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REGULARIZATION METHODS FOR MATHEMATICAL MODEL OF LASER BEAMS

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Abstract The radiation field of a laser (a collimated laser beam) in a bounded domain is considered. The paper concerns reconstruction of this field from measurements made on a part of the domain boundary. The relevant model problem of the physical system is described by the Cauchy problem for the Helmholtz equation on a rectangle in the case when noisy data are given on one side of the rectangle only. In the general case when the beam is not axially symmetric, a convergent series representation of the solution is derived. This representation is the starting point for formulation of different regularization methods. An example of a spectral type regularization method is formulated and analyzed. An error bound for the method is presented.

Key words: Helmholtz equation, Cauchy problem on rectangle, Fourier series, regularization, stability, error estimation

AMS Mathematics Subject Classification: 65M30, 65T40, 35R25

1 Introduction

In optoelectronics, determination of the radiation field surrounding a source of radiation (e.g. a laser or a light emitting diode) is a problem of frequent occurrence. As a rule, experimental determination of the whole radiation field is not possible. Practically, we are able to measure the electromagnetic field only on some subset of physical space (e.g. on some surfaces). So, the problem arises how to reconstruct the radiation field from such experimental data (see for instance [3, 17]).

Let us consider collimated light beams generated by some sources. In this case the sources generate the electromagnetic field in the whole space $\mathbb{R}^3$ outside of the sources, but field values become very small, practically vanish far from the beam axis.

We consider a simplified mathematical model (for a stationary case) in which each component of the field in an open bounded domain D outside of the sources is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{ in } D, \tag{1}$$

with a given real wave number k. The problem consists in reconstruction of the solution u of (1) in a subdomain $\Omega \subset D$ from measurement data, i.e. from inexact values of u and its normal derivatives given on $\Gamma \subset \partial\Omega$, $\Gamma \neq \partial\Omega$. The above-mentioned problem is an example of the ill-posed Cauchy problems for elliptic equations. In the recent literature many aspect of regularizing these problems with noisy data have been studied. For an overview see e.g. [1, 5, 6, 8]

For simplicity we restrict our consideration to the case of rectangular domain Ω in $\mathbb{R}^2$ where the solution of the Helmholtz equation has to be reconstructed. The obtained results can also be extended to the case of a cuboid. With respect to real experiments for a collimated laser beam it is reasonable to assume that measurement data are given only on the one side of rectangle (cuboid) most distant from the sources. This is the main difference between this paper and previous ones, cf. [4, 10, 11, 21] where additional homogeneous or periodic boundary conditions are assumed on the sides parallel to the beam axis. However, the homogeneous boundary conditions have no clear physical meaning, and periodic boundary conditions can be applied only in the case of symmetric beams. The model considered in this paper is more general.

In [7, 15, 16, 19, 20] the Cauchy problem for the Helmholtz equation (1) was considered on the infinite strip $\mathbb{R}^2 \times (0, d)$ (or $\mathbb{R} \times (0, d)$) with data given on a one strip side. The approach applied there consisted in application of Fourier transform with respect to the two variables in $\mathbb{R}^2$ (or the one variable in $\mathbb{R}$) which yields to the equivalent formulation of the problem in the form of an operator equation in the frequency space. It was shown in [16] that some spectral type methods give the optimal or order optimal error bounds on certain source sets. This approach cannot be directly applied for the case of rectangle or cuboid because the related Fourier series are not termwise differentiable (as it is in the case of homogeneous boundary conditions on the sides parallel to the beam axis). However, using the idea described in [12, 9], we replace the nonhomogeneous boundary value problem by the auxiliary one such that the eigenfunction expansion method can be applied for it. This yields to the infinite system of differential equations which is satisfied by the Fourier coefficients of the solution expansion.

In Section 2 we derive a series representation of the solution which is the starting point to formulation of different regularization methods. An example of a spectral type regularization method is formulated in Section 3. Error bounds for regularized solutions are obtained. These estimations depend on the regularization parameter, a measurement error and a priori bounds for certain norms of the solution trace on the rectangle sides where no measurements exist.

2 Cauchy problem on a rectangle

Let us consider the two dimensional model problem presented schematically on Fig.(1). Assume that $u \in H^2(D)$ satisfies the Helmholtz equation on an open domain $D \subset \mathbb{R}^2$. Measurements are available on $\Gamma = (0, a) \times \{0\} \subset D$. Let g and h be the exact values of the solution u and its derivative $\frac{\partial u}{\partial y}$ on the set Γ. Therefore, u is also a solution of the Cauchy problem on $\Omega = (0, a) \times (0, b) \subset D$

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \; ; \\ u(x, 0) = g(x); \;\; u_y(x, 0) = h(x), & x \in (0, a). \end{cases} \tag{2}$$

The problem (2) is ill posed in $L^2(\Omega)$: the solution does not depend continuously on the boundary data and it is also possible that no solution exists even for arbitrary smooth functions $\widetilde{g} \sim g$, $\widetilde{h} \sim h$. However, if g, h determine a solution of (2), then they determine exactly one solution (see [8], Chapter 3). This uniqueness result is shown in [2], Theorem 4.1, for the case of an arbitrary Lipschitz domain in $\mathbb{R}^d$ under the

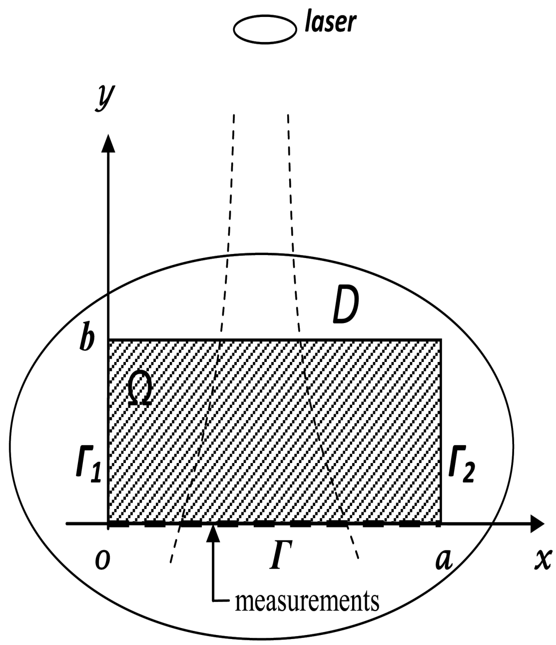

Figure 1: Scheme of the model problem with measurements available on Γ only

assumption, that $\exists z \in \Gamma$ and $\exists r > 0$ such that $\Gamma \supset B(z,r) \cap \partial\Omega$ where $B(z,r)$ denotes the ball with the center z and the radius r.

Problem P1. Given noisy data $g^\delta(x)$ and $h^\delta(x)$ on Γ satisfying

$$\|g - g^\delta\|_{L^2(0,a)} \le \delta, \quad \|h - h^\delta\|_{L^2(0,a)} \le \delta \tag{3}$$

for a given data error bound δ. For any fixed $y \in (0,b]$ find a function $u^\delta(\cdot,y) \in L^2(0,a)$ which is an approximation of the exact solution $u(\cdot,y)$ for (2).

Let $\Gamma_1 := \{0\} \times (0,b)$, $\Gamma_2 := \{a\} \times (0,b)$ and $f_i := u|_{\Gamma_i}$, $i = 1,2$. We make the following assumptions on the problem under consideration

A1 : The exact solution u is small on $\Gamma_1,\ \Gamma_2$, i.e. $\exists \varepsilon$

$$\|f_i\|_{H^2(0,b)} \le \varepsilon,\ i = 1,2; \tag{4}$$

A2 : A constant $M < \infty$ is known such that

$$\|u'_x(\cdot,b)\|_{L^2(0,a)} \le M \tag{5}$$

2.1 Auxiliary problem

Let us consider the following auxiliary problem:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega; \\ u(x,0) = \widetilde{g}(x);\ u_y(x,0) = \widetilde{h}(x), & x \in (0,a)\ ; \\ u(0,y) = 0,\ u(a,y) = 0, & y \in (0,b). \end{cases} \tag{6}$$

Let the functions $\widetilde{g}$ and $\widetilde{h}$ be such that the solution $\widetilde{u}$ of (6) exists in $H^2(\Omega)$. Using the method of separation of variables we easily find

$$\widetilde{u}(x,y) = \sum_{n=1}^{\infty} U_n(y) \sin \frac{n\pi x}{a} \tag{7}$$

where

$$U_n(y) = \begin{cases} \widetilde{g}_n \cosh y\zeta_n + \frac{1}{\zeta_n}\widetilde{h}_n \sinh y\zeta_n, & \text{if } n \neq \frac{ak}{\pi}\ ; \\ \widetilde{g}_n + \widetilde{h}_n y, & \text{if } n = \frac{ak}{\pi}. \end{cases} \tag{8}$$

and $\zeta_n := \sqrt{\frac{n^2\pi^2}{a^2} - k^2}$, $\widetilde{g}_n$ and $\widetilde{h}_n$ denote the Fourier coefficients of the odd $2a$- periodic functions equal to $\widetilde{g}$ and $\widetilde{h}$ on the interval $(0,a)$, respectively.

For simplicity we will assume subsequently that $a \neq \frac{n\pi}{k}$.

2.2 Series representation in general case

Problem (2) is equivalent to

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega\ ; \\ u(x,0) = g(x);\ u_y(x,0) = h(x), & x \in (0,a); \\ u(0,y) = f_1(y); u(a,y) = f_2(y), & y \in (0,b). \end{cases} \tag{9}$$

with unknown data f_1 and f_2.

In order to use the variable separation and the eigenfunction expansion methods, as it was done for (6), we reduce the nonhomogeneous boundary conditions to the homogeneous case (cf. [12], [9], sec.6.6). We choose P as an interpolating polynomial with respect to x, i.e. $P(x,y) = p_0(y) + p_1(y)x$ and $P(0,y) = f_1(y)$, $P(a,y) = f_2(y)$. Thus

$$P(x,y) = f_1(y) + \frac{f_2(y) - f_1(y)}{a} x. \tag{10}$$

Clearly, if u is a solution to (2), then the function

$$v(x,y) := u(x,y) - P(x,y) \tag{11}$$

is a solution to the following initial boundary value problem

$$\begin{cases} \Delta v + k^2 v = \psi, & \text{in } \Omega\ ; \\ v(x,0) = g(x) - P(x,0);\ v_y(x,0) = h(x) - P_y(x,0), & x \in (0,a); \\ v(0,y) = v(a,y) = 0, & y \in (0,b), \end{cases} \tag{12}$$

where

$$\psi(x,y) = \psi_0(y) + \psi_1(y)x, \tag{13}$$

$$\psi_0(y) = -f_1''(y) - k^2 f_1(y),\ \psi_1(y) = \frac{f_1''(y) - f_2''(y)}{a} + k^2 \frac{f_1(y) - f_2(y)}{a}.$$

Let

$$f_{i,0} := \lim_{y\to 0} f_i(y),\ f_{i,1} := \lim_{y\to 0} f_i'(y),\ i = 1,2.$$

Theorem 2.1. *If $u \in H^2(D)$ and $u_{|\Omega}$ is the solution to (9), then for any fixed $y \in (0,b)$ u has the following convergent representation twice differentiable term by term*

$$u(x,y) = P(x,y) + \sum_{n=1}^{\infty} V_n(y) \sin \frac{n\pi x}{a}, \tag{14}$$

with P given by (10) and

$$V_n(y) = \widetilde{g}_n \cosh y\zeta_n + \frac{\widetilde{h}_n}{\zeta_n} \sinh y\zeta_n + \frac{1}{\zeta_n}\int_0^y \sinh((\tau - y)\zeta_n)\varphi_n(\tau)d\tau, \tag{15}$$

where

$$\zeta_n := \sqrt{\frac{n^2\pi^2}{a^2} - k^2}, \;\; \varphi_n = \frac{2a}{n\pi}\left((-1)^{n+1}(\psi_0 + \psi_1) + \psi_0\right),$$

$$\widetilde{g}_n = g_n - \frac{2}{n\pi}\left(f_{1,0} + f_{2,0}(-1)^{n+1}\right), \;\; \widetilde{h}_n = h_n - \frac{2}{n\pi}\left(f_{1,1} + f_{2,1}(-1)^{n+1}\right),$$

and g_n, h_n are Fourier coefficients of the odd $2a$ periodic functions equal to g and h on $(0,a)$.

Proof. We split (12) into the well posed nonhomogeneous Dirichlet problem

$$\begin{cases} \Delta s + k^2 s = \psi, \\ s(x,0) = s_y(x,b) = 0, \\ s(0,y) = s(a,y) = 0, \end{cases} \tag{16}$$

and the ill-posed Cauchy problem

$$\begin{cases} \Delta w + k^2 w = 0, \\ w(x,0) = \widetilde{g}(x), \;\; w_y(x,0) = \widetilde{h}_1(x), \\ w(0,y) = w(a,y) = 0, \end{cases} \tag{17}$$

with $\widetilde{g} = g - P(\cdot,0)$, $\widetilde{h}_1 = h - P_y(\cdot,0) - s_y(\cdot,0)$. The solution w to (17) exists, since $w = v - s$ and v exists by the assumption. Let $S_n(y)$ and $W_n(y)$ denote the Fourier series coefficients for $s(\cdot,y)$ and $w(\cdot,y)$, respectively. From (8) it follows that

$$W_n(y) = \widetilde{g}_n \cosh y\zeta_n + \frac{1}{\zeta_n}\widetilde{h}_{1n} \sinh y\zeta_n. \tag{18}$$

Moreover, it can easily be found that

$$S_n(y) = \frac{1}{\zeta_n}\int_0^y \sinh(y-\tau)\zeta_n \varphi_n(\tau)d\tau - \frac{\sinh y\zeta_n}{\zeta_n \sinh b\zeta_n}\int_0^b \sinh(b-\tau)\zeta_n\varphi_n(\tau)d\tau. \tag{19}$$

For a detailed proof see [13]. □

3 Identification u from inexact boundary data

The representation (14) of u depends on g, h as well as on the unknown traces of u onto Γ_1 and Γ_2. Moreover, if $g,\ h$ are replaced by noisy data $g^\delta,\ h^\delta$, then this series is generally not convergent. For approximate solving Problem P1 we propose a spectral type regularization method which does not use unknown f_i, $i = 1, 2$.

Let $\alpha \in (0, 1)$ and

$$n_\alpha := \max\{n : \cosh b\sqrt{\frac{n^2\pi^2}{a^2} - k^2} \le \frac{1}{\alpha}\}. \tag{20}$$

Let a regularization solution be defined as follows:

$$u_\alpha^\delta := \sum_{n=1}^{n_\alpha} W_n^\delta(y) \sin\frac{n\pi x}{a}, \tag{21}$$

where

$$W_n^\delta = g_n^\delta \cosh y\zeta_n + \frac{1}{\zeta_n} h_n^\delta \sinh y\zeta_n, \quad \text{and} \quad \zeta_n := \sqrt{\frac{n^2\pi^2}{a^2} - k^2}. \tag{22}$$

Let us observe that for any $n_\alpha < \infty$ the function u_α^δ is well defined.

In order to estimate the distance between u_α^δ and u we introduce the auxiliary function

$$u_\alpha := P(x, y) + \sum_{n=1}^{n_\alpha} V_n(y) \sin\frac{n\pi x}{a}. \tag{23}$$

Its convergence to u $\forall y \in (0, b]$ follows from convergence of (14).

Proposition 3.1. *If the assumptions A1 and (3) are satisfied and $k \ne n\frac{\pi}{a}$, then $\forall y \in (0, b]$ and $\alpha \in (0, 1)$*

$$\|u_\alpha^\delta(\cdot, y) - u_\alpha(\cdot, y)\|_{L^2(0,a)} \le c_1\frac{\delta}{\alpha} + c_2\frac{\varepsilon}{\alpha}, \tag{24}$$

where $c_1 = \sqrt{3 + 3b^2}$, $c_2 = 2\sqrt{2}(1 + b) + 4\sqrt{b^3}(a + 1) + \sqrt{a}$.

Proof. According to (21) and (23)

$$\|u_\alpha^\delta(\cdot, y) - u_\alpha(\cdot, y)\| \le \|P(\cdot, y)\| + \left(\sum_{n=1}^{n_\alpha} |V_n(y) - W_n^\delta(y)|^2\right)^{\frac{1}{2}}.$$

Since for $n \le n_\alpha$

$$\frac{\sinh y\zeta_n}{y\zeta_n} \le \frac{1}{\alpha},$$

from(22) and (15) it follows

$$|V_n(y) - W_n^\delta(y)| \le \frac{1}{\alpha}\left[|\widetilde{g}_n - g_n^\delta| + y|\widetilde{h}_n - h_n^\delta| + y^{\frac{3}{2}}\|\varphi_n\|_{L^2(0,b)}\right].$$

By (3) and A1 we have

$$|\widetilde{g}_n - g_n^\delta| \le |g_n - g_n^\delta| + \frac{4\varepsilon}{n\pi}, \quad |\widetilde{h}_n - h_n^\delta| \le |h_n - h_n^\delta| + \frac{4\varepsilon}{n\pi},$$

$$\sum_{n=1}^{\infty} |g_n - g_n^\delta|^2 \le \delta^2, \quad \sum_{n=1}^{\infty} |h_n - h_n^\delta|^2 \le \delta^2.$$

Moreover, according to the definition of φ_n

$$\|\varphi_n\|_{L^2(0,b)} \le \frac{2\sqrt{2}}{n\pi}\left[(2a+1)\|f_1\|_{H^2(0,b)} + \|f_2\|_{H^2(0,b)}\right] \le \frac{\varepsilon}{n}\frac{4\sqrt{2}}{\pi}(a+1).$$

Thus, for $C(a,b) = \frac{4}{\pi}\left(1 + b + \sqrt{2b^3}(a+1)\right)$

$$\sum_{n=1}^{n_\alpha} |V_n(y) - V_n^\delta(y)|^2 \le \frac{3}{\alpha^2}\left((1+b^2)\delta^2 + \varepsilon^2 C^2(a,b)\sum_{n=1}^{n_\alpha}\frac{1}{n^2}\right) \le$$

$$\le \frac{\delta^2}{\alpha^2} 3\left(1+b^2\right) + \frac{\varepsilon^2}{\alpha^2} C^2 \frac{\pi^2}{2}.$$

Finally, the estimation

$$\|P(\cdot,y)\|^2_{L^2(0,a)} = \frac{a}{3}\left(f_1(y)^2 + f_1(y)f_2(y) + f_2(y)^2\right) \le a\varepsilon^2$$

ends the proof. $\square$

Now we are going to estimate an order of convergence of u_α to u. We have

$$\|u(\cdot,y) - u_\alpha(\cdot,y)\| \le \left(\sum_{n>n_\alpha} S_n^2(y)\right)^{\frac{1}{2}} + \left(\sum_{n>n_\alpha} W_n^2(y)\right)^{\frac{1}{2}}, \tag{25}$$

where S_n and W_n are given by formulas (19) and (18), respectively. Due to definition (20), if $n > n_\alpha$, then $n > k\frac{a}{\pi}$, i.e. $\zeta_n > 0$ and

$$0 < \frac{1}{\sinh b\zeta_n} < 2\alpha.$$

We have two auxiliary lemmas.

Lemma 3.1. *If S_n are the Fourier series coefficients for the solution $s(\cdot,y)$ to (16), then for $n > n_\alpha$*

$$|S_n(y)| \le \frac{2\sqrt{b}}{\zeta_n}\|\varphi_n\|_{L^2(0,b)}.$$

Lemma 3.2. *Let W_n be the Fourier series coefficients for the solution w to (17). If A1 is satisfied, then for $n > n_\alpha$*

$$|W_n(y)|^2 \le |W_n(b)|^2 + g_n^2 + \frac{16\varepsilon^2}{n^2\pi^2} + \frac{1}{\zeta_n^2}\left(h_n^2 + 2\frac{16\varepsilon}{n^2\pi^2} + 2b\|\varphi_n\|^2\right).$$

We omit here the technical proofs of Lemmas. They are presented in details in [13].

Proposition 3.2. *Let $k \neq n\frac{\pi}{a}$. If the assumptions A1 and A2 are satisfied, then* $\exists C_1, C_2\ \forall y \in (0, b]$

$$\|u(\cdot, y) - u_\alpha(\cdot, y)\|_{L^2(0,a)} \leq C_1(arcosh\frac{1}{\alpha})^{-1} + C_2\varepsilon(arcosh\frac{1}{\alpha})^{-\frac{1}{2}} \tag{26}$$

and the constants C_1, C_2 depend on a, b and k.

Proof. From (20) for $n > n_\alpha$

$$\frac{1}{\zeta_n} < \frac{b}{arcosh\frac{1}{\alpha}} \quad \text{and} \quad \frac{1}{n} < \frac{\pi}{a}\frac{b}{arcosh\frac{1}{\alpha}}.$$

Taking into account Lemma 1 and the estimation

$$\|\varphi_n\| \leq \frac{1}{n}\frac{2a}{\pi}(2\|\psi_0\| + \|\psi_1\|) \leq \frac{\varepsilon}{n}\frac{6a}{\pi},$$

we get

$$\sum_{n>n_\alpha} S_n^2(y) \leq 4b \sum_{n>n_\alpha} \frac{1}{\zeta_n^2}\|\varphi_n\|^2 \leq (5a\sqrt{b})^2 \frac{\varepsilon^2}{(arcosh\frac{1}{\alpha})^2}.$$

For estimating the second term of (25) we use Lemma 2 and the assumption A2, i.e. $\|u_x'(\cdot, b)\| \leq M$. After some calculations we get

$$\sum_{n>n_\alpha} W_n^2(b) \leq \frac{1}{n_\alpha^2 + 1} \sum_{n>n_\alpha} n^2 W_n^2(b) \leq \frac{b^2\pi^2}{a^2 arcos^2\frac{1}{\alpha}} M_\varepsilon.$$

Let G and H be upper bounds: $\|g'\|_{L^2(0,a)} \leq G$ and $\|h\|_{L^2(0,a)} \leq H$. We get

$$\sum_{n>n_\alpha} g_n^2 \leq \frac{1}{n_\alpha^2 + 1} \sum_{n>n_\alpha} n^2 g_n^2 \leq \frac{G^2 b^2 \pi}{a^2} \left(\frac{1}{arcosh\frac{1}{\alpha}}\right)^2,$$

$$\sum_{n>n_\alpha} \frac{1}{\zeta_n^2} h_n^2 \leq H^2 b^2 \left(\frac{1}{arcosh\frac{1}{\alpha}}\right)^2,$$

$$\sum_{n>n_\alpha} \frac{\varepsilon^2}{n^2} \leq C\frac{\varepsilon^2}{arcosh\frac{1}{\alpha}},$$

which completes the proof. □

Summarizing the above results we come to the following error estimation:

Theorem 3.1. *Let $u \in H^2(D)$ be the exact solution of (2) and u_α^δ be the regularized solution defined by (21) for noisy data (3). If the assumptions A1 and A2 are satisfied, then there exist constants C_1, C_2 such that $\forall y \in (0, b]$*

$$\|u(\cdot, y) - u_\alpha^\delta(\cdot, y)\|_{L^2(0,a)} \leq C_1\frac{\delta + \varepsilon}{\alpha} + C_2\frac{1}{arcosh\frac{1}{\alpha}}\left(1 + \varepsilon\sqrt{arcosh\frac{1}{\alpha}}\right) \tag{27}$$

An open question is how to choose the regularization parameter α in order to minimize the above error bound for given δ and ε. Naturally, because of ε, we have no convergence, when the data error bound δ tends to 0. However, in the model considered, ε decreases, when the length of Γ increases. So, we may formulate the following remark:

Remark. Let D be an infinite strip and $\Omega := (x_-, x_+) \times (0, d) \subset D$. Let us assume that $\forall \varepsilon \exists x_-(\varepsilon), x_+(\varepsilon)$ such that

$$\|u(x_\pm.\cdot)\|_{H^2(o,b)} \leq \varepsilon.$$

Thus, if $\|u'(\cdot, b)\|_{L^2(\mathbb{R})} \leq M$ and $\varepsilon = \delta$, then

$$\|u(\cdot, y) - u_\alpha^\delta(\cdot, y)\|_{L^2(x_-, x_+)} \leq \widetilde{C}_1 \frac{\delta}{\alpha} + \widetilde{C}_2 \frac{1}{arcosh \frac{1}{\alpha}} \left(1 + \delta \sqrt{arcosh \frac{1}{\alpha}} \right).$$

However, the constants $\widetilde{C}_1$, $\widetilde{C}_2$ depend now on the length of $\Gamma = \Gamma(\varepsilon)$, i.e. on $x_+(\varepsilon) - x_-(\varepsilon)$.

4 Conclusion

The difference between our formulation of the Cauchy problem for the Helmholtz equation on a rectangle and the previous ones consists in the fact that here data are given only on one side of the rectangle. In previous formulations, additional homogeneous or periodic boundary conditions on the sides parallel to the beam axis have been applied. However, they had no clear physical meaning. In such a case, usually, the problem was formulated on infinite strip which allowed to apply the Fourier transform.

The approach presented in this paper is an alternative way to analyze such a problem. Under the assumption that the collimated laser beam is such that A1 is satisfied, we propose a series expansion approach which yields to series representation of the exact solution. This representation can be used for formulation of different regularization methods. An example of such a method is proposed and its stability and error bound are shown. The problem of choice of regularization parameter for this method is not undertaken here and will be a subject of a subsequent paper.

References

[1] G. Alessandrini, L. Rondi, E. Rosset and S. Vessella, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems, Vol. 25, Issue 12 (2009).

[2] W. Arendt, and T. Regińska, *Cauchy problem for the Helmholtz equation on Lipschitz domains in R^d*, Journal of Inverse and Ill-Posed Problems Vol. 17 (2009), pp. 703–711.

[3] A. Belsky, T. Korneychik, and A. Khapalyuk, *Space structure of laser radiation*, Minsk, 1982 (in Russian).

[4] Hao Cheng, Chu-Li Fu, Xiao-Li Feng, *An optimal filtering method for the Cauchy problem of the Helmholtz equation*, Appl. Math. Lett., Vol. 24, Issue 6 (2011), pp. 958-964.

[5] L. Eldén and F. Berntsson, *A stability estimate for a Cauchy problem for an elliptic partial differential equation*, Inverse Problems, Vol. 21(2005), pp.1643–1653.

[6] H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, 1996.

[7] Chu-Li Fu, Xiao-Li Feng, Zhi Qian, *The Fourier regularization for solving the Cauchy problem for the Helmholtz equation*, Appl. Numer. Math., Vol. 59, Issue 10 (2009), pp. 2625-2640.

[8] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, 2006.

[9] Y. Pinchover, and J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge University Press, 2007.

[10] Ai-Lin Qian, Xiang-Tuan Xiong, Yu-Jiang Wu, *On a quasi-reversibility regularization method for a Cauchy problem of the Helmholtz equation*, J. Comput. Appl. Math., Vol. 233, Issue 8 (2010), pp. 1969-1979.

[11] H.H. Qin, T. Wei, *Modified regularization method for the Cauchy problem of the Helmholtz equation*, Appl. Math. Modelling, Vol. 33, Issue 5 (2009), pp. 2334-2348.

[12] A.G. Ramm, *Series that can be differentiated term-wise m times if the function is m-smooth*, Mathematica Aeterna, Vol. 1, Issue 3(2011), pp.137–148.

[13] T. Regińska, *Regularization methods for mathematical model of laser beams*, IMPAN Preprint 731 (2011).

[14] T. Regińska and K. Regiński, *A Cauchy problem for the Helmholtz equation: application to analysis of light propagation in solids*, Tech. Report 06-04, Univ. Kiel, 2006.

[15] T. Regińska and K. Regiński, *Approximate solution of a Cauchy problem for the Helmholtz equation*, Inverse Problems, Vol. 22 (2006), pp. 975–989.

[16] T. Regińska and U. Tautenhahn, *Conditional stability estimates and regularization with applications to Cauchy problems for the Helmholtz equation*, Numer. Funct. Anal. and Optimiz., Vol. 30, Issue 9-10 (2009), pp. 1065–1097.

[17] A. Siegman, *Lasers*, Oxford University Press, 1986.

[18] U. Tautenhahn, *Optimal stable solution of Cauchy problems for elliptic equations*, Z. Anal. Anw., Vol. 4(1996), pp. 961–984.

[19] Xiang-Tuan Xiong, Chu-Li Fu, *Two approximate methods of a Cauchy problem for the Helmholtz equation*, Comput. Appl. Math., Vol. 26, Issue 2(2007), pp. 285–307.

[20] Xiang-Tuan Xiong, *A regularization method for a Cauchy problem of the Helmholtz equation*, J. Comput. Appl. Math., Vol. 233, Issue 8 (2010), pp. 1723–1732.

[21] H.W. Zhang, H.H. Qin, T. Wei, *A quasi-reversibility regularization method for the Cauchy problem of the Helmholtz equation*, Int. J. Comput. Math., Vol. 88, Issue 4(2011), pp. 839–850.

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