H_2(t_2,s)-H_2(t_1,s)]f(s)\,ds - \int_{\alpha(t_1)}^{\alpha(t_2)} H_2(t_2,s)f(s)\,ds, \quad \text{if } t_1<\alpha(t_2)(\le t_2). \tag{4.23}$$

The assertion easily follows from formulae (4.21), (4.22), (4.23). □

Corollary 4.1. *The linear operators L_6 and L_7 defined by (4.19) and (4.20) satisfy the following estimates:*

$$\|L_6(u_0,h(0))\|_{C^\beta([0,\tau];X)} \le J_4(K_6,\tau,X,X)\|h(0)\| + J_4(K_8,\tau,Y,X)\|u_0\|_Y,$$

$$\|L_7(u_0,h(0))\|_{C^\beta([0,\tau];X)} \le J_5(K_7,\tau,X,X)\|h(0)\| + J_5(K_9,\tau,Y,X)\|u_0\|_Y.$$

Proof. It immediately follows from Lemma 4.1. □

By virtue of Theorem 3.2 and recalling that $\alpha'(0)K_3(0)h(0)=0$ (cf. (3.11)), even though $h(0)\ne 0$, we can now estimate $\widetilde{g}$ in $C^\beta([0,T];X)$ (cf. (4.18));

$$\|\widetilde{g}\|_{C^\beta([0,\tau];X)} \le [1-J_1(K_3,\tau_0)]^{-1}\Big\{\|\alpha' K_3\|_{C^\beta([0,\tau];X)}\|h(0)\| + \|g-g(0)\|_{C^\beta([0,\tau];X)} + \sum_{j=1}^{2}\big[J_6(K_{5+j},\tau,X,X)\|h(0)\| + J_6(K_{7+j},\tau,Y,X)\|u_0\|_Y\big]\Big\},$$

where

$$J_6(K,\tau,X_1,X_2) = J_4(K,\tau,X_1,X_2) + J_5(K,\tau,X_1,X_2).$$

4.2 Solving the Cauchy problem

By virtue of theorem 6.1.3 in [7] we deduce that the solution w to problem (4.13), (4.14), with $Z = L_1 = L_2 = L_3 = O$, belongs to

$$\mathcal{Z}^\beta(\tau) = C^{1+\beta}([0,\tau];X) \cap C^\beta([0,\tau];Y), \quad \tau \in (0,T],$$

and satisfies the estimate

$$\|w\|_{\mathcal{Z}^\beta(\tau)} \le C(T)\|\widetilde{g}\|_{C^\beta([0,\tau];X)}, \quad \tau \in (0,T],$$

the positive constant $C(T)$ being independent of $\tau \in (0,T]$.
Then we notice that problem (4.13), (4.14) is equivalent to the integral equation

$$v(t) = w(t) + \int_0^t G(t,s)Z(K_4(v\circ\alpha))(s)\,ds + \int_0^t G(t,s)(I-M)^{-1}(L_1(v\circ\alpha))(s)\,ds$$
$$+ \int_0^t G(t,s)(I-M)^{-1}(L_2 v)(s)\,ds + \int_0^t G(t,s)(I-M)^{-1}(L_3 v)(s)\,ds, \quad (4.24)$$

for any $t \in [0,T]$, where G denotes the evolution operator associated with the family of operators $\{K_4(t)\}_{t\in[0,T]}$ (cf. corollary 6.1.8 in [7]) and

$$w(t) = \int_0^t G(t,s)(I-M)^{-1}\widetilde{g}(s)\,ds, \quad t \in [0,T].$$

We recall that the kernel G satisfies the estimates

$$\|G(t,s)\|_{\mathcal{L}(X)} \le C_0, \quad \|D_t G(t,s)\|_{\mathcal{L}(X)} + \|K_4(t)G(t,s)\|_{\mathcal{L}(X)} \le C_0(t-s)^{-1},$$

for any $0 < s < t < T$ and some positive constant C_0 independent of $(t,s) \in \overline{\omega_1(T)} \cup \overline{\omega_2(T)}$.
Finally, observe that, owing to definition (3.18), properties (3.9), (3.12), and inequality (3.41), it is not difficult to show that K_4 belongs to $C^\beta([0,T];\mathcal{L}(Y;X))$. More precisely, setting $\widetilde{K}_{2,j}(t) = K_{2,j}(t,t)$, $t \in [0,T]$ and $j = 0,1$, we have

$$\|K_4\|_{C^\beta([0,T];\mathcal{L}(Y;X))} \le \|[L_0 + \widetilde{K}_{2,0}]^{-1}\|_{C^\beta([0,T];\mathcal{L}(X))}$$
$$\times \big[\|L_0'\|_{C^\beta([0,T];\mathcal{L}(Y;X))} + \|\widetilde{K}_{2,1}\|_{C^\beta([0,T];\mathcal{L}(Y;X))}\big].$$

Then we need the following theorem.

Theorem 4.1. *The linear operator*

$$\mathcal{G}f(t) = \int_0^t G(t,s)f(s)\,ds, \quad t \in [0,T], \quad (4.25)$$

maps $C_0^\beta([0,\tau];X) = \{f \in C^\beta([0,\tau];X) : f(0) = 0\}$ *continuously into* $\mathcal{Z}^\beta(\tau)$ *for all* $\beta \in (0,1)$ *and* $\tau \in (0,T]$, *and satisfies the estimate*

$$\|\mathcal{G}f\|_{\mathcal{Z}^\beta(\tau)} \le C(\beta,T)\|f\|_{C^\beta([0,\tau];X)}, \quad \tau \in (0,T], \quad (4.26)$$

for some positive constant $C(\beta,T)$ *independent of* $\tau \in (0,T]$.

For reader's convenience a sketch of the proof will be given in Appendix.

To show that equation (4.24) is solvable in $\mathcal{Z}^\beta(\tau)$ for small enough τ it suffices to apply the contraction mapping principle. For this purpose we observe that from (3.44), (4.26) and Corollary 4.2, for τ satisfying (3.29), we easily deduce the following estimates (cf. (4.3), (4.4)):

$$\begin{aligned}&\|\mathcal{G}(I-M)^{-1}L_{1+j}v\|_{\mathcal{Z}^\beta(\tau)} \le C(\beta,T)\|(I-M)^{-1}L_{1+j}v\|_{C^\beta([0,\tau];X)}\\ &\le J_1(K_3,\tau)\|L_{1+j}v\|_{C([0,\tau];X)} \le J_1(K_3,\tau)J_{6+j}(\tau)\|v\|_{\mathcal{Z}^\beta(\alpha^{2-j}(\tau))}, \quad j=1,2, \qquad (4.27)\end{aligned}$$

where

$$\begin{aligned}J_7(\tau) &= J_1(K_3,\tau)\big[J_4(K_6,\tau,X,X)\|h(0)\| + J_4(K_8,\tau,Y,X)\|u_0\|_Y\big],\\ J_8(\tau) &= J_1(K_3,\tau)\big[J_5(K_7,\tau,X,X)\|h(0)\| + J_5(K_9,\tau,Y,X)\|u_0\|_Y\big],\end{aligned}$$

while from Theorems 3.1 and definition (3.19) we get

$$\begin{aligned}&\|\mathcal{G}(I-M)^{-1}L_1(v\circ\alpha)\|_{\mathcal{Z}^\beta(\tau)} \le C(\beta,T)\|(I-M)^{-1}L_1(v\circ\alpha)\|_{C^\beta([0,\tau];X)}\\ &\le C(\beta,T)J_1(K_3,\tau)\|\alpha' K_5(v\circ\alpha)\|_{C^\beta([0,\tau];\mathcal{L}(X)}\\ &\le C(\beta,T)J_1(K_3,\tau)\Big[\|\alpha' K_5\|_{C^\beta([0,\tau];\mathcal{L}(X))}\|v\|_{C([0,\alpha(\tau)];X)}\\ &\quad + \|\alpha' K_5\|_{C([0,\tau];\mathcal{L}(X))}\|\alpha'\|^\beta_{C([0,\alpha(\tau)];\mathbb{R})}|v|_{C^\beta([0,\alpha(\tau)];X)}\Big]\\ &\le C(\beta,T)J_1(K_3,\tau)\max\big\{\|\alpha' K_5\|_{C^\beta([0,\tau];\mathcal{L}(X))} + \|\alpha' K_5\|_{C([0,\tau];\mathcal{L}(X))}\\ &\quad \times \|\alpha'\|^\beta_{C([0,\alpha(\tau)];\mathbb{R})}\big\}\|v\|_{C^\beta([0,\alpha(\tau)];X)}.\end{aligned}$$

Moreover, since $\|\alpha'\|_{C([0,\tau];\mathbb{R})} < 1$ (cf (3.11)), we deduce

$$\begin{aligned}&\|\mathcal{G}\big[Z(K_4(v)\circ\alpha\big]\|_{\mathcal{Z}^\beta(\tau)} \le C(\beta,T)\|Z(K_4(v)\circ\alpha\|_{C^\beta([0,\tau];X)}\\ &\le C(\beta,T)\big[\|Z(K_4 v)\|_{C([0,\alpha(\tau)];X)} + |Z(K_4 v)|_{C^\beta([0,\alpha(\tau)];X)}\|\alpha'\|^\beta_{C([0,\tau];\mathbb{R})}\big]\\ &\le C(\beta,T)\|Z(K_4 u)\|_{C^\beta([0,\alpha(\tau)];X)} \le C(\beta,T)\frac{J_1(K_3,\tau)(\alpha(\tau))}{1-J_1(K_3,\tau)(\alpha(\tau))}\|K_4 v\|_{C^\beta([0,\alpha(\tau)];X)}\\ &\le C(\beta,T)\frac{J_1(K_3,\tau)(\alpha(\tau))}{1-J_1(K_3,\tau)(\alpha(\tau))}\|v\|_{\mathcal{Z}^\beta([0,\alpha(\tau)];X)}.\end{aligned}$$

We choose now $\tau\in(0,\tau_0]$ so as to satisfy the inequality

$$C(\beta,T)\frac{J_1(K_3,\tau)(\alpha(\tau))}{1-J_1(K_3,\tau)(\alpha(\tau))} \quad\Longleftrightarrow\quad J_1(K_3,\tau)(\alpha(\tau)) < \big[1+C(\beta,T)\big]^{-1}. \qquad (4.28)$$

We now observe that, according to (3.11),

$$\lim_{\tau\to 0+} J_1(K_3,\tau)(\alpha(\tau)) = \alpha'(0)^{1+\beta}\|K_3(0)\|_{\mathcal{L}(X)} = 0. \qquad (4.29)$$

Consequently, inequality (4.28) is solvable.
Finally, from (4.27)-(4.28) we can conclude that there exists $\tau_1 \in (0,T]$ such that the operator

$$I - \mathcal{G}[Z(K_4(\cdot \circ \alpha)) + (I-M)^{-1}(L_1 + L_2 + L_3)]$$

is invertible from $\mathcal{Z}^\beta([0,\tau];X)$ into itself for any $\tau \in (0,\tau_1]$, since, owing to the previous estimate (4.29), its norm tends to 0 as $\tau \to 0+$. Consequently, we easily deduce that equation (4.24) admits a unique solution $v \in \mathcal{Z}^\beta([0,\tau];X)$ for any $\tau \in (0,\tau_1]$. Such a solution continuously depends on w with respect to the metrics of the same space. Indeed, it satisfies the following estimate

$$\|v\|_{\mathcal{Z}^\beta(\tau)} \le C(T)\big\{\|u_0\|_Y + \|f'\|_{C^\beta([0,\tau];X)} + \|H(u_0,f)\|_{(Y;X)_{\beta,\infty}}\big\}, \quad \tau \in (0,\tau_1],$$

where

$$H(u_0,f) = [L_0(0) + K_{2,0}(0,0)]^{-1}\big\{L_0'(0) + K_{2,1}(0,0) + \alpha'(0)[K_{2,0}(0,0) - K_{1,0}(0,0)]\big\}u_0 + f'(0),$$

which shows how the solution v depends on the data (u_0,f).
To show that such our local solution can be extended to a *global* one, introduce, as in (3.35), the finite sequence of positive real points defined by

$$\tau_{j+1} = \alpha^{-1}(\tau_j), \quad j = 1,\dots,n, \quad (\tau_n \le T < \tau_{n+1}).$$

We proceed by recurrence and assume to have shown that equation (4.24) admits a solution v_j defined in the interval $[0,\tau_j]$. Consider then the following function ζ_{j+1} in the interval $[\tau_j,\tau_{j+1}]$:

$$\begin{aligned}
\zeta_{j+1}(t) \;=\;& v_j(\tau_j) + \int_{\tau_j}^t \alpha'(s)G(t,s)\big[K_2(s,\alpha(s))v_j'(\alpha(s)) + K_4(s,\alpha(s))v_j(\alpha(s))\big]\,ds \\
&+ \int_{\tau_j}^t G(t,s)\,ds \int_{\alpha(s)}^{\tau_j}[D_tK_2(s,\sigma)v_j'(\sigma) + D_tK_4(s,\sigma)v_j(\sigma)]\,d\sigma \\
&+ \int_{\tau_j}^t G(t,s)\,ds \int_{\tau_j}^t[D_tK_2(s,\sigma)\zeta_{j+1}'(\sigma) + D_tK_4(s,\sigma)\zeta_{j+1}(\sigma)]\,d\sigma \\
&+ g(t).
\end{aligned} \tag{4.30}$$

According to our assumptions (3.5)-(3.12) we conclude that equation (4.30) admits a unique *global* solution $\zeta_{j+1} \in C([\tau_j,\tau_{j+1}];Y)$ such that $\zeta_{j+1}(\tau_j) = \zeta_j(\tau_j)$. Consequently, the function

$$v_{j+1}(t) = \begin{cases} v_j(t), & t \in [0,\tau_j], \\ \zeta_{j+1}(t), & t \in [\tau_j,\tau_{j+1}], \end{cases}$$

solves problem (4.24) in the interval $[0,\tau_{j+1}]$ for any $j \in \{1,\dots,n\}$. Hence v_n is our *global* solution in $[0,T]$ satisfying the estimate

$$\|v\|_{\mathcal{Z}^\beta(T)} \le C(T)\big\{\|u_0\|_Y + \|f'\|_{C^\beta([0,T];X)} + \|H(u_0,f)\|_{(Y;X)_{\beta,\infty}}\big\}. \ \square$$

5 Some applications to integro-differential problems

The elliptic problem. Consider the integro-differential boundary value problem

$$\int_0^{\alpha(t)} \Big\{ -\sum_{i,j=1}^n D_{x_i}[a_{i,j}^{(1)}(t,s,x)D_{x_j}u(s,x)] + a_{0,0}^{(1)}(t,s,x)u(s,x)\Big\}ds$$

$$+\int_{\alpha(t)}^t \Big\{ -\sum_{i,j=1}^n D_{x_i}[a_{i,j}^{(2)}(t,s,x)D_{x_j}u(s,x)] + a_{0,0}^{(2)}(t,s,x)u(s,x)\Big\}ds = f(t,x),$$

$$(t,x)\in[0,T]\times\Omega, \qquad (5.1)$$

$$u(t,x)=0, \quad (t,x)\in[0,T]\times\partial\Omega, \qquad (5.2)$$

where function α belongs to $C^1([0,T],\mathbb{R})$ and enjoys the properties (1.3), (1.4). Further assume (cf. (1.4), (1.5))

$$\sum_{i,j=1}^n a_{i,j}^{(2)}(t,t,x)\xi_i\xi_j \ge \mu|\xi|^2, \quad t\in[0,T],\ x\in\overline{\Omega},\ \xi\in\mathbb{R}^n \quad \text{for some } \mu>0, \qquad (5.3)$$

$$a_{0,0}^{(2)}(t,t,x)\ge 0, \quad (t,x)\in[0,T]\times\Omega, \qquad (5.4)$$

$$a_{i,j}^{(k)} \in C^1(\overline{\omega_k(T)};W^{1,\infty}(\Omega)), \quad i,j=1,\ldots,n, \quad a_{0,0}^{(k)}\in C^1(\overline{\omega_2(T)};L^\infty(\Omega)), \quad k=1,2, \qquad (5.5)$$

$$D_t a_{i,j}^{(k)} \in C(\overline{\omega_k(T)};W^{1,\infty}(\Omega)), \quad i,j=1,\ldots,n, \quad k=1,2. \qquad (5.6)$$

Then we assume that either of the following conditions holds true:

$$\alpha'(0)=0, \qquad (5.7)$$

or *there exists a* $\lambda\in[0,2]$ *such that*

$$a_{i,j}^{(1)}(0,0,x)=\lambda a_{i,j}^{(2)}(0,0,x), \quad a_{0,0}^{(1)}(0,0,x)=\lambda a_{0,0}^{(2)}(0,0,x), \quad i,j=1,\ldots,n. \qquad (5.8)$$

Finally, assume

$$f\in C^1([0,T];L^2(\Omega)), \quad f(0,\cdot)=0. \qquad (5.9)$$

Theorem 5.1. *Under assumptions (5.3)-(5.9) the integro-differential problem (5.1) admits a unique solution* $u\in C([0,T];H_0^1(\Omega)\cap H^2(\Omega))$ *continuously depending on the datum* f.

Proof. Set

$$K_j(t,s) = -\sum_{i,j=1}^n D_{x_i}[a_{i,j}^{(j)}(t,s,x)D_{x_j}] + a_{0,0}^{(j)}(t,s,x), \quad j=1,2.$$

Observe that $(t,s)\to K_j(t,s)\in C(\overline{\omega_j(T)};\mathcal{L}(Y,X))$ where $Y=H_0^1(\Omega)\cap H^2(\Omega)$ and $X=L^2(\Omega)$. Moreover, from well-known regularity results for elliptic boundary value problems (cf. [9, Chapt. 4]), we deduce that $K_2(t,t)$ is invertible for any $t\in[0,T]$ and

$$\|K_2(t,t)^{-1}\|_{\mathcal{L}(X;Y)} \le C, \quad t\in[0,T].$$

Consequently

$$t \to K_2(t,t)^{-1} \in C([0,T];\mathcal{L}(L^2(\Omega);H_0^1(\Omega)\cap H^2(\Omega))).$$

Indeed, for all $t, t_0 \in [0,T]$ we have

$$\begin{aligned}
&\|K_2(t,t)^{-1} - K_2(t_0,t_0)^{-1}\|_{\mathcal{L}(X;Y)} \\
= \ &\|K_2(t_0,t_0)^{-1}[K_2(t,t) - K_2(t_0,t_0)]K_2(t,t)^{-1}\|_{\mathcal{L}(X;Y)} \\
\leq \ &\|K_2(t_0,t_0)^{-1}\|_{\mathcal{L}(X;Y)}\|K_2(t,t)^{-1}\|_{\mathcal{L}(X;Y)}\|K_2(t,t) - K_2(t_0,t_0)\|_{\mathcal{L}(Y;X)}.
\end{aligned}$$

Finally, note that $D_tK_j \in C(\overline{\omega_j(T)};\mathcal{L}(H^2(\Omega)\cap H_0^1(\Omega);L^2(\Omega)))$, $j = 1,2$, according to assumption (5.6). Moreover, condition (2.4) is trivially satisfied if $\alpha'(0) = 0$, while under (5.8) we get

$$\begin{aligned}
\|K_2(0,0)^{-1}[K_2(0,0) - K_1(0,0)]\|_{\mathcal{L}(Y;X)} = &\|K_2(0,0)^{-1}(1-\lambda)K_2(0,0)\|_{\mathcal{L}(Y;X)} \\
\leq &|1-\lambda| \leq 1.
\end{aligned}$$

Since $0 < \alpha'(0) < 1$, condition (2.4) is satisfied also in this case.
This concludes the proof. □

The parabolic problem. Set $\alpha_0(t) = 0$, $\alpha_1(t) = \alpha(t)$, $\alpha_2(t) = t$, $t \in [0,T]$.
Consider the integro-differential initial and boundary value problem

$$\begin{aligned}
&\ell_0(t,x)u(t,x) + \sum_{h=1}^{2}\int_{\alpha_{h-1}(t)}^{\alpha_h(t)}\Big\{\rho^h(t,s,x)D_su(s,x) - \sum_{i,j=1}^{d} a_{i,j}^h(t,s,x)D_{x_i}D_{x_j}u(s,x) \\
&-\sum_{j=1}^{d} a_{0,j}^h(t,s,x)D_{x_j}u(s,x) - a_{0,0}^h(t,s,x)u(s,x)\Big\}ds = f(t,x), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [0,T]\times\Omega, \qquad (5.10)
\end{aligned}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{5.11}$$

$$B(x,D_x)u(t,x) = 0, \quad (t,x) \in [0,T]\times\partial\Omega. \tag{5.12}$$

We make the following assumptions for some $\delta \in (0,1)$ and some positive constants μ_1 and μ_2:

$$\rho^h \in C^{1+\delta,0}(\overline{\omega_h(T)};\mathbb{R}), \quad h = 1,2, \tag{5.13}$$

$$\ell_0 \in C^{1+\delta,0}(\overline{\omega_h(T)};\mathbb{R}), \quad m(\Omega_0) \geq 0, \quad \Omega_0 = \{x \in \Omega : \ell_0(0,x) = 0\}, \tag{5.14}$$

$$|\ell_0(t,x) + \rho^2(t,t,x)| \geq \mu_1, \quad t \in [0,T],\ x \in \Omega, \tag{5.15}$$

$$\begin{aligned}
&a_{0,0}^h,\ a_{0,j}^h,\ a_{i,j}^h,\ D_ta_{0,0}^h,\ D_ta_{0,j}^h,\ D_ta_{i,j}^h \in C^{\delta,0}(\overline{\omega_h(T)};\mathbb{R}), \\
&i,j = 1,\dots,d,\ h = 1,2, \qquad (5.16)
\end{aligned}$$

$$[\ell_0(t,x) + \rho^2(t,t,x)]^{-1}\sum_{i,j=1}^{n} a_{i,j}^2(t,t,x)\xi_i\xi_j \geq \mu_2|\xi|^2,$$

$$(t,x,\xi) \in [0,T]\times\overline{\Omega}\times\mathbb{R}^n. \tag{5.17}$$

As far as the linear differential operator $B(x,D_x)=\sum_{j=1}^N d_j(y)D_{y_k}+d_0(y)=0$, of order not exceeding 1 and standing for Dirichlet or Neumann or Robin boundary conditions, is concerned, we assume that, when $B(x,D_x)\neq I$, the coefficients $d_j\in C^1(\overline{\Omega})$, $j=0,\ldots,N$, satisfy the uniform *non-tangentiality condition* $|\sum_{j=0}^N \nu_j(y)d_j(y)|\geq\mu_3>0$ for all $y\in\partial\Omega$ and some positive constant μ_3 (cf. [7, Chapter 3]), ν standing for the unit exterior normal to $\partial\Omega$, as well as the inequality $d_0(y)\geq 0$ for all $y\in\partial\Omega$.

Remark 1. We could also deal with any $2m$-th order linear differential elliptic operator $A=\sum_{|\alpha|\leq 2m}a_\alpha(y)D_y^\alpha$ satisfying conditions (4.2.2) and (4.2.3) in [7, p. 112]) and endowed with m boundary conditions related to m boundary linear differential elliptic operators satisfying conditions (4.2.5), (4.2.6) in [7, pp. 112-113]. In this case Ω stands for a bounded open set in $\mathbb{R}^N$ with a C^{2m}-boundary.

Functions α and f and u_0 satisfy, respectively, properties (4.3), (3.6) and

$$f\in C^{1+\beta}([0,T];L^p(\Omega)),\quad f(0,\cdot)=0,\quad u_0\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega),\quad u_0=0 \text{ in } \Omega\setminus\Omega_0,$$

for some $p\in(1,+\infty)$ and $\beta\in(0,1)\setminus\{1/(2p)\}$.
Finally, we assume

$$\begin{aligned}&\big[\ell_0(0,\cdot)+\rho^2(0,0,\cdot)\big]^{-1}\Big[D_t\ell_0(0,\cdot)+\sum_{i,j=1}^n a_{i,j}^2(0,0,\cdot)D_{x_i}D_{x_j}u_0\\&+\sum_{j=1}^n a_{0,j}^2(0,0,\cdot)D_{x_j}u_0+a_{0,0}^2(0,0,\cdot)u_0\Big]\\&+\alpha'(0)[\rho^2(0,0,\cdot)-\rho^1(0,0,\cdot)]+D_tf(0,\cdot)\in\mathcal{W}^{2\beta,p}(\Omega),\quad \beta\in(0,\delta),\end{aligned}\tag{5.18}$$

where the intermediate space $\mathcal{W}^{2\beta,p}(\Omega)$, $\beta\neq 1/(2p)$, is defined (cf. [18, p. 420] by

$$\mathcal{W}^{2\beta,p}(\Omega)=\begin{cases}W^{2\beta,p}(\Omega), & 0<\alpha<1/(2p),\\ \{\varphi\in W^{2\beta,p}(\Omega):\varphi=0 \text{ on } \partial\Omega\}, & 1/(2p)<\beta<1.\end{cases}$$

We can state the result of this subsection.

Theorem 5.2. *Under assumptions (3.5), (5.13)-(5.18) problem (5.10)-(5.12) admits a unique solution $u\in C^{1+\beta}([0,T];L^p(\Omega))\cap C^\beta([0,T];W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))$ continuously depending on the data (f,u_0) with respect to the metrics pointed out.*

Proof. First we introduce the linear operators

$$K_{h,0}(t,s)u(x)=\rho^h(t,s,x)u(x),$$

$$K_{h,1}(t,s)u(x)=\sum_{i,j=1}^n a_{i,j}^h(t,s,x)D_{x_i}D_{x_j}u(x)+\sum_{j=1}^n a_{0,j}^h(t,s,x)D_{x_j}u(x)$$

$$+a_{0,0}^h(t,s,x)u(x).$$

We observe that

$$K_{h,0},\ D_tK_{h,0} \in C^\delta(\overline{\omega_h(T)};\mathcal{L}(X)), \quad K_{h,1},\ D_tK_{h,1} \in C^\delta(\overline{\omega_h(T)};\mathcal{L}(Y;X)),$$

where the open sets $\omega_h(T)$ are defined by (1.4) and

$$Y = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \quad X = L^p(\Omega).$$

We observe that, according to our assumptions on ℓ_0, the kernel of the linear multiplication operator defined by $\ell_0(0,\cdot)$ coincides with the vector space

$$N(L_0(0)) = \{w \in Y : w(x) = 0,\ x \in \Omega \setminus \Omega_0\}, \quad \Omega_0 = \{x \in \Omega : \ell_0(0,x) = 0\}.$$

In particular, we observe that

$$[\ell_0(t,x)I + K_{2,0}(t,t,x)]u(x) = [\ell_0(t,x) + \rho^2(t,t,x)]u(x), \quad t \in [0,T],\ x \in \overline{\Omega},$$

is continuously invertible from $C^\beta([0,\tau];X)$ into itself for *any* $\tau \in (0,T]$. Then the linear operators $K_3(t) \in \mathcal{L}(X)$ and $K_4(t) \in \mathcal{L}(Y;X)$ are defined by

$$K_3(t)u(x) = [\ell_0(t,x) + \rho^2(t,t,x)]^{-1}(\rho^2 - \rho^1)(t,\alpha(t),x)u(x),$$

$$K_4(t)u(x) = [\ell_0(t,x) + \rho^2(t,t,x)]^{-1}\Big[\sum_{i,j=1}^n a_{i,j}^2(t,t,x)D_{x_i}D_{x_j}u(x)$$

$$+\sum_{j=1}^n a_{0,j}^2(t,t,x)D_{x_j}u(x) + a_{0,0}^2(t,t,x)u(x)\Big],$$

$$K_5(t)u(x) = [\ell_0(t,x) + \rho^2(t,t,x)]^{-1}[\rho^2(t,t,\alpha(t)) - \rho^1(t,t,\alpha(t))].$$

We note that, according to (5.17), $K_4(t)$ is uniformly elliptic for all $t \in [0,T]$ with positive constant μ_2. As a consequence, the family of operators $\{K_4(t)\}_{t\in[0,T]}$ satisfies the assumptions of theorem 6.1.3 in [7]. Therefore it generates the evolution operators $G(t,s)$, $0 \le s \le t \le T$.
Furthermore, the properties H1 and H2 in Section 3 involving the family $K_4(t)_{t\in[0,T]}$ are satisfied according to assumptions (5.13)-(5.17) and the results on pp. 140-144 in [10].
Finally, condition (3.13) simplifies to (5.18). □

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A Appendix

Here we outline the proof of Theorem 3.4. We recall that the family of operators $\{K_4(t)\}_{t\in[0,T]}$ is assumed to satisfy properties (4.15)-(4.17).

We recall now that the evolution operator G admits the representation (cf. [7, Chapter 3])

$$G(t,s) = e^{(t-s)K_4(t)} + W(t,s), \quad 0 \le s \le t \le T, \quad W(t,t) = O, \quad t \in [0,T],$$

where

$$\|W(t,s)\|_{\mathcal{L}(X)} \le c_3, \quad \|D_t W(t,s)\|_{\mathcal{L}(X)} \le c_4(t-s)^{-1+\beta}, \quad 0 \le s \le t \le T.$$

for some constants c_3 and c_4.
For any $\varepsilon \in (0,1)$, we consider the following formulae holding for all $t \in [0,T]$:

$$D_t \int_0^{\varepsilon t} G(t,s)f(s)\,ds = G(t,\varepsilon t)f(\varepsilon t) + \int_0^{\varepsilon t} D_t G(t,s)f(s)\,ds,$$

$$= G(t,\varepsilon t)f(t) + G(t,\varepsilon t)[f(\varepsilon t) - f(t)] + \int_0^{\varepsilon t} D_t G(t,s)f(s)\,ds,$$

$$\int_0^{\varepsilon t} D_t G(t,s)f(s)\,ds = \int_0^{\varepsilon t} D_t G(t,s)[f(s) - f(t)]\,ds + \int_0^{\varepsilon t} D_t G(t,s)f(t)\,ds,$$

$$\int_0^{\varepsilon t} D_t G(t,s)f(t)\,ds = -\int_0^{\varepsilon t} D_s[e^{(t-s)K_4(t)}]f(t)\,ds + \int_0^{\varepsilon t} D_t W(t,s)f(t)\,ds$$

$$= -e^{t(1-\varepsilon)K_4(t)}f(t) + e^{tK_4(t)}f(t) + \int_0^{\varepsilon t} D_t W(t,s)f(t)\,ds$$

$$\to -f(t) + e^{tK_4(t)}f(t) + \int_0^{t} D_t W(t,s)f(t)\,ds, \quad \text{as } \varepsilon \to 1- \,.$$

and

$$D_t \int_0^{\varepsilon t} G(t,s)f(s)\,ds = G(t,\varepsilon t)f(t) + G(t,\varepsilon t)[f(\varepsilon t) - f(t)] + \int_0^{\varepsilon t} D_t G(t,s)f(s)\,ds$$

$$\to e^{tK_4(t)}f(t) + \int_0^{t} D_t W(t,s)f(t)\,ds, \quad t \in [0,T], \text{ as } \varepsilon \to 1-,$$

$$K_4(t)\int_0^{\varepsilon t} G(t,s)f(s)\,ds = \int_0^{\varepsilon t} K_4(t)G(t,s)f(s)\,ds = \int_0^{\varepsilon t} D_t G(t,s)f(s)\,ds$$

$$\to -f(t) + e^{tK_4(t)}f(t) + \int_0^{t} D_t W(t,s)f(t)\,ds, \quad t \in [0,T], \text{ as } \varepsilon \to 1- \,. \tag{A.1}$$

To derive the formulae involving t-increments, we need the unbounded curve $\gamma(\eta)$, $|\eta| < \phi$, oriented from $\infty \mathrm{e}^{-i\eta}$ to $\infty \mathrm{e}^{i\eta}$ and defined by

$$\gamma(\eta) \;=\; \{r\mathrm{e}^{-i\eta} \in \mathbb{C} : r \ge 0\} \cup \{r\mathrm{e}^{i\eta} \in \mathbb{C} : r \ge 0\}.$$

Then we observe that for any pair $0 \le t_1 < t_2 \le T$ we have

$$I_1(t_1,t_2) := e^{t_2 K_4(t_2)} - e^{t_1 K_4(t_1)} = \frac{1}{2\pi i}\int_{\gamma(\eta)} e^{t_1\lambda}\big\{\big[\lambda - K_4(t_2)\big]^{-1} - \big[\lambda - K_4(t_1)\big]^{-1}\big\}\,d\lambda$$

$$= \frac{1}{2\pi i}\int_{\gamma(\eta)} e^{t_1\lambda}\big[\lambda - K_4(t_2)\big]^{-1}\big\{\big[K_4(t_2)-K_4(t_1)\big]K_4(t_1)^{-1}\big\}K_4(t_1)\big[\lambda - K_4(t_1)\big]^{-1}\,d\lambda.$$

For any $\lambda \in \gamma(\eta)$ we have $\operatorname{Re}\lambda \le 0$, so that the following inequalities hold:

$$\begin{aligned}
|\mathrm{e}^{t_2\lambda} - e^{t_1\lambda}| &= \Big|\int_{t_1}^{t_2} \lambda \mathrm{e}^{\lambda s}\,ds\Big| \le |\lambda| \int_{t_1}^{t_2} \mathrm{e}^{s\operatorname{Re}\lambda}\,ds \le |\lambda|(t_2-t_1)^{\beta}\Big[\int_{t_1}^{t_2} \mathrm{e}^{s\operatorname{Re}\lambda/(1-\beta)}\,ds\Big]^{1-\beta}\\
&\le |\lambda|(t_2-t_1)^{\beta}\Big[\int_{0}^{+\infty} \mathrm{e}^{s\operatorname{Re}\lambda/(1-\beta)}\,ds\Big]^{1-\beta}\\
&\le (1-\beta)^{1-\beta}|\lambda||\operatorname{Re}\lambda|^{-1+\beta}(t_2-t_1)^{\beta}\\
&\le C(\beta,\eta)|\lambda|^{\beta}(t_2-t_1)^{\beta}, \quad \text{if } \lambda\in\gamma(\eta).
\end{aligned}$$

Whence we deduce the estimates

$$\|I_1(t_1,t_2)\|_{\mathcal{L}(X)} \le \frac{c_0(1+c_0)}{2\pi}|K|_{C^{\beta}([0,T];\mathcal{L}(X))}|t_2-t_1|^{\beta}\int_{\lambda_0+\gamma(\varepsilon t^{-1},\eta)} |\lambda|^{-1}e^{t_1\operatorname{Re}\lambda}\,|d\lambda|$$

$$\le \frac{c_0^2(1+c_0)}{2\pi}c_5(\varepsilon,\eta)e^{T\lambda_0}|K|_{C^{\beta}([0,T];\mathcal{L}(X))}|t_2-t_1|^{\beta} =: c_6|t_2-t_1|^{\beta}, \quad t_1,t_2\in[0,T],$$

Consequently, if $f \in C_0^{\beta}([0,T];X)$, we deduce

$$\|e^{t_2K_4(t_2)}f(t_2) - e^{t_1K_4(t_1)}f(t_1)\| \le \|e^{t_2K_4(t_2)}\|_{\mathcal{L}(X)}\|f(t_2)-f(t_1)\|$$
$$+\|e^{t_2K_4(t_2)} - e^{t_1K_4(t_1)}\|_{\mathcal{L}(X)}\|f(t_1)\| \le |t_2-t_1|^{\beta}|f|_{C^{\beta}([0,T];X)}\big(c_0+c_6T^{\beta}\big).$$

Finally, it is well-known that the integral involving D_tW, e.g. in the last side in (5.7), defines a function in $C^{\beta}([0,T];X)$.
This concludes our task.

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