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REFORMULATION OF AN INVERSE PROBLEM STATEMENT THAT REDUCES COMPUTATIONAL COSTS

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Abstract The paper shows how we can reformulate the inverse problem, which is solved by optimization method. The approach is demonstrated on four models of inverse problems. It is shown that when the inverse problem is the coefficient problem, the use of parallel computing can reduce almost half of the time of the inverse problem's numerical solution calculating, as solving direct and conjugate problems can be searched in parallel way. If the inverse problem is linear (the unknown boundary condition or the right side of an equation are being looked for), then the inverse problem can be reduced to the numerical solution of the moment problem, for which all the necessary functions can be computed in advance by the known data of the inverse problem. To illustrate the proposed approach, we represent the numerical solution of the Cauchy problem for an elliptic equation on data obtained from the physical experiment.

Key words: coefficient hyperbolic inverse problem, retrospective inverse problem of heat conduction, the Cauchy problem for an elliptic equation, definition of the function of an elliptic equation source, optimization method, conjugate problem, residual functional, problem of moments

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1 Introduction

In practice, quite often inverse problems are solved using the optimization method. Gradient of the residual functional is sought by solving the conjugate problem (see, e.g., [1, 2]). The computational complexity of this approach is that the computation of the gradient requires solutions of direct and conjugate problems, the conjugate problem can be solved only after the direct problem solution is found. Thus, a certain amount of time is spent on one iteration of minimization process.

In this paper we proposed a reformulation of the original statement of the inverse problem, which allows us to search for direct and conjugate problems solutions in parallel way, which reduces the computation time. Since the article is partly methodological, so the essence of the proposed approach is demonstrated on four models of the inverse problem as follows:

- coefficient hyperbolic inverse problem;
- retrospective inverse problem of heat conduction;
- the Cauchy problem for the Laplace equation that can be reduced to the inverse problem of search for the unknown boundary condition;
- problem of determining the right side of the elliptic equation.

Let us show, that in the first and the second cases we can achieve almost double gain in the computation time for one iteration of the minimization and, consequently, for the solution of inverse problems in general. In the third and the fourth cases let us show, that the computation of the gradient of the residual functional does not require direct and conjugate problems solution at all.

The idea of the proposed approach is related to the technique of constructing the sensitivity operator (see, e.g., [6, 7]) and the technology of residual functional gradient's output for the numerical solution of the inverse problem (see, e.g., [1]), but in these publications the problem of reducing computation time was not considered.

2 Coefficient hyperbolic inverse problem

2.1 The traditional way of solving the inverse problem

Let us consider the following statement of the inverse problem

1st inverse problem: Find the unknown function $q(x)$, if it is known for the solution of the direct problem

$$u_{tt} = u_{xx} - qu, \quad (x,t) \in \{0 < x < T, x < t < 2T - x\}, \tag{1}$$
$$u_x|_{x=0} = 0, \quad t \in [0, 2T], \tag{2}$$
$$u|_{t=x} = 1/2, \quad x \in [0, T], \tag{3}$$

the following additional information

$$u|_{x=0} = f(t), \quad t \in [0, 2T]. \tag{4}$$

This statement of the inverse problem is well understood, for example, in [5].

The numerical solving of the inverse problem (1)-(4) by using optimization method assumes that the minimum of the residual functional is being sought:

$$J[q] = \int_0^{2T} [u(0,t) - f(t)]^2 dt. \tag{5}$$

The gradient of the residual functional (5) can be obtained in the following form

$$J'[q](x) = \int_x^{2T-x} u(x,\tau)\psi(x,\tau)d\tau \tag{6}$$

where the function $\psi(x,\tau)$ is the solution of conjugate problem

$$\psi_{tt} = \psi_{xx} - q\psi, \quad (x,t) \in \{0 < x < T, x < t < 2T - x\}, \tag{7}$$
$$\psi_x|_{x=0} = 2[u(0,t) - f(t)], \quad t \in [0, 2T], \tag{8}$$
$$\psi|_{t=2T-x} = 0, \quad x \in [0, T]. \tag{9}$$

We believe that the technique of obtaining the gradient of the residual functional is well known, so we omit the intermediate calculations. The description of this technique with a simple example can be found in [1].

Thus, one iteration of the minimization of the residual functional (5) contains three basic steps as follows:

1st step: solving of the direct problem (1)-(3);
2nd step: solving of the conjugate problem (7)-(9);
3d step: calculating of the gradient (6).

As we note, step 2 can not be done in parallel way with step 1, as there is function $u(0,t)$ in the statement of the conjugate problem (7)-(9), which can be obtained only after step 1. Solving of the direct and conjugate problems requires the same computation time.

2.2 Transformation of the 1st inverse problem's statement

Let us consider the following statement of direct problems:

$$v_{tt} = v_{xx} - qv, \quad (x,t) \in \{0 < x < T, x < t < 2T - x\}, \tag{10}$$
$$v_x|_{x=0} = \alpha(t,\nu_k), \quad t \in [0,2T], \quad k = 0,1,2,..., \tag{11}$$
$$v|_{t=2T-x} = \beta, \quad x \in [0,T]. \tag{12}$$

The differential operator of the equation (10) is conjugate to the differential operator in (1), the function $\alpha(t,\nu_k)$ and the constant β will be defined later, ν_k is a known parameter, as it will be defined later.

To emphasize the function $v(x,t)$ dependence on the parameter ν_k, where it will be needed, we will write $v(x,t;\nu_k)$.

Let us take an integral of the identical equation

$$\begin{aligned}
0 \;\equiv\; & \int_0^T \int_\xi^{2T-\xi} \Big(u_{tt} - u_{xx} + qu\Big)\, v\, d\tau d\xi - \int_0^T \int_\xi^{2T-\xi} \Big(v_{tt} - v_{xx} + qv\Big)\, u\, d\tau d\xi \\
= \; & \int_0^T \int_\xi^{2T-\xi} \left(\frac{\partial}{\partial t}\Big(u_t v - v_t u\Big) - \frac{\partial}{\partial x}\Big(u_x v - v_x u\Big)\right) d\tau d\xi \\
= \; & -\int_0^T \left(\frac{d}{d\xi}\left[u(\xi,2T-\xi)\right] v(\xi,2T-\xi) - u(\xi,2T-\xi)\frac{d}{d\xi}\left[v(\xi,2T-\xi)\right]\right) d\xi \\
- \; & \int_0^T \left(\frac{d}{d\xi}\left[u(\xi,\xi)\right] v(\xi,\xi) - u(\xi,\xi)\frac{d}{d\xi}\left[v(\xi,\xi)\right]\right) d\xi \\
+ \; & \int_0^{2T} \Big(u_x(0,\tau)v(0,\tau) - u(0,\tau)v_x(0,\tau)\Big) d\tau.
\end{aligned}$$

Hence, in view of (1)-(4), (10)-(12) the following par implies:

$$v(0,0;\nu_k) = 2\left(\beta \cdot f(2T) - \int\limits_0^{2T} f(\tau)\alpha(\tau,\nu_k)\,d\tau\right). \tag{13}$$

Since the choice of the function $\alpha(t,\nu_k)$ is in our hands, the integral in (13) can be computed in advance. As a function $\alpha(t,\nu_k)$, we can take, for example, $\cos(\nu_k t)$, where $\nu_k = k\pi/T$. Then the integral in (13) is the coefficient f_k of function expansion $f(t)$ in the Fourier series on the interval $[0,2T]$. The constant β in this case can be arbitrary, however, if the boundary condition (3) would be different, then we could get a condition on the choice of β.

Thus, we have a series of statements of direct problems (10)-(12) and the relations (13), depending on the parameter ν_k $(k = 0,1,...)$, and hence we can formulate:

2nd inverse problem: Find the unknown function $q(x)$, if there is the additional information (13) for the solution of the direct problem (10)-(12). The inverse problem (10)-(13) can be solved numerically using the minimization of the residual functional

$$\Phi[q] = \sum_k [v(0,0;\nu_k) - v_0(\nu_k)]^2, \tag{14}$$

where

$$v_0(\nu_k) = 2\left(\beta \cdot f(2T) - \int\limits_0^{2T} f(\tau)\alpha(\tau,\nu_k)\,d\tau\right).$$

To find the gradient of the residual functional (14), we use the well-known techniques (see, e.g., [1, 2]). We obtain

$$\Phi'[q](x) = \sum_k \int\limits_x^{2T-x} v(x,\tau;\nu_k)w(x,\tau;\nu_k)\,d\tau, \tag{15}$$

where the functions $w(x,t;\nu_k)$ are the solutions of the conjugate problems

$$\begin{aligned}
&w_{tt} = w_{xx} - qw, \quad (x,t) \in \{0 < x < T, x < t < 2T - x\},\\
&w_x|_{x=0} = 0, \quad t \in [0,2T],\\
&w|_{t=x} = -2[v(0,0;\nu_k) - v_0(\nu_k)], \quad x \in [0,T].
\end{aligned}$$

It is easy to see that the function $w(x,\tau;\nu_k)$ can be represented as

$$w(x,\tau;\nu_k) = -4[v(0,0;\nu_k) - v_0(\nu_k)] \cdot u(x,t),$$

where $u(x,t)$ is the solution of the starting direct problem (1)-(3).

So the gradient of the residual functional (14) can be written in the following form:

$$\Phi'[q](x) = \int\limits_x^{2T-x} V(x,\tau)u(x,\tau)\,d\tau, \tag{16}$$

where

$$V(x,t) = \sum_k 2[v(0,0,\nu_k) - v_0(\nu_k)]v(x,t;\nu_k).$$

2.3 The conclusions of the transformation results

The functions $\alpha(t, \nu_k)$ can be chosen as the basis functions on the interval $[0, 2T]$. Thus, we deal with the Fourier series for function $f(t)$. When it is being solved numerically, this Fourier series can be interrupted. The following reasons may serve this:

- The first elements of the Fourier series can describe the behavior of function $f(t)$ with good accuracy.
- With increasing index number the Fourier coefficient tends to zero. Consequently, in (13) value $v(0, 0; \nu_k)$ tends to a constant, and the variations of the coefficients with high index numbers have, practically, no influence on the behavior of the residual functional (14).
- In practice, the function $f(t)$ is known with some error. As the rule, the noise has a high frequency component, so that the coefficients corresponding to the high-frequency harmonics are calculated with a great error, much greater than the first coefficients of the series. Consequently, they do not contain information about the behavior of the function $f(t)$.

Thus, for the numerical solution of the inverse problem we can limit ourselves to a finite set of functions $\alpha(t, \nu_k)$ and parameters ν_k $(k = \overline{0, N})$. The number of elements of the Fourier series can be chosen from the condition

$$\int_0^{2T} \left(f(t) - \sum_{k=1}^{N} f_k \alpha(t, \nu_k) \right)^2 dt = \delta^2,$$

where δ is the error level, with what the values of function $f(t)$ were measured.

The capabilities of modern programming languages and computers allow to compute $v(x, t; \nu_k)$ $(k = \overline{0, N})$ and $u(x, t)$ in a parallel way. Therefore, to make one iteration of minimization of the residual functional (14), we need to do two basic steps:

1st step: solving of the direct problems (10)-(12) $v(x, t; \nu_k)$ $(k = \overline{0, N})$, and the solution of the direct problem (1)-(3) $u(x, t)$;

2nd step: calculating of the gradient (16).

Steps 1 and 3 in the case of solving the 1st inverse problem and the steps 1-2 in the case of solving the 2nd inverse problem will require the same computation time, so we obtain a double gain in the computation time for the implementation of one iteration of the minimization process.

3 Retrospective inverse problem of heat conduction

3.1 The traditional way of solving the inverse problem

Let us assume the following statement of the inverse problem

3d inverse problem: Find the unknown function $\lambda(x)$, if it is known for the solution of the direct problem

$$u_t = (\lambda u_x)_x, \quad (x,t) \in \{0 < x < L, 0 < t < T\}, \tag{17}$$
$$u_x|_{x=0} = 0, \quad u_x|_{x=L} = 0, \quad t \in [0,T], \tag{18}$$
$$u|_{t=0} = a(x), \quad x \in [0,L], \tag{19}$$

the following additional information

$$u|_{t=T} = b(x), \quad x \in [0,L]. \tag{20}$$

Discussion of this problem can be found, for example, here [2].

This definition is interesting to us, because in this case, additional information is not measured at the point $x = 0$ but at $t = T$.

Numerical solving of the inverse problem (17)-(20) reduces to finding the minimum of the residual functional:

$$J[\lambda] = \int_0^L [u(x,T) - b(x)]^2 dx. \tag{21}$$

The gradient of the residual functional (21) has the form:

$$J'[\lambda](x) = -\int_0^T u_x(x,\tau)\psi_x(x,\tau)d\tau \tag{22}$$

where the function $\psi(x,\tau)$ is the solution of the conjugate problem

$$-\psi_t = (\lambda \psi_x)_x, \quad (x,t) \in \{0 < x < L, 0 < t < T\}, \tag{23}$$
$$\psi_x|_{x=0} = 0, \quad \psi_x|_{x=L} = 0, \quad t \in [0,T], \tag{24}$$
$$\psi|_{t=T} = 2[u(x,T) - b(x)], \quad x \in [0,L]. \tag{25}$$

As in the case of the 1st inverse problem we deal with the same problem: one iteration of the minimization of the residual functional (21) contains three basic steps, the conjugate problem (23)-(25) can not be solved in a parallel way with the solving of the direct problem (17)-(19), the solving of the conjugate problem requires as much time as of the direct problem.

3.2 Transformation of the statement of the 3d inverse problem

Let us consider the following statement of direct problems:

$$-v_t = (\lambda v_x)_x, \quad (x,t) \in \{0 < x < L, 0 < t < T\}, \tag{26}$$
$$v_x|_{x=0} = 0, \quad v_x|_{x=L} = 0, \quad t \in [0,T], \tag{27}$$
$$v|_{t=T} = \gamma(x;\nu_k), \quad x \in [0,L], \tag{28}$$

where $\gamma(x;\nu_k)$ are known functions, depending on parameter ν_k $(k=0,1,...)$, and the conditions $\gamma_x|_{x=0}=\gamma_x|_{x=L}=0$ are to be satisfied.

Taking an integral of identical equation

$$0 \equiv \int_0^T\int_0^L \big(u_t-(\lambda u_x)_x\big)v\,dxdt-\int_0^T\int_0^L\big(-v_t-(\lambda v_x)_x\big)u\,dxdt$$

and taking into account (17)-(20) and (26)-(28), we obtain the equality

$$\int_0^L b(x)\gamma(x;\nu_k)\,dx=\int_0^L a(x)v(x,0;\nu_k)\,dx \tag{29}$$

Thus, we can state:

4th inverse problem: Find the unknown function $\lambda(x)$, if it is known for the solution of the direct problem (26)-(28) the additional information (29).

For the numerical solving of the inverse problem (26)-(29), we will find the minimum of the residual functional

$$\Phi[\lambda]=\sum_k\left[\int_0^L a(x)v(x,0;\nu_k)\,dx-b_k\right]^2, \tag{30}$$

where

$$b_k=\int_0^L b(x)\gamma(x;\nu_k)\,dx,$$

that can be calculated beforehand. Calculating the gradient of the residual functional (30), we find that the conjugate problem will be:

$$\begin{aligned}
&w_t=(\lambda w_x)_x,\quad (x,t)\in\{0<x<L,0<t<T\},\\
&w_x|_{x=0}=0,\quad w_x|_{x=L}=0,\quad t\in[0,T],\\
&w|_{t=0}=2a(x)\cdot\left[\int_0^L a(x)v(x,0;\nu_k)\,dx-b_k\right],\quad x\in[0,L],
\end{aligned}$$

where it is easily seen that

$$w(x,t;\nu_k)=2\left[\int_0^L a(x)v(x,0;\nu_k)\,dx-b_k\right]\cdot u(x,t).$$

Given this, the gradient of the residual functional (30) can be obtained as follows

$$\Phi'[q](x)=-\int_0^T V_x(x,\tau)u_x(x,\tau)\,d\tau, \tag{31}$$

where

$$V(x,t) = \sum_k \left[\int_0^L a(x)v(x,0;\nu_k)\,dx - b_k \right] \cdot v(x,t;\nu_k).$$

Obviously, all the conclusions drawn in section 2.3, occur in this case too.

4 The Cauchy problem for the Laplace equation

4.1 The traditional way of solving the problem

Let us consider the Cauchy problem for the Laplace equation:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad (x,y) \in \{0 < x < L, 0 < y < M\}, \\ &u_x|_{x=0} = 0, \quad u_x|_{x=L} = 0, \quad y \in [0,M], \\ &u|_{y=0} = 0, \quad u_y|_{y=M} = b(x), \quad x \in [0,L]. \end{aligned}$$

This statement of the problem can be reformulated as an inverse problem to find the unknown boundary condition [3, 4]. This can be done in several ways, we formulate the inverse problem as follows:

5th inverse problem: Find the unknown function $p(x)$, if it is known for the solution of the direct problem

$$u_{xx} + u_{yy} = 0, \quad (x,y) \in \{0 < x < L, 0 < y < M\}, \tag{32}$$

$$u_x|_{x=0} = 0, \quad u_x|_{x=A} = 0, \quad y \in [0,M], \tag{33}$$

$$u|_{y=0} = 0, \quad u_y|_{y=B} = p(x), \quad x \in [0,L]. \tag{34}$$

the following additional information

$$u_y|_{y=0} = b(x), \quad x \in [0,L]. \tag{35}$$

This statement of the inverse problem is interesting for us, because the aim of restoring is the unknown boundary condition on the parts of the boundary.

For the numerical solution of the 5th inverse problem the minimum of the following residual functional can be sought

$$J[p] = \int_0^L [u_y(x,0) - b(x)]^2 dx, \tag{36}$$

which gradient has the form as follows

$$J'[p](x) = -\psi(x,M), \tag{37}$$

where the function $\psi(x,y)$ is the solution of the conjugate problem

$$\psi_{xx} + \psi_{yy} = 0, \quad (x,t) \in \{0 < x < L, 0 < y < M\}, \tag{38}$$

$$\psi_x|_{x=0} = 0, \quad \psi_x|_{x=L} = 0, \quad y \in [0,M], \tag{39}$$

$$\psi|_{y=0} = 2[u_y(x,0) - b(x)], \quad \psi_y|_{y=M} = 0, \quad x \in [0,L]. \tag{40}$$

As in the case of inverse problems 1 and 3, we are faced with the same problem: one iteration of the minimization of the residual functional (36) contains three basic steps, the conjugate problem (38)-(40) can not be solved in a parallel way with the solving of the direct problem (32)-(34), the solving of the conjugate problem requires as much time as of the direct problem.

4.2 Transformation of the statement of the 5th inverse problem

Let us consider the following statements of the direct problems:

$$v_{xx} + v_{yy} = 0, \quad (x,t) \in \{0 < x < L, 0 < y < M\}, \tag{41}$$

$$v_x|_{x=0} = 0, \quad v_x|_{x=L} = 0, \quad y \in [0, M], \tag{42}$$

$$v_y|_{y=M} = 0, \quad v|_{y=0} = \theta(x; \nu_k), \quad x \in [0, L], \tag{43}$$

where $\theta(x; \nu_k)$ are known functions, depending on parameter ν_k $(k = 0, 1, ...)$, and the conditions $\theta_x|_{x=0} = \theta_x|_{x=L} = 0$ are to be satisfied.

Taking an integral of identical equation

$$0 \equiv \int_0^M \int_0^L \big(u_{xx} + u_{yy}\big) v \, dxdy - \int_0^M \int_0^L \big(u_{xx} + u_{yy}\big) v \, dxdy$$

and taking into account (32)-(35) and (41)-(43), we obtain the following equality:

$$\int_0^L p(x) v(x, M; \nu_k) \, dx = \int_0^L b(x) \theta(x; \nu_k) \, dx. \tag{44}$$

We note, that

- problems (41)-(43) are correctly formulated and can be solved in advance, i.e. prior to the search of the unknown boundary condition $p(x)$;
- for finding the unknown function $p(x)$ from (44), we obtained the relations

$$\int_0^L p(x) K(x, \nu_k) \, dx = b_k, \tag{45}$$

  where

$$K(x, \nu_k) = v(x, M; \nu_k), \quad b_k = \int_0^L b(s) \theta(x; \nu_k) \, dx, \quad k = 0, 1, 2....$$

Thus, in this case, the proposed transformation reduced statement of the 5th inverse problem to the solution of the moment problems.

Remark. As we found out, the reduction of the numerical solution of the Cauchy problem for an elliptic equation to the solution of the moment problem has been proposed in [8] and further developed in the papers [10]-[14], that represent the theoretical results, and various options for the numerical solution of this problem on the simulated data.

If for finding the function $p(x)$ of the relations (45), we apply optimization method, we will seek the minimum of the residual functional

$$\Phi[p] = \sum_k \left[\int_0^L p(x) K(x, \nu_k)\, dx - b_k \right]^2,$$

which gradient, as it is easily seen, has the form

$$\Phi'[p](x) = \sum_k 2 \left[\int_0^L p(x) K(x, \nu_k)\, ds - b_k \right] \cdot K(x, \nu_k),$$

and does not require the solving of direct and conjugate problems.

Often in practice it is required to solve the problem (32)-(35) many times for the same type of data $b(x)$. The proposed procedure can help to reduce the computation time significantly, because the problems (41)-(43) are solved in advance.

If the domain has a simple form, such as in our case, then the appropriate choice of functions $\theta(x, \nu_k)$ can be obtained by the solution of the problem (41)-(43) in an analytical form. If, for example,

$$\theta(x, \nu_k) = \cos(\nu_k x), \quad \nu_k = \frac{k\pi}{L}, \quad k = 0, 1, ...,$$

then it is easy to obtain a solution of (41)-(43)

$$v(x, y; 0) = 1, \quad v(x, y; \nu_k) = \frac{\mathrm{e}^{-\nu_k M} \mathrm{e}^{\nu_k (y-M)} + \mathrm{e}^{-\nu_k y}}{1 + \mathrm{e}^{-2\nu_k M}} \cos(\nu_k x), \quad k = 1, 2, ...$$

Substituting the corresponding expressions in (44), we obtain

$$p_0 = b_0, \quad p_k = \frac{1}{2} b_k \, \mathrm{e}^{\nu_k M} \left(1 + \mathrm{e}^{-2\nu_k M}\right), \quad k = 1, 2, ..., \tag{46}$$

where p_k, b_k and a_k are the Fourier series coefficients for the related functions. Then

$$p(x) = \frac{1}{2} p_0 + \sum_{k=1}^{\infty} p_k \cos(\nu_k x). \tag{47}$$

Using found $p(x)$, we solve the problem (32)-(34) and find $u(x, y)$.

Remark. Impropriety of the Cauchy problem becomes apparent in the presence of the factor $\mathrm{e}^{\nu_k M}$ in the expression for p_k (see (46)), which increases with rising of k.

In the numerical calculating of series (47) the Tikhonov regularization [17] should be used.

4 Determining the right side of the elliptic equation

4.1 The traditional way of solving the inverse problem

Let us consider the following inverse problem: **6th inverse problem**: Find the unknown function $f(x)$, if it is known for the solution of the direct problem

$$u_{xx} + u_{yy} = f(x)g(y), \quad (x, y) \in \{0 < x < L, 0 < y < M\}, \tag{48}$$
$$u_x|_{x=0} = 0, \quad u_x|_{x=L} = 0, \quad y \in [0, M], \tag{49}$$
$$u|_{y=0} = 0, \quad u_y|_{y=M} = 0, \quad x \in [0, L], \tag{50}$$

the following additional information

$$u|_{y=M} = p(x), \quad x \in [0, L], \tag{51}$$

where the function $g(y) \not\equiv 0$ is known.

For the numerical solving of the inverse problem (48)-(51) the minimum of the residual functional can be sought

$$J[p] = \int_0^L [u(x, M) - p(x)]^2 dx, \tag{52}$$

which gradient has the form as follows

$$J'[p](x) = -\int_0^M g(y)\psi(x, y)\, dy, \tag{53}$$

where the function $\psi(x, y)$ is the solution of the following conjugate problem

$$\psi_{xx} + \psi_{yy} = 0, \quad (x, t) \in \{0 < x < L, 0 < y < M\}, \tag{54}$$
$$\psi_x|_{x=0} = 0, \quad \psi_x|_{x=L} = 0, \quad y \in [0, M], \tag{55}$$
$$\psi_y|_{y=M} = 2[u(x, M) - p(x)], \quad \psi|_{y=0} = 0, \quad x \in [0, L]. \tag{56}$$

As in the case of inverse problems 1, 3 and 5 we face the same problems: the conjugate problem (54)-(56) can be solved only after the solving the direct problem (48)-(50); numerical solving of the direct and conjugate problems requires the same amount of computation time.

4.2 Transformation of the statement of 6th inverse problem

Let us consider the following statements of direct problems:

$$v_{xx} + v_{yy} = 0, \quad (x, t) \in \{0 < x < L, 0 < y < M\}, \tag{57}$$
$$v_x|_{x=0} = 0, \quad v_x|_{x=L} = 0, \quad y \in [0, M], \tag{58}$$
$$v|_{y=0} = 0, \quad v_y|_{y=M} = \mu(x; \nu_k), \quad x \in [0, L], \quad k = 0, 1, 2, ... \tag{59}$$

where $\mu(x;\nu_k)$ are known functions, depending on parameter ν_k $(k=0,1,...)$.

Taking an integral of the related equality and respecting (48)-(51) and (54)-(56) we obtain the equation

$$-\int_0^L p(x)\mu(x;\nu_k)\,dx = \int_0^L\int_0^M f(x)g(y)v(x,y;\nu_k)\,dydx.$$

Let us note, that

- as in the previous case, the problems (57)-(59) can be solved in advance;
- for finding the unknown function $f(x)$, we obtain relations

$$\int_0^L f(x)K(x,\nu_k)\,dx = p_k, \tag{60}$$

where

$$K(x,\nu_k) = \int_0^M g(y)v(x,y;\nu_k)\,dy, \quad p_k = -\int_0^L p(x)\mu(x;\nu_k)\,dx.$$

As in the previous case, the problem (60) can be solved using the minimization of the related residual functional using some gradient method. As in the previous case, to calculate the gradient of the residual functional, solutions of direct and conjugate problems will not be required.

If the domain has a simple form, such as in our case, then the appropriate choice of functions $\mu(x,\nu_k)$ can be obtained by solving the direct problems (57)-(59) in an analytical form. For example,

$$\mu(x,\nu_k) = \cos(\nu_k x), \quad \nu_k = k\pi/L, \quad k=0,1,...,$$

then solutions of problems (57)-(59) will have the form

$$v(x,y;0) = y, \quad v(x,y;\nu_k) = \frac{1}{\nu_k}\cdot\frac{\mathrm{e}^{\nu_k(y-M)} - \mathrm{e}^{-\nu_k(y+M)}}{1+\mathrm{e}^{-2\nu_k M}}\cos(\nu_k x), \quad k=1,2,....$$

Substituting the corresponding expressions in (60), we obtain

$$f_0 = \frac{1}{q_0}\left[Mb_0 - p_0\right], \quad f_k = \frac{p_k}{q_k}\left(1+\mathrm{e}^{-2\nu_k M}\right), \quad k=1,2,...,$$

where f_k and p_k are the Fourier series coefficients for the corresponding functions and

$$q_0 = \int_0^L g(y)y\,dy, \quad q_k = \int_0^L g(y)\left(\mathrm{e}^{\nu_k(y-M)} - \mathrm{e}^{-\nu_k(y+M)}\right)\,dy, \quad k=1,2,....$$

Then for the following expression

$$f(x) = \frac{1}{2}f_0 + \sum_{k=1}^{\infty} f_k\cos(\nu_k x),$$

the numerical calculating can be obtained using the Tikhonov regularization.

6 Approach approbation

Calculations were carried out to solve the following problem. One or more drops of the liquid rest on the upper surface of the heated foil (see Fig. 1), temperature is measured on the bottom surface. It is required to determine the heat flux on the upper surface.

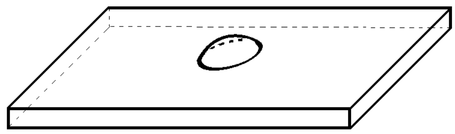

Figure 1: Foil with a drop.

Now we present a mathematical statement of the problem.

Stationary process of heat transfer in foil is described by the equation

$$\mathrm{div}\Big(\lambda \nabla T\Big) = -q_V, \qquad (x,y,z) \in V \equiv \Big\{(0,A)\times(0,B)\times(0,h)\Big\}. \tag{61}$$

Here λ is conductivity coefficient, q_V is density of bulk heat sources, $h \ll A$ and $h \ll B$, because foil is considered as thin matter, A, B and h are linear dimensions of the foil.

On the part of boundary available for measurements we have condition

$$T|_{z=0} = T_0(x,y). \tag{62}$$

Also, we believe that on the surface the convective heat flux is determined by the Newton's law

$$\lambda \frac{\partial T}{\partial z}\bigg|_{z=0} = \alpha\big(T|_{z=0} - T_\infty\big), \tag{63}$$

here T_∞ is the temperature of environment.

Sidewalls of foil are heat insulated

$$\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = 0, \qquad \lambda \frac{\partial T}{\partial x}\bigg|_{x=A} = 0, \qquad \lambda \frac{\partial T}{\partial y}\bigg|_{y=0} = 0, \qquad \lambda \frac{\partial T}{\partial y}\bigg|_{y=B} = 0. \tag{64}$$

Here q_V, λ, α and T_∞ are taken as known constants. The problem (61)-(64) is the Cauchy problem for an elliptic equation.

It is required to define $q(x,y) = -\lambda T_z|_{z=h}$.

To solve the problem (61)-(64) the proposed approach was used.

Since the domain has the shape of a parallelepiped, as known functions and parameters we can choose

$$\alpha(x,y,\nu_x,\nu_y) = \cos(x\nu_x)\cos(y\nu_y), \quad \nu_x = \frac{\pi k}{A}, \quad \nu_x = \frac{\pi l}{B}, \quad k,l = 0,1,2,..., \tag{65}$$

therefore the corresponding conjugate problems can be solved analytically, and for the required flow values we have the expression:

$$\begin{aligned} q(x,y) &= \frac{1}{4}\hat{q}_{00} + \frac{1}{2}\sum_{k=1}^{\infty}\hat{q}_{k0}\cos\left(\frac{\pi k}{A}x\right) + \frac{1}{2}\sum_{l=1}^{\infty}\hat{q}_{0l}\cos\left(\frac{\pi l}{B}y\right) \\ &+ \sum_{k,l=1}^{\infty}\hat{q}_{kl}\cos\left(\frac{\pi k}{A}x\right)\cos\left(\frac{\pi l}{B}y\right). \end{aligned} \tag{66}$$

where values of q_{kl} are obtained from the analytical expressions analogical to (46).

Data collected during the physical experiment were smoothed. This was done in two steps:

- smoothing of values obtained by wrong measured pixels: if the value at some point exceeded by more than 15 % of the average value of the neighboring points, the value at this point was taken as the average value at neighboring points;
- smoothing using the moving average of 25 points, the extreme values were smoothed by 9 and 3 points respectively.

Since the solution of the Cauchy problem (61)-(64) is unstable, the Fourier series (66) was taken finite. The number of elements of series N_x and N_y was defined from the condition

$$\int_0^A\int_0^B \left(T_0(x,y) - \mathcal{F}[T_0]\right)^2 dxdy = \delta^2,$$

where $\mathcal{F}_N[T_0]$ is the finite Fourier series for system of functions (65), δ is estimated measurement error of the function $T_0(x,y)$, the Tikhonov regularization [17] was used for the summing.

The numerical solution was tested by solving the problem (61)-(64) using the optimization method [3, 4].

Two results of calculation series are shown on Fig. 2 (Fig. 2 - (a, b) – one drop, Fig. 2-(c,d) — two drops).

6 Conclusions

The essence of the reformulation of an inverse problem statement in solving it using the optimization method was demonstrated on four models of inverse problems as follows:

- coefficient hyperbolic inverse problem;
- retrospective inverse problem of heat conduction;
- the Cauchy problem for the Laplace equation that can be reduced to the inverse problem of search for the unknown boundary condition;
- problem of determining the right side of the elliptic equation.

It is shown, that when the inverse problem is the coefficient problem (the first and second example), then the use of parallel computing can reduce almost half of the time of the numerical solution calculating of the inverse problem as solving direct and

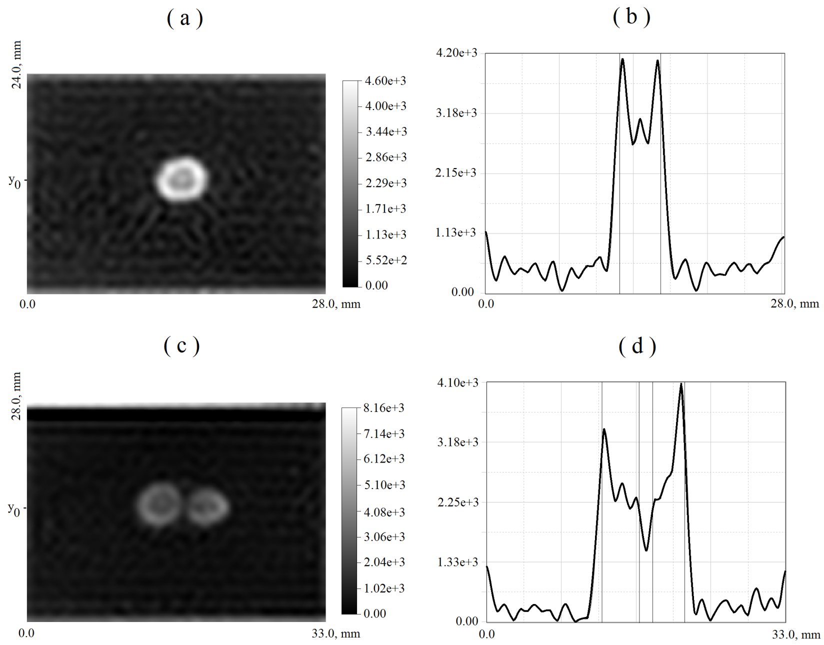

Figure 2: Calculated heat flux. Left: the flow values $q(x, y, z = h)$ are given, on Oy axis coordinate y_0 is marked; right: the flow values $q(x, y = y_0, z = h)$ are given, the vertical lines indicate the boundaries of the drop. Parameters: $h = 25 \cdot 10^{-6}$ m, $\lambda = 23$ W/(m Љ), $q_V = 2.09 \cdot 10^{12}$ W/m^3, (a)-(b) $\alpha = 16.2$ W/(m^{2}K), $T_\infty = 28.75\ ^oC$; (c)-(d) $\alpha = 16.0$ W/(m^{2}K), $T_\infty = 28.8\ ^oC$.

conjugate problems can be searched in a parallel way. If the inverse problem is linear (as it is in third example about search for an unknown boundary condition, and in the fourth example about search for the right side of an equation), then the inverse problem can be reduced to the numerical solution of the moment problem, for which all the necessary functions can be computed in advance by the known data of direct and inverse problems.

To illustrate the proposed approach the numerical solution of the Cauchy problem for an elliptic equation on data obtained from the physical experiment is presented.

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