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INVERSION OF SOME SPHERICAL CAP RADON TRANSFORMS

T. T. Truong

Abstract Radon transforms on spheres have contributed to the field of partial differential equations as solutions to Darboux equation's and, under prescribed conditions, have served as imaging principles for Sonar, Radar, photo-acoustic imaging and the like. In this paper the inverses of Radon transforms defined on three classes of spherical caps (surfaces with boundary) are derived. Through the so-called Harmonic Component Decomposition of functions and the application of Funk-Hecke's theorem, the inversion procedure follows a route established long ago for the Radon transform on spheres intersecting a fixed point. These results may open the way to new three-dimensional imaging processes, based on ionizing radiation properties

Key words: Radon Transforms, Gegenbauer Transforms, Integral Geometry, Three-dimensional Imaging

AMS Mathematics Subject Classification: 44A12, 45Q05, 65R10, 65R32

1 Introduction

In his seminal paper of 1917 [1], J Radon had posed the problem of reconstructing a function from the knowledge of its integrals on a sufficiently dense set $\mathcal{M}$ of smooth sub-manifolds of $\mathbb{R}^n$. When $\mathcal{M}$ is the set of planes in $\mathbb{R}^n$, J Radon had given an explicit solution. The extension of Radon's problem to more general sets of sub-manifolds has been a tantalizing problem over the years. Such generalized Radon transforms originate generally either from measurements of physical quantities or from pure mathematical considerations. In transport theory, conservation laws in particle binary collisions imply the appearance of generalized Radon transform on arbitrary surfaces in parameter space [2]. Seismic imaging makes use of Radon transforms on isochronic surfaces [3] (see also [4]). In 1987 A M Cormack had formulated the problem of defining a Radon transform on a special class of surfaces displaying rotational symmetry about an axis, unfortunately he could find an explicit inversion only in the cases of paraboloids and cardioids of revolution [5]. In the end, it appears that the set of spheres emerges as the most interesting surfaces for theoretical and practical purposes.

Radon transforms on $\mathbb{R}^3$-spheres have appeared in numerous applied fields such as Synthetic Aperture Radar (SAR) [6, 7], Sound Navigation and Ranging (SONAR) [8], diffraction tomography [9] and more recently thermo-acoustic imaging [10, 11]. As a sphere in $\mathbb{R}^n$ is specified by $(n+1)$ parameters (the coordinates of its center and its radius), particularized classes of spheres may be obtained by imposing one constraint on its $(n+1)$ parameters, then corresponding Radon transforms are precisely functions of n variables. Known cases are spheres centered on a plane [12, 13, 14], spheres tangent to a surface [15][1], spheres centered on a given sphere [10, 16] and spheres passing through a fixed point [9, 17]. It is remarkable to observe that not all classes of spheres admit an inverse formula for the corresponding Radon transform which is independent of n. Details can be found in excellent review articles, see *e.g.* chapter 7 of [18].

In various applications, collected data is generally expressed as Radon transform on spheres of a physical density function. This is due to measurements of wave energy resulting from propagation and interaction with a traversed medium. But in other instances, the measurement is limited to spherical caps which are portions of the surface of a sphere, as advocated in gamma-ray astrophysics by [19]. Now the main problem consists of recovering this physical density function from the set of its integrals on spheres or on spherical caps. This is an inverse problem, highly strategic for imaging systems in medicine, in seismic, in astrophysics, in non-destructive material testing, etc, which need to be solved. In mathematics, spherical means constitute a special topic in Gelfand's integral geometry [20] and an efficient tool for investigating partial differential equations [21].

On the other hand, in a large number of articles dealing with this subject, the name of spherical Radon transforms (or spherical means) occurs frequently. However this denomination is not universal since it has been also used by some authors to designate the Funk-Minkowski transform [22], which integrates functions on a sphere along its great circles [2]. Therefore, in this paper, to avoid possible confusion, by Radon transforms on spherical caps we mean transforms that integrate functions on a spherical cap in its natural measure - without dividing by the value of its area.

To get an intuitive picture, let us consider first spherical caps in $\mathbb{R}^3$. Geometrically spherical caps are areas on the surface of a sphere bounded by a *circular rim*, having a rotational symmetry axis which goes through the coordinate system origin O. Three interrelated classes of such spherical caps will be described. They are classified by their position with respect to a *reference* sphere (Γ_q) of radius q and centered at O as follows:

- (Σ)-spherical caps: they are located on spheres passing through the origin of coordinates O and lying away from (or externally to) (Γ_q),
- $(\Sigma_{(1,\epsilon)})$-spherical caps: they are on spheres *intersecting* (Γ_q) along one of its great circles and lying either inside (Γ_q) $(\epsilon = -1)$ or outside (Γ_q) $(\epsilon = +1)$,
- $(\Sigma_{(2,\epsilon)})$-spherical caps: they are on spheres *orthogonal* to (Γ_q) [3]. $\epsilon = -1$ (resp. $\epsilon = +1$) labels the cap inside (resp. outside) (Γ_q).

The intersection of (Γ_q) with any of the three classes of spheres is a circular rim (C_q), contained in a plane (P_q) which is orthogonal to the rotational symmetry axis of any of the three classes of spherical caps. The (Σ)- and the $(\Sigma_{(1,\epsilon)})$-spherical caps are situated away from (P_q), whereas the $(\Sigma_{(2,\epsilon)})$-spherical caps are each one (for given ϵ) on either side of (P_q).

[1] cited by D Finch and Rakesh in [16]

[2] see *e.g.* [23, 24, 25]. This may cause some confusion and has been pointed out in [26]. Also it is often said that this transform is "diagonalized" by spherical harmonics, *e.g.* see page 297 of [25].

[3] In $\mathbb{R}^3$-geometry, these spheres form a *complex* of spheres such that O is their *radical* point [27].

We observe that for $q \to 0$, the limit of

- the (Σ)-spherical cap or
- the $(\Sigma_{(1,+1)})$-spherical cap or
- the $(\Sigma_{(2,+1)})$-spherical cap

is a sphere intersecting the coordinate origin O, since their circular rim (C_q) shrinks to the point O. Moreover, for given j the spherical caps $(\Sigma_{(j,\epsilon)})$ are geometrically inverse of each other in the inversion of center O and modulus q, see subsection 3.1. These observations may be then appropriately extended to $\mathbb{R}^n$.

Off-hand it may look curious why such fancy spherical caps $(\Sigma_{(j,\epsilon)})$ come about in defining new Radon transforms. This is because they are natural higher dimensional extension of new arcs of circles in $\mathbb{R}^2$, on which Radon transforms are found to be invertible [28]. The idea of treating Radon transforms on such spherical caps stems from the work of A M Cormack [5], in which he found a class of particular surfaces (which generalize a class of his special curves in $\mathbb{R}^2$), for which the Radon problem, through spherical harmonic component decomposition of functions, is converted into the inversion of a special Gegenbauer integral transform [29]. However an explicit inversion for the spherical components of the sought function has only been found for paraboloids (or cardioids) of revolution with freely swinging rotational symmetry axis around the origin of coordinates. For this last case a global closed form for the reconstructed function in $\mathbb{R}^3$ has been given later in [30]. In 1992 A Kurusa showed the invertibility of the Radon transform on abstract rotational manifolds of real type, which generalizes the Cormack's result without giving explicit inversion formulas [31]. Kurusa established his results especially for sub-manifolds obtained by rotating a geodesic around an orthogonal geodesic joining its closest point to a base point and also for the case of spaces of constant curvature. Because of the assumed rotational symmetry, both A M Cormack and A Kurusa have used the spherical harmonics decomposition of functions to achieve their goal.

In this paper, following the pioneering works [5, 31], we derive explicit inversion formulas for the $\Sigma_{(j,\epsilon)}$-spherical cap Radon transforms. Also as a by-product of the function spherical component decomposition and as a result of Funk-Hecke theorem, we show that the inversion procedure in this case can be reduced to the inversion procedure of a standard Radon transform on spheres going through the coordinate system origin [17]. Therefore explicit formulas can be written down for all n. However these results do not necessarily imply that Radon transforms on whole spheres $(\Sigma_j) = (\Sigma_{(j,+)}) \bigcup (\Sigma_{(j,-)})$ are automatically invertible with the same approach.

The paper is organized as follows. In view of the set objective, section 2 is devoted to reviewing the Radon transform on (Σ)-spherical caps, as worked out in [5, 17, 32, 33]. The next section 3 describes the crucial steps which convert the expression of the Radon transforms on $(\Sigma_{(j,\epsilon)})$ with $(j = 1, 2)$ into the form of the Radon transform on (Σ)-spherical caps. In section 4, we discuss possible three-dimensional imaging processes using the Compton effect undergone by ionizing radiation in matter. A short conclusion summarizes the results and suggests some future research perspectives.

Notations and assumptions

Throughout the text, we shall work with functions $f(\mathbf{r})$ in $\mathcal{S}(\mathbb{R}^n)$. These Schwartz functional spaces do have such nice properties which turn out to be very convenient when it comes to make changes in the order of integration and to perform repeated integration by parts [34]. Incidentally this assumption has been adopted by A M Cormack in [17, 35].

As the symmetry axis of these spherical caps is assumed to swing freely around the coordinate origin O, it is appropriate to work with a spherical coordinate system centered at O. Specifically we shall write $f(\mathbf{r}) = f(r\mathbf{k}) = f(r, \Omega_{\mathbf{k}})$ with $r \in \mathbb{R}_+$ and $\Omega_{\mathbf{k}}$ is the set of angular coordinates of the unit vector $\mathbf{k} \in \mathbb{S}^{n-1}$. Moreover we shall make use of the expansion of $f(r, \Omega_{\mathbf{k}}) \in \mathcal{S}(\mathbb{R}^n)$ in $\mathbb{R}^n$-spherical harmonics $S_{lm}(\Omega_{\mathbf{k}})$ (see [36])

$$f(r, \Omega_{\mathbf{k}}) = \sum_{lm} f_{lm}(r)\, S_{lm}(\Omega_{\mathbf{k}}). \tag{1.1}$$

Let $\lambda = (n-2)/2$ and for given $l \in \mathbb{N}$, there are $h(\lambda, l)$ spherical harmonic functions $S_{lm}(\Omega_{\mathbf{k}})$, see [36] with

$$h(\lambda, l) = \frac{2(l+\lambda)\,\Gamma(l+2\lambda)}{\Gamma(1+l)\Gamma(1+2\lambda)}. \tag{1.2}$$

Then we have also $f_{lm}(r) \in \mathcal{S}(\mathbb{R}_+)$.

2 The Radon transform on (Σ)-spherical caps

In this section, the Radon transform on(Σ)-spherical caps, its properties and inversion formula are presented. It will serve as a basic framework for deriving properties and inversion formulas for the Radon transform on the $(\Sigma_{(j,\epsilon)})$-spherical caps. Thus whatever property known or established for this case can be smoothly transferred to the new cases.

2.1 Definition

For a sphere of diameter p, such that the coordinate origin O lies on its surface, its center Ω is specified by the vector $(p/2)\,\mathbf{n}$, with $\mathbf{n} \in \mathbb{S}^{n-1}$ and $O\Omega = p/2$. In a spherical coordinate system centered at O, its equation is $r = p\,(\mathbf{k}\cdot\mathbf{n})$. For $0 < q < p$, this sphere intersects (Γ_q) and the part of this sphere away from (Γ_q) (*i.e.* not containing O), is called a Σ-spherical cap. The normalized delta function concentrated on this (Σ)-spherical cap, with respect to the volume element is $r^{n-1}\, dr\, d\Omega_{\mathbf{k}}$, is given by

$$\delta(\Sigma) = \frac{p}{r}\,\delta(r - p\,(\mathbf{k}\cdot\mathbf{n})), \tag{2.1}$$

where $d\Omega_{\mathbf{k}}$ is the measure on the unit sphere and $q < r < p$, see eq. (A2) in Appendix A of [5]. Note that $(\mathbf{k}\cdot\mathbf{n})$ is the cosine of an angle and consequently $0 < r < p$.

Definition 2.1. *Let $f(\mathbf{r}) = f(r, \Omega_{\mathbf{k}}) \in \mathcal{S}(\mathbb{R}^n)$. Its Radon transform on (Σ)-spherical caps $\widehat{f}(p, \Omega_{\mathbf{n}})$, for $p > q$, is given by*

$$\widehat{f}(p, \Omega_{\mathbf{n}}) = \int_{\mathbb{R}^n} r^{n-1}\, dr\, d\Omega_{\mathbf{k}}\, \frac{p}{r}\,\delta(r - p(\mathbf{k}\cdot\mathbf{n}))\; f(r, \Omega_{\mathbf{k}}). \tag{2.2}$$

Evidently for $p < q$, $\widehat{f}(p, \Omega_{\mathbf{n}}) = 0$ [4]. Note that, since both r and p are positive, the equation of the Σ-sphere implies that $\mathbf{k}$ can be at most orthogonal to $\mathbf{n}$.

2.2 The integral transform for function spherical components

Proposition 2.2. *Let $f_{lm}(r)$ (resp. $\widehat{f}_{lm}(p)$ be the spherical components of $f(r, \Omega_{\mathbf{r}})$ (resp. $\widehat{f}(p, \Omega_{\mathbf{n}})$. Then they are related by the following Gegenbauer integral transform*

$$\widehat{f}_{lm}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_q^p dr\; C_l^\lambda(r/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda - 1/2} r^{2\lambda}\, f_{lm}(r), \tag{2.3}$$

Proof. When the spherical component decomposition (1.1) of $f(r, \Omega\mathbf{k})$ is inserted in eq. (2.2), $\widehat{f}(p, \Omega_{\mathbf{n}})$ can be written as an (lm)-sum of terms $\widehat{f}_{lm}(p, \Omega_{\mathbf{n}})$ of the form

$$\widehat{f}_{lm}(p, \Omega_{\mathbf{n}}) = \int_{\mathbb{R}^n} r^{n-1}\, dr\, d\Omega_{\mathbf{k}}\, \frac{p}{r}\, \delta(r - p(\mathbf{k}\cdot\mathbf{n}))\; f_{lm}(r)\, S_{lm}(\Omega_{\mathbf{k}}). \tag{2.4}$$

After performing the r-integration, eq. (2.4) can now be written as

$$\widehat{f}_{lm}(p, \Omega_{\mathbf{n}}) = \int_{\mathbb{S}^{n-1}} d\Omega_{\mathbf{k}}\, S_{lm}(\Omega_{\mathbf{k}})\, G_{lm}^{2\lambda}((\mathbf{k}\cdot\mathbf{n})), \tag{2.5}$$

with

$$G_{lm}^{2\lambda}((\mathbf{k}\cdot\mathbf{n})) = p\,(p(\mathbf{k}\cdot\mathbf{n}))^{2\lambda}\, f_{lm}(p(\mathbf{k}\cdot\mathbf{n})). \tag{2.6}$$

As $G_{lm}^{2\lambda}((\mathbf{k}\cdot\mathbf{n}))$ is solely a function of the inner product of two unit vectors in $\mathbb{S}^{n-1}$, the Funk-Hecke theorem [36] gives the value of the $\Omega_{\mathbf{k}}$-integral in eq. (2.5) as $\widehat{f}_{lm}(p, \Omega_{\mathbf{n}}) = \beta_{lm}^{2\lambda}(p)\, S_{lm}(\Omega_{\mathbf{n}})$, where $\beta_{lm}^{2\lambda}(p)$ is

$$\beta_{lm}^{2\lambda}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_{(q/p)}^1 dt\; C_l^\lambda(t) \left(1 - t^2\right)^{\lambda - 1/2} p\,(p\,t)^{2\lambda}\, f_{lm}(p\,t), \tag{2.7}$$

where $\omega_{n-2} = 2\pi^{\frac{n-1}{2}}/\Gamma(\frac{n-1}{2})$ is the area of the unit sphere $\mathbb{S}^{n-2}$ and $C_l^\lambda(t)$ the (l, λ)-Gegenbauer polynomial of variable t. Thus we can identify $\beta_{lm}^{2\lambda}(p)$ as the spherical component $\widehat{f}_{lm}(p)$ of the Radon transform $\widehat{f}(p, \Omega_{\mathbf{n}})$. The integral equation connecting the spherical component of f to the spherical component of $\widehat{f}$ is therefore

$$\widehat{f}_{lm}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_q^p dr\; C_l^\lambda(r/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda - 1/2} r^{2\lambda}\, f_{lm}(r), \tag{2.8}$$

when rewritten as an integral on r. This integral is well-defined for $f_{lm}(r) \in \mathcal{S}(\mathbb{R}_+)$ and eq. (2.8) represents a Gegenbauer transform considered in [37, 38] [5].

[4] In a sense this Radon problem for $q > 0$ is akin to the exterior Radon problem on hyper-planes lying outside a sphere (Γ_q) see *e.g.* Ludwig [29].

[5] Because the integral kernel depends only on l, the index m is generally dropped, see *e.g.* eq. (2.8) [17, 32]

When $p \to \infty$ inspection of eq. (2.8) shows that

$$\lim_{p\to\infty} \widehat{f}_{lm}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)}\, C_l^0(0) \int_q^\infty dr\, r^{2\lambda}\, f_{lm}(r),$$

the last integral being finite for $f_{lm}(r) \in \mathcal{S}(\mathbb{R}_+)$ and

$$\begin{aligned} C_l^0(0) &= 0 \quad \text{for} \quad l \text{ odd} \\ &= (-1)^l\, \frac{\Gamma(\lambda + l/2)}{\Gamma(\lambda)\Gamma(1+l/2)} \quad \text{for} \quad l \text{ even} \end{aligned}$$

However the Radon data depends implicitly on q, the lower bound of r, as can be seen from the q-derivative of $\widehat{f}_{lm}(p)$

$$\frac{d}{dq}\, \widehat{f}_{lm}(p) = -\frac{\omega_{n-2}}{C_l^\lambda(1)}\, C_l^\lambda(q/p) \left(1 - \frac{q^2}{p^2}\right)^{\lambda-1/2} q^{2\lambda}\, f_{lm}(q). \tag{2.9}$$

2.3 Null space for $q = 0$

For $q \neq 0$, the existence and the nature of the null space of this spherical cap Radon transform remains open for the time being. However for $q = 0$, there exists a result due to E T Quinto in [32] for spherical means which can be cited as follows:

Theorem 2.3. *Let B be a ball centered at the origin of $\mathbb{R}^n$. The null space of the Radon transform on the (Σ)-spheres for $\mathcal{L}^2(B)$ functions is the closure of the span of functions of the form $r^{k+2-n}\, S_{lm}(\Omega_{\mathbf{k}})$, where $(n-4)/2 < k < l$, with $(l-k)$ even and $S_{lm}(\Omega_{\mathbf{k}})$, a homogeneous spherical harmonic of degree l. The map $f \longmapsto \widehat{f}$ in $\mathcal{L}^2(B) \to \mathcal{L}^2(B)$ is one to one.*

This means that for functions such as $f_{lm}(r) = r^k$, where $k \in \mathbb{N}$

$$\widehat{(r^k)}_{lm}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_0^p dr\; C_l^\lambda(r/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda-1/2} r^{2\lambda}\, r^k = 0, \tag{2.10}$$

if n is even (resp. odd) then k is even (resp. odd); l is even and $l > k + n - 2$.

2.4 Consistency conditions on the Radon data

As it is current in the approach by spherical components, one may ask which values of $k \in \mathbb{N}$ annihilates the integral

$$\int_q^\infty dp\; p^{-k}\; \widehat{f}_{lm}(p) = 0. \tag{2.11}$$

Proposition 2.4. *For $\widehat{f}_{lm}(p, \Omega_{\mathbf{n}})$ given by eq. (2.8), the integral (2.11) vanishes for $k = 2, 3, ..., l, (l+1)$.*

Proof. To compute these values of k, we use (2.8) and Fubini's theorem to get

$$\int_q^\infty dp\, p^{-k}\, \widehat{f}_{lm}(p) = \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_q^\infty dr\, r^{2\lambda}\, f_{lm}(r) \int_r^\infty dp\, p^{-k}\, C_l^\lambda(r/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda-1/2}, \tag{2.12}$$

which becomes after a change of variable

$$\int_q^\infty dp\ p^{-k}\, \widehat{f}_{lm}(p) = \frac{\omega_{2\lambda}}{C_l^\lambda(1)} \int_q^\infty dr\, r^{2\lambda}\, f_{lm}(r)\ r^{1-k} \int_0^1 dt\, t^{k-2}\, C_l^\lambda(t) \left(1 - t^2\right)^{\lambda-1/2}. \tag{2.13}$$

The value of the t-integral in (2.13) can be found in [39] as

$$\int_0^1 dt\, t^{k-2}\, C_l^\lambda(t) \left(1-t^2\right)^{\lambda-1/2} = \frac{\Gamma(2\lambda+l)}{\Gamma(2\lambda)} \frac{\Gamma(\lambda+1/2)}{2^{l+1}\, l!} \frac{\Gamma((k-l-1)/2)}{\Gamma(\lambda+(k+l)/2)} \frac{\Gamma(k-1)}{\Gamma(k-l-1)}. \tag{2.14}$$

The two first fractions on the right-hand-side do not depend on k, the third fraction in general has no zeros. The last fraction is

$$\frac{\Gamma(k-1)}{\Gamma(k-l-1)} = (k-2)(k-3)...(k-l)(k-l-1) \tag{2.15}$$

Hence the integral (2.11) vanishes only for $k = 2, 3, ..., l, (l+1)$ (l values of k), which is expected from the orthogonality of the Gegenbauer polynomials. □

2.5 Inversion formula

Before starting the inversion procedure, we prove the lemma

Lemma 2.5. *For $g(s) \in \mathcal{C}^\infty(\mathbb{R})$, we have*

$$\left(\frac{d}{dt}\right)^{m+1} \int_0^t ds\ g(s)\ (t-s)^m = m!\ g(t). \tag{2.16}$$

Proof. For $m = 1$, this is explicitly verified

$$\left(\frac{d}{dt}\right)^2 \int_0^t ds\ g(s)\ (t-s) = \frac{d}{dt} \int_0^t ds\ g(s) = 1!\ g(t) \tag{2.17}$$

For $m > 1$, as $(t-s)^m = \sum_{l=0}^m \begin{pmatrix} m \\ l \end{pmatrix} (-s)^l t^{m-l}$ the left-hand-side of eq. (2.16) is

$$\left(\frac{d}{dt}\right)^{m+1} \int_0^t ds\ g(s)\ (t-s)^m = \left(\frac{d}{dt}\right)^m \sum_{l=0}^m \begin{pmatrix} m \\ l \end{pmatrix} (-1)^l (m-l) t^{m-l-1} \int_0^t ds\ g(s)\ s^l,$$

(2.18)

since the second term from the d/dt is $t^m g(t) \sum_{l=0}^{m} \binom{m}{l} (-1)^l = 0$. By rearranging the binomial expansion coefficient and factorizing m, the rest is just the binomial expansion coefficient of $(t-s)^{m-1}$. Hence

$$\left(\frac{d}{dt}\right)^{m+1} \int_0^t ds\; g(s)\; (t-s)^m = m \left(\frac{d}{dt}\right)^{m} \int_0^t ds\; g(s)\; (t-s)^{m-1} = m\,(m-1)!\; g(t), \tag{2.19}$$

by recursion. □

Theorem 2.6. *The spherical component* $f_{lm}(r)$ *of* $f(r, \Omega_{\mathbf{k}}) \in \mathcal{S}(\mathbb{R}^n)$ *is recovered via the spherical component* $\widehat{f}_{lm}(p)$ *of* $\widehat{f}(p, \Omega_{\mathbf{n}}) \in \mathcal{S}(\mathbb{R}^n)$ by

$$f_{lm}(r) = \frac{1}{K(\lambda, l)}\; \frac{1}{r^{2\lambda}} \left(\frac{d}{dr}\right)^{2\lambda+1} r^{2\lambda-1} \int_q^r dp\; C_l^\lambda(r/p) \left(1 - \frac{p^2}{r^2}\right)^{\lambda-1/2} \widehat{f}_{lm}(p). \tag{2.20}$$

where

$$K(\lambda, l) = \frac{\omega_{n-2}}{C_l^\lambda(1)}\; \frac{\pi}{2^{2\lambda-1}} \left\{\frac{\Gamma(l+2\lambda)}{\Gamma(\lambda)\,\Gamma(1+l)}\right\}^2. \tag{2.21}$$

Moreover the reconstructed $f(r, \Omega_{\mathbf{k}}) \in \mathbb{R}^n$ *does not depend on* q*.*

Proof. Eq. (2.8) is a Gegenbauer transform which can be inverted in a standard way. To this end we multiply on both sides of eq. (2.8) by $C_l^\lambda(s/p)((s/p)^2-1)^{\lambda-1/2}\,p^{2\lambda-1}$ and integrate over p from q to s

$$\int_q^s dp\; C_l^\lambda(s/p) \left(\frac{s^2}{p^2} - 1\right)^{\lambda-1/2} p^{2\lambda-1}\, \widehat{f}_{lm}(p) =$$

$$\int_q^s dp\; C_l^\lambda(s/p) \left(\frac{s^2}{p^2} - 1\right)^{\lambda-1/2} p^{2\lambda-1}\, \frac{\omega_{n-2}}{C_l^\lambda(1)} \int_q^p dr\; C_l^\lambda(r/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda-1/2} r^{2\lambda}\, f_{lm}(r), \tag{2.22}$$

the right-hand-side becomes after exchange of integration order

$$\frac{\omega_{n-2}}{C_l^\lambda(1)} \int_q^s dr\; r^{2\lambda}\, f_{lm}(r) \int_r^s dp\, p^{2\lambda-1}\, C_l^\lambda(r/p) C_l^\lambda(s/p) \left(1 - \frac{r^2}{p^2}\right)^{\lambda-1/2} \left(\frac{s^2}{p^2} - 1\right)^{\lambda-1/2}. \tag{2.23}$$

The p-integral can be exactly evaluated (formula (30) in [37]) and is equal to

$$\frac{\pi}{2^{2\lambda-1}} \left(\frac{\Gamma(l+2\lambda)}{\Gamma(l+1)\Gamma(\lambda)}\right)^2 \frac{(s-r)^{2\lambda}}{\Gamma(2\lambda+1)}. \tag{2.24}$$

Then we get

$$\frac{K(\lambda,l)}{\Gamma(1+2\lambda)}\int_q^s dr\,(s-r)^{2\lambda}\,r^{2\lambda}\,f_{lm}(r) = s^{2\lambda-1}\int_q^s dp\;C_l^\lambda(s/p)\left(1-\frac{p^2}{s^2}\right)^{\lambda-1/2}\widehat{f}_{lm}(p). \tag{2.25}$$

To extract $f_{lm}(r)$ from the above eq. (3.2), we use the lemma 2.5.1.

Noting that $2\lambda = (n-2)$ is an integer since $n \in \mathbb{N}$, application of formula (2.16) leads to the reconstruction formula for the spherical components $f_{lm}(r)$

$$f_{lm}(r) = \frac{1}{K(\lambda,l)}\,\frac{1}{r^{2\lambda}}\left(\frac{d}{dr}\right)^{2\lambda+1} r^{2\lambda-1}\int_q^r dp\;C_l^\lambda(r/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2}\widehat{f}_{lm}(p). \tag{2.26}$$

Thus $f_{lm}(r)$ is well-defined for all $q<p<r$, since $\widehat{f}_{lm}(p)$ is of the form $\widehat{f}_{lm}(p) = p^{2\lambda+l+1}\times h_{lm}^\lambda(p)$ and reconstructed with the Radon data in $q<p<r$. At this point there is no need to use the consistency conditions.

We now show that the reconstructed $f_{lm}(r)$ does no depend on q. For this we compute first the q-derivative of the p-integral

$$\frac{d}{dq}\int_q^r dp\;C_l^\lambda(r/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2}\widehat{f}_{lm}(p) =$$

$$-\,C_l^\lambda(q/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2}\widehat{f}_{lm}(q) + \int_q^r dp\;C_l^\lambda(r/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2}\frac{d}{dq}\,\widehat{f}_{lm}(p), \tag{2.27}$$

As $\widehat{f}_{lm}(q) = 0$, we only have to evaluate the p-integral in eq. (2.27) with the help of eq. (2.9) and the result of eq. (2.24)

$$-\frac{\omega_{n-2}}{C_l^\lambda(1)}q^{2\lambda}\,f_{lm}(q)\int_q^r dp\;C_l^\lambda(r/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2}C_l^\lambda(q/p)\left(1-\frac{q^2}{p^2}\right)^{\lambda-1/2}, \tag{2.28}$$

which is

$$-\frac{\omega_{n-2}}{C_l^\lambda(1)}\,q^{2\lambda}\,f_{lm}(q)\,\frac{1}{r^{2\lambda-1}}\,\frac{\pi}{2^{2\lambda-1}}\left(\frac{\Gamma(l+2\lambda)}{\Gamma(l+1)\Gamma(\lambda)}\right)^2\frac{(r-q)^{2\lambda}}{\Gamma(2\lambda+1)}. \tag{2.29}$$

Thus to recover $f_{lm}(r)$, one should apply the differential operator (see eq. (2.24)

$$\left(\frac{d}{dr}\right)^{2\lambda+1} r^{2\lambda-1}$$

to the previous result (2.29), which is evidently 0 since it is the $(n-1)$th r-derivative of an r-polynomial $(r-q)^{n-2}$ of order $(n-2)$ in r. Hence the reconstruction of the spherical components of $f(r,\Omega_{\mathbf{k}})$ is independent of q. □

2.6 An alternative form of the inversion formula

Because eq. (2.26) may not be numerically stable for $q = 0$, an alternative inversion formula can be derived following the approach proposed by A M Cormack in [40]. We observe that the p-integral in (2.26) can be transformed in the following way to avoid a possible divergence of the integrand for $p = q \to 0$, when $\widehat{f}_{lm}(p)$ decrease fast enough.

Proposition 2.7. *The product of $C_l^\lambda(r/p)$ with $(1-p^2/r^2)^{\lambda-1/2}$ can be represented as the sum of a polynomial and of a convergent power series for $p < r$*

$$C_l^\lambda(r/p)\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2} = N_l(r/p)\ \left(\mathcal{Q}_l^\lambda(r^2/p^2) + \mathcal{R}_l^\lambda(p^2/r^2)\right)$$

with

$$\begin{aligned} \mathcal{Q}_l^\lambda(r^2/p^2) &= \sum_{\kappa=0}^{[l/2]} Q_{l,\kappa}^\lambda(-1/4)\left(\frac{4r^2}{p^2}\right)^{[l/2]-\kappa} \\ \mathcal{R}_l^\lambda(p^2/r^2) &= \sum_{\kappa=1}^{\infty} R_{l,\kappa}^\lambda(-1/4)\left(\frac{p^2}{4r^2}\right)^{\kappa}, \end{aligned} \tag{2.30}$$

where $N_l(x) = 1$ if $l =$ even, and $N_l(x) = 2x$ if $l =$ odd, and

$$\begin{aligned} Q_{l,\kappa}^\lambda(z) &= 4^\kappa \sum_{j=0}^{\kappa}\left(\frac{(\lambda)_{l-\kappa-j}(-\lambda+1/2)_j}{(l-2\kappa-2j)!}\right)\frac{z^{\kappa-j}}{(\kappa-j)!}, \\ R_{l,\kappa}^\lambda(z) &= 4^{\kappa+[l/2]}\sum_{i=0}^{[l/2]}\left(\frac{(\lambda)_{l-i}(-\lambda+1/2)_{[l/2]+\kappa-i}}{(l-2i)!}\right)\frac{z^i}{i!}, \end{aligned} \tag{2.31}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$, the Pochhammer symbol for a, $[x]$ the largest integer in x.

Proof. The proof consists in using the power expansions for $p/r < 1$ of the Gegenbauer polynomial

$$C_l^\lambda(r/p) = \sum_{\kappa=0}^{[l/2]}(-1)^\kappa\ \frac{(\lambda)_{l-\kappa}}{\kappa!(l-2\kappa)!}\left(2\frac{r}{p}\right)^{l-2\kappa}, \tag{2.32}$$

and of the binomial

$$\left(1-\frac{p^2}{r^2}\right)^{\lambda-1/2} = \sum_{n=0}^{\infty}(-\lambda+1/2-n)_n\left(\frac{p^2}{r^2}\right)^n, \tag{2.33}$$

and reordering the product as an increasing power series in (p/r). $Q_{l,\kappa}^\lambda(z)$ and $R_{l,\kappa}^\lambda(z)$ are special hypergeometric polynomials of the variable z.

Eq. (2.26) can now be put under the form

$$f_{lm}(r) = \frac{1}{K(\lambda,l)}\frac{1}{r^{2\lambda}}\left(\frac{d}{dr}\right)^{2\lambda+1} r^{2\lambda-1}\int_q^r dp\, N_l(r/p)\left(\mathcal{Q}_l^\lambda(r^2/p^2)+\mathcal{R}_l^\lambda(p^2/r^2)\right)\widehat{f}_{lm}(p).$$

(2.34)

The first integral with $\mathcal{Q}_l^\lambda(r^2/p^2)$ in the integrand may be transformed after taking into account the consistency conditions (2.14,2.15) into

$$\int_q^r dp\, N_l(r/p)\, Q^\lambda_{l,[l/2]}(-1/4)\, \widehat{f}_{lm}(p). \tag{2.35}$$

But this is a constant for $l =$ even and a linear function in r for $l =$ odd. Hence upon application of $(d/dr)^{2\lambda+1} r^{2\lambda-1}$, these contributions vanish. Therefore we have a new form of the reconstruction formula for the spherical component of $f(r, \Omega_{\mathbf{k}})$, namely

$$f_{lm}(r) = \frac{1}{K(\lambda,l)} \frac{1}{r^{2\lambda}} \left(\frac{d}{dr}\right)^{2\lambda+1} r^{2\lambda-1} \times$$

$$\left[\int_q^r dp\, N_l(r/p) \mathcal{R}_l^\lambda(p^2/r^2) \widehat{f}_{lm}(p) - \int_r^\infty dp\, N_l(r/p) \mathcal{Q}_l^\lambda(r^2/p^2) \widehat{f}_{lm}(p)\right]. \tag{2.36}$$

□

- We observe that $\mathcal{R}_l^{\lambda=1/2}(p^2/r^2) = 0$ and $N_l(r/p)\mathcal{Q}_l^{\lambda=1/2}(r^2/p^2) = P_l(r/p)$.
- Up to this stage, it is not possible to write down a closed form inversion formula by reformulating the summation $f(r,\omega_{\mathbf{k}}) = \sum_{lm} f_{lm}(r)\, S_{lm}(\omega_{\mathbf{k}})$. This is due to the fact for general $n > 3$, the p-integrals in eq. (2.36) cannot be evaluated explicitly.

3 The Radon transform on the $(\Sigma_{(j,\epsilon)})$-spherical caps in $\mathbb{R}^n$

In this section we show that the Radon transform on the $(\Sigma_{(j,\epsilon)})$-spherical caps, for $\epsilon = \pm 1$ and $j = 1, 2$, can be brought down to the form of the Radon transform on (Σ)-spherical caps presented in section 2 for a new function of a new variable. We first give a description of the $(\Sigma_{(j,\epsilon)})$-spherical caps. Then the expression of the related Radon transform is introduced. Next when spherical component of functions are used, this Radon transform takes the form of a special Gegenbauer integral transform, which, amazingly through a succession of proper change of variables and functions, regains the form of eq. (2.8). Such a feat has been already noticed by A M Cormack for $\mathbb{R}^2$ in [40]. Thus upon establishment of this correspondence, it is straightforward to write down the explicit inversion formulas.

3.1 The Radon transform on $(\Sigma_{j,\epsilon})$-spherical caps with $\epsilon = \pm 1$ and $j = 1, 2$

Spherical cap's equation

In a $\mathbb{R}^n$-spherical coordinate system, where a point $\mathbf{r} = r\,\mathbf{k}$ is labeled by $(r, \Omega_{\mathbf{k}})$, ($r = |\mathbf{r}|$ and $\Omega_{\mathbf{k}}$ is the set of angles defining the unit vector $\mathbf{k} \in \mathbb{S}^{n-1}$), a $(\Sigma_{(j,\epsilon)})$-spherical cap with axis of rotational symmetry going through the coordinate system origin O in the direction of the unit vector $\mathbf{n}$ is given by

$$r_{(j,\epsilon)} = r_{(j,\epsilon)}(\tau, t) = q(\epsilon^{j-1}\sqrt{\tau^2 t^2 + (-1)^{j-1}} + \epsilon^j \tau\, t), \tag{3.1}$$

where $t = (\mathbf{n}\cdot\mathbf{k})$ and τ a real parameter. Thus the $r_{(j,\epsilon)}$ are positive for $\tau > 0$ and $0 < t < 1$ if $j = 1$ and for $t > \tau^{-1} > 1$ if $j = 2$. It can be checked that $r_{(j,+)} r_{(j,-)} = q^2$. This shows that, for given j, the spherical caps $(\Sigma_{(j,\epsilon)})$ are inverse of each other in an inversion of center O and modulus q. Moreover $r_{(j,\epsilon)}$ are the positive solutions of

$$t = \frac{\epsilon^j}{2\tau}\left(\frac{r_{(j,\epsilon)}}{q} + (-1)^j \frac{q}{r_{(j,\epsilon)}}\right). \tag{3.2}$$

Definition

Definition 3.1. *The Radon transform on a $(\Sigma_{(j,\epsilon)})$-spherical cap maps a function $f(r, \Omega_{\mathbf{k}}) \in \mathcal{S}(\mathbb{R}^n)$ onto $\widehat{f}^{(j,\epsilon)}(\tau, \Omega_{\mathbf{n}})$ given by*

$$\widehat{f}^{(j,\epsilon)}(\tau, \Omega_{\mathbf{n}}) = \int_0^\infty r^{n-1}\, dr \int_{\mathbb{S}^{n-1}} d\Omega_{\mathbf{k}}\, \delta(\Sigma_{j,\epsilon})\, f(r, \Omega_{\mathbf{k}}), \tag{3.3}$$

where $\delta(\Sigma_{(j,\epsilon)})$ is the normalized delta function on these spherical caps

$$\delta(\Sigma_{(j,\epsilon)}) = \sqrt{\frac{\tau^2 + (-1)^{j-1}}{\tau^2 t^2 + (-1)^{j-1}}}\, \delta\left(r - q(\epsilon^{j-1}\sqrt{\tau^2 t^2 + (-1)^{j-1}} + \epsilon^j \tau\, t)\right) H_j(t). \tag{3.4}$$

where

$$\begin{aligned} H_j(t) &= \delta_{j\,1} \quad \text{for} \quad t > 0 \\ &= \delta_{j\,2} \quad \text{for} \quad t > \tau^{-1}). \end{aligned} \tag{3.5}$$

The square root normalization factor is necessary so that the integral over all $\mathbb{R}^n$ of $\delta(\Sigma_{(j,\epsilon)})$ is the area ω_{n-1} of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Remark

We observe that the delta functions concentrated on spheres on which the considered spherical caps $(\Sigma_{(j,\epsilon)})$ are located can be expressed as

- $\delta(\sqrt{r^2 + \epsilon 2r\,\tau q\, \cos\gamma + \tau^2 q^2} - q\sqrt{\tau^2 + 1})$, for spheres (Σ_1^ϵ) on which $(\Sigma_{(1,\epsilon)})$ lies,
- $\delta(\sqrt{r^2 - 2r\,\tau q\, \cos\gamma + \tau^2 q^2} - q\sqrt{\tau^2 - 1})$, for spheres $(\Sigma_2) = (\Sigma_{2,-}) \bigcup (\Sigma_{2,+})$.

They may be used to define Radon transform on such spheres and not on spherical caps, as we do in this work.

3.2 The associated Gegenbauer integral transform for the spherical components

We now give successively several equivalent forms of this Radon transform which are necessary for its inversion.

Original form

The first form of the integral equation linking the spherical components $f_{lm}(r)$ and $\widehat{f}_{lm}^{(j,\epsilon)}(\tau)$ is given by

Proposition 3.2. *The Radon transform on $(\Sigma_{j,\epsilon})$-spherical caps, defined as the mapping $f(r,\Omega_k) \mapsto \widehat{f}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}})$, may be equivalently expressed as the mapping of the function spherical components: $f_{lm}(r) \mapsto \widehat{f}_{lm}^{(j,\epsilon)}(\tau)$ with*

$$\widehat{f}_{lm}^{(j,\epsilon)}(\tau) = \frac{\omega_{n-2}}{C_l^\lambda(1)}$$

$$\int_{-1}^{1} dt\, C_l^\lambda(t)\,(1-t^2)^{\lambda-1/2}\sqrt{\frac{\tau^2+(-1)^{j-1}}{\tau^2(\mathbf{n}\cdot\mathbf{k})^2+(-1)^{j-1}}}\,H_j(t)\;\left(r^{n-1}\,f_{lm}(r)\right)_{r=r_{(j,\epsilon)}(\tau,t)}, \tag{3.6}$$

where $H_j(t)$ is defined by eq. (3.5).

Proof. If $f(r,\Omega_{\mathbf{k}})$ is replaced by its decomposition in spherical components (see eq. (1.1)), then its $(\Sigma_{(j,\epsilon)})$-spherical cap appears as the sum $\widehat{f}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}}) = \sum_{lm}\widehat{f}_{lm}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}})$ where, after performing the r-integration in eq. (3.3)

$$\widehat{f}_{lm}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}}) = \int_{\mathbb{S}^{n-1}} d\Omega_{\mathbf{k}}\, S_{lm}(\Omega_{\mathbf{k}})\, G_{lm}^{(j,\epsilon)}(\tau,\mathbf{n}\cdot\mathbf{k}), \tag{3.7}$$

with

$$G_{lm}^{(j,\epsilon)}(\tau,\mathbf{n}\cdot\mathbf{k}) = \sqrt{\frac{\tau^2+(-1)^{j-1}}{\tau^2(\mathbf{n}\cdot\mathbf{k})^2+(-1)^{j-1}}}\,H_j(t)\;\left(r^{n-1}\,f_{lm}(r)\right)_{r=r_{(j,\epsilon)}(\tau,\mathbf{n}\cdot\mathbf{k})}. \tag{3.8}$$

Funk-Hecke theorem gives the value of $\widehat{f}_{lm}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}})$ in eq. (3.16). Hence we may identify $\beta_{lm}^{(j,\epsilon)}(\tau)$ with the lm-spherical component of $\widehat{f}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}})$. Transcribing the expression of $\beta_{lm}^{(j,\epsilon)}(\tau)$ given in [36], we obtain an integral equation

$$\widehat{f}_{lm}^{(j,\epsilon)}(\tau) = \frac{\omega_{n-2}}{C_l^\lambda(1)}\int_{-1}^{1} dt\, C_l^\lambda(t)\,(1-t^2)^{\lambda-1/2}\, G_{lm}^{(j,\epsilon)}(\tau,t), \tag{3.9}$$

linking the spherical components. Then eq. (3.6) is obtained after replacing $G_{lm}^{(j,\epsilon)}(\tau,t)$ by its expression as given in eq. (3.8). This is also a Gegenbauer-type integral equation. □

Second form of eq. (3.6) using the $r_{(j,\epsilon)}$-variable

We now go through a few steps to bring eq. (3.9) to the form of eq. (2.8). Then the inversion procedure of the Σ-spherical cap Radon transform can be applied to obtain the inversion of the $(\Sigma_{(j,\epsilon)})$-spherical cap Radon transforms.

Proposition 3.3. *Eq. (3.6) can be alternatively reformulated as*

$$\widehat{f}_{lm}^{(j,\epsilon)}(\tau) = \frac{\omega_{n-2}}{C_l^\lambda(1)}\,\epsilon\,\frac{\sqrt{\tau^2+(-1)^{j-1}}}{\tau}\int_q^{q(\epsilon^{j-1}\sqrt{\tau^2+(-1)^{j-1}}+\epsilon^j\tau)}\frac{dr_{(j,\epsilon)}}{r_{(j,\epsilon)}}\times$$
$$C_l^\lambda\left(\frac{\epsilon^j}{2\tau}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)\right)\left(1-\frac{1}{4\tau^2}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)\right)^{\lambda-1/2} r_{(j,\epsilon)}^{n-1} f_{lm}(r_{(j,\epsilon)}). \tag{3.10}$$

Proof. To this end we switch back to the variable $r_{(j,\epsilon)}$ as integration variable. From eq. (3.1), one may establish that

$$\frac{dr_{(j,\epsilon)}}{r_{(j,\epsilon)}} = \epsilon\tau\frac{dt}{\sqrt{\tau^2t^2+(-1)^{j-1}}}. \tag{3.11}$$

The factor $H_j(t)$ fixes the lower t-integration bound to be $t_{0j} = (j-1)\delta_{j1}+\tau^{-1}\delta_{j2}$, the corresponding values of $r_{(j,\epsilon)}(t_{0j}) = q$. The upper bound of the t-integration is always 1, this implies that the $r_{(j,\epsilon)}(1)$-upper bound is $r_{(j,\epsilon)}(1) = q(\epsilon^{j-1}\sqrt{\tau^2+(-1)^{j-1}}+\epsilon^j\tau)$.

The integral equation (3.6) takes now the form of eq. (3.10). □

Third form of eq. (3.6) using the $s_{(j,\epsilon)}$-variable

We now perform another change of variables in order to have

Proposition 3.4. *An alternative way to express the Gegenbauer transform $\widehat{f}_{lm}^{(j,\epsilon)}(\tau)$ of $f_{lm}(r)$ is*

$$\frac{\tau\,\widehat{f}_{lm}^{(j,\epsilon)}(\tau)}{\sqrt{\tau^2+(-1)^{j-1}}} = \frac{\omega_{n-2}}{C_l^\lambda(1)}\,\epsilon^2\int_{(j-1)}^{\tau}\frac{ds_{(j,\epsilon)}}{\sqrt{s_{(j,\epsilon)}^2+(-1)^{j-1}}}$$
$$C_l^\lambda(s_{(j,\epsilon)}/\tau)\left(1-\frac{s_{(j,\epsilon)}^2}{\tau^2}\right)^{\lambda-1/2}\left(r_{(j,\epsilon)}^{n-1}f_{lm}(r_{(j,\epsilon)})\right)_{r_{(j,\epsilon)}=q\left(\epsilon^{j-1}\sqrt{s_{(j,\epsilon)}^2+(-1)^{j-1}}+\epsilon^j s_{(j,\epsilon)}\right)}. \tag{3.12}$$

Proof. Let

$$s_{(j,\epsilon)} = \frac{\epsilon^j}{2}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right), \tag{3.13}$$

or conversely $r_{(j,\epsilon)}$ may be solved in terms of $s_{(j,\epsilon)}$ as

$$r_{(j,\epsilon)} = q\left(\epsilon^{j-1}\sqrt{s_{(j,\epsilon)}^2+(-1)^{j-1}}+\epsilon^j s_{(j,\epsilon)}\right). \tag{3.14}$$

Then a relation between the differentials $ds_{(j,\epsilon)}$ and $dr_{(j,\epsilon)}$ may be found

$$\frac{dr_{(j,\epsilon)}}{r_{(j,\epsilon)}} = \epsilon\,\frac{ds_{(j,\epsilon)}}{\sqrt{s_{(j,\epsilon)}^2+(-1)^{j-1}}}. \tag{3.15}$$

This implies new boundary values for the $s_{(j,\epsilon)}$-integration:

- First bound: if $r_{(j,\epsilon)} = q$ then $s_{(j,\epsilon)} = (j-1)\,\delta_{j1} + \delta_{j2}$,
- Second bound: if $r_{(j,\epsilon)} = q(\epsilon^{j-1}\sqrt{\tau^2 + (-1)^{j-1}} + \epsilon^j \tau)$ then $s_{(j,\epsilon)} = \tau$,

Note that

$$\frac{\tau\, \widehat{f}^{(j,\epsilon)}_{lm}(\tau)}{\sqrt{\tau^2 + (-1)^{j-1}}} = 0 \quad \text{for} \quad (j = 1, \tau = 0) \quad \text{or} \quad (j = 2, \tau = 1).$$

Putting these elements together one gets a third form for eq. (3.6), which is eq. (3.15). Note that $\epsilon^2 = 1$ appears in this expression. □

Final form of eq. (3.6)

Proposition 3.5. *Through the substitutions*

$$\widehat{F}^{(j,\epsilon)}_{lm}(\tau) = \frac{\tau\, \widehat{f}^{(j,\epsilon)}_{lm}(\tau)}{\sqrt{\tau^2 + (-1)^{j-1}}}, \tag{3.16}$$

$$s^{2\lambda}_{(j,\epsilon)}\, F^{(j,\epsilon)}_{lm}(s_{(j,\epsilon)}) = \frac{\left(r^{n-1}_{(j,\epsilon)} f_{lm}(r_{(j,\epsilon)})\right)_{r_{(j,\epsilon)} = q\left(\epsilon^{j-1}\sqrt{s^2_{(j,\epsilon)} + (-1)^{j-1}} + \epsilon^j s_{(j,\epsilon)}\right)}}{\sqrt{s^2_{(j,\epsilon)} + (-1)^{j-1}}}, \tag{3.17}$$

eq. (3.12) takes the form of eq. (2.8) obtained in the case of the Radon transform on the Σ*-spherical cap of section (1)*

$$\widehat{F}^{(j,\epsilon)}_{lm}(\tau) = \frac{\omega_{n-2}}{C^\lambda_l(1)} \int_{(j-1)}^{\tau} ds_{(j,\epsilon)}\, C^\lambda_l(s_{(j,\epsilon)}/\tau) \left(1 - \frac{s^2_{(j,\epsilon)}}{\tau^2}\right)^{\lambda - 1/2} s^{2\lambda}_{(j,\epsilon)}\, F^{(j,\epsilon)}_{lm}(s_{(j,\epsilon)}). \tag{3.18}$$

Proof. The proof is straightforward. Note that, as $f_{lm}(r_{(j,\epsilon)}) \in \mathcal{S}(\mathbb{R}^+)$, eq. (3.17) shows that $F^{(j,\epsilon)}_{lm}(s_{(j,\epsilon)})$ has the same decrease rate as $f_{lm}(r_{(j,\epsilon)})$ for $r_{(j,\epsilon)} \to \infty$. □

This result shows that in a space with spherical coordinates $(s_{(j,\epsilon)}, \Omega_{\mathbf{k}})$, deduced from the original spherical coordinates $(r, \Omega_{\mathbf{k}})$ by eq. (3.13), the function $F(s_{(j,\epsilon)}, \Omega_{\mathbf{k}})$ is transformed into $\widehat{F}^{(j,\epsilon)}(\tau, \Omega_{\mathbf{n}})$ by a Radon transform on (Σ)-spherical caps. Both $F(s_{(j,\epsilon)}, \Omega_{\mathbf{k}})$ and $\widehat{F}^{(j,\epsilon)}(\tau, \Omega_{\mathbf{n}})$ are related to $f(r, \Omega_{\mathbf{k}})$ and to $\widehat{f}(\tau, \Omega_{\mathbf{n}})$ by eqs. (3.16,3.17). This observation is similar to the one made by A M Cormack for his Radon transform on α- and β-curves in the plane [5]. This remarkable feat, although implicit in [31], is explicitly demonstrated for the $(\Sigma_{(j,\epsilon)})$-spherical caps.

3.3 Null spaces

The problem is the determination of the values of $k \in \mathbb{N}$, for $F_{lm}^{(j,\epsilon)}(s_{(j,\epsilon)}) = (s_{(j,\epsilon)})^k$ which makes the integral (3.18) vanishes, *i.e.*

$$\frac{\omega_{n-2}}{C_l^\lambda(1)} \int_{(j-1)}^{\tau} ds_{(j,\epsilon)}\, C_l^\lambda(s_{(j,\epsilon)}/\tau) \left(1 - \frac{s_{(j,\epsilon)}^2}{\tau^2}\right)^{\lambda-1/2} s_{(j,\epsilon)}^{2\lambda}\,(s_{(j,\epsilon)})^k = 0 \tag{3.19}$$

A change of variable $t = s_{(j,\epsilon)}/\tau$ leads to the integral

$$\frac{\omega_{n-2}}{C_l^\lambda(1)}\,\tau^{2\lambda+1+k} \int_{(j-1)/\tau}^{1} dt\, C_l^\lambda(t)\,(1-t^2)^{\lambda-1/2}\, t^{2\lambda+k}, \tag{3.20}$$

which is calculable only for $j = 1$ and has the value (see eq. (2.14))

$$\frac{\Gamma(2\lambda+l)}{\Gamma(2\lambda)}\,\frac{\Gamma(\lambda+1/2)}{2^{l+1}\,l!}\,\frac{\Gamma((2\lambda+1+k-l)/2)}{\Gamma(2\lambda+1+(k+l)/2)}\,\frac{\Gamma(2\lambda+1+k)}{\Gamma(2\lambda+1+k-l)}.$$

This expression vanishes for $k = (2\lambda+k)(2\lambda+k-1)...(2\lambda+k+1-l)$. Consequently the null space for $j = 2$ remains yet to be determined.

3.4 Consistency conditions on the Radon data

The question is to determine $k \in \mathbb{N}$, such that

$$\int_{(j-1)}^{\infty} d\tau\,\tau^{-k}\,\widehat{F}_{lm}^{(j,\epsilon)}(\tau) = 0. \tag{3.21}$$

Setting the expression of $\widehat{F}_{lm}^{(j,\epsilon)}(\tau)$ given by Eq. (3.18) in eq. (3.21), we perform the calculation of the integral as in subsection (2.4) and end up with the values of k given by (2.15), *i.e.* $k = 2, 3, ..., l, (l+1)$.

3.5 Inversion of the Gegenbauer integral transform for spherical components

We are now in a position to state the following theorem:

Theorem 3.6. *The reconstructed spherical component $f_{lm}(r)$ is given in terms of $\widehat{f}_{lm}^{(j,\epsilon)}(\tau)$ by*

$$\frac{r^{n-1}\,f_{lm}(r)}{\frac{1}{2}\left|\left(\frac{r}{q}+(-1)^{j-1}\frac{q}{r}\right)\right|} = \frac{1}{K(\lambda,l)}\left(\frac{\epsilon\, r}{\frac{1}{2}\left|\left(\frac{r}{q}+(-1)^{j-1}\frac{q}{r}\right)\right|}\frac{d}{dr}\right)^{2\lambda+1}\left(\frac{1}{2}\left|\left(\frac{r}{q}+(-1)^j\frac{q}{r}\right)\right|\right)^{2\lambda-1}\times$$

$$\int_{(j-1)}^{\frac{\epsilon^j}{2}(r/q+(-1)^j q/r)} d\tau\, C_l^\lambda\left(\frac{\epsilon^j}{2}(r/q+(-1)^j q/r)\right)\left(1-\frac{\tau^2}{\frac{1}{4}\left(\frac{r}{q}+(-1)^j\frac{q}{r}\right)^2}\right)^{\lambda-1/2}\frac{\tau\,\widehat{f}_{lm}^{(j,\epsilon)}(\tau)}{\sqrt{\tau^2+(-1)^{j-1}}}, \tag{3.22}$$

where $r = r_{(j,\epsilon)}$ for short.

Proof. Transposing the inverse formula for the Σ-spherical cap Radon transform (2.26) to eq. (3.18) yields the formula

$$F_{lm}^{(j,\epsilon)}(s) = \frac{1}{K(\lambda,l)}\,\frac{1}{s^{2\lambda}}\left(\frac{d}{ds}\right)^{2\lambda+1} s^{2\lambda-1}\int_{(j-1)}^{s} d\tau\, C_l^{\lambda}(s/\tau)\left(1-\frac{\tau^2}{s^2}\right)^{\lambda-1/2}\widehat{F}_{lm}^{(j,\epsilon)}(\tau), \tag{3.23}$$

where $s = s_{(j,\epsilon)}$ to save writing. We now invert eqs. (3.16,3.17) to retrieve the final reconstruction formula for the spherical components $f_{lm}(r)$ of $f(r,\Omega_{\mathbf{k}})$. □

Of course a closed form of this inversion is not obtainable at present. The independence of the result of the parameter q may be argued on the basis of scale invariance: only r/q occurs in the calculation as in Cormack's work. We also should check that the integrals are well-defined as we have changed functions. But our goal has been achieved with eq. (3.22).

4 Radon transforms on $(\Sigma_{(j,\epsilon)})$-spherical caps in $\mathbb{R}^3$

In this section the case $\lambda = 1/2$ (or $n = 3$) is considered because there may be potential applications in three-dimensional imaging processes based on the detection of Compton scattered ionizing radiation. Then the Gegenbauer polynomial becomes the Legendre polynomial $C_l^{1/2}(x) = P_l(x)$. The spherical caps may be viewed as generated by the rotation of circular arcs of references [41, 28] around their axis of reflection symmetry. We go over first the main results for (Σ)-spherical caps on $\mathbb{R}^3$-spheres going through the origin and then gives the main inversion formulas for Radon transforms on $(\Sigma_{(j,\epsilon)})$-spherical caps.

4.1 Radon transform on (Σ)-spherical caps

A (Σ)-spherical cap is fully defined by a restriction on its radial distance range $q < r < p$ or equivalently by the restriction on the angular range $0 < \gamma < \gamma_0$, where $\cos\gamma_0 = q/p$. When $\gamma_0 = \pi/2$ (or $q = 0$), the spherical cap is the whole sphere going through the origin O.

A function $f(r,\Omega_{\mathbf{k}})$, where $\Omega_{\mathbf{k}}$ represents the azimuthal and co-latitude angles (ϕ,θ) of $\mathbf{k}$ on $\mathbb{S}^2$, is transformed into $\widehat{f}(p,\Omega_{\mathbf{n}})$ given by the integral

$$\widehat{f}(p,\Omega_{\mathbf{n}}) = \int_{\mathbb{R}^3} r^2\, dr\, d\Omega_{\mathbf{k}}\,\frac{p}{r}\,\delta\left(r - p\,(\mathbf{k}\cdot\mathbf{n})\right)\, f(r,\Omega_{\mathbf{k}}). \tag{4.1}$$

The Radon data depends on q. The plane (P_q) in which is the boundary circle of the (Σ)-spherical cap lies, for given p, is at a distance $q\sin\gamma_0$ from the origin O.

Following section 2, the spherical components $f_{lm}(r)$ of $f(r,\Omega_{\mathbf{k}})$ are related to $\widehat{f}_{lm}(p)$, the spherical components of $\widehat{f}(p,\Omega_{\mathbf{n}})$ by

$$\widehat{f}_{lm}(p)^q = 2\pi\int_q^p dr\, P_l\left(\frac{r}{p}\right) r\, f_{lm}(r). \tag{4.2}$$

This data is subjected to the following consistency conditions

- for $l = 2l'$, then

$$\int_0^\infty dp\ \widehat{f}^q_{lm}(p)\ \frac{1}{p^{2k}} = 0 \quad \text{for} \quad k = 1, 2, ..., l' \quad \text{but} \quad \int_0^\infty dp\ \widehat{f}^q_{lm}(p) \neq 0, \tag{4.3}$$

- for $l = 2l' + 1$, then

$$\int_0^\infty dp\ \widehat{f}^q_{lm}(p)\ \frac{1}{p^{2k+1}} = 0 \quad \text{for} \quad k = 1, 2, ..., l' \quad \text{but} \quad \int_0^\infty dp\ \widehat{f}^q_{lm}(p)\ \frac{1}{p} \neq 0. \tag{4.4}$$

The reconstruction formula for the spherical component which takes into account the consistency conditions is simply

$$f_{lm}(r) = -\frac{1}{2\pi\, r}\,\frac{d^2}{dr^2}\int_r^\infty dp\ P_l\left(\frac{r}{p}\right)\ \widehat{f}^q_{lm}(p). \tag{4.5}$$

We may also verify that this reconstruction is independent of q. Using the closure relation for the Legendre polynomials [30]

$$\sum_{l\in\mathbb{Z}}(2l+1)\,P_l(\mathbf{k}\cdot\mathbf{n})\,P_l(r/p) = 2\ \delta\left(\frac{r}{p} - (\mathbf{k}\cdot\mathbf{n})\right). \tag{4.6}$$

a closed form of the reconstruction formula may written down

$$f(r,\Omega_{\mathbf{k}}) = -\int_{\mathbb{S}^2} d\Omega_{\mathbf{n}}\left\{\frac{1}{2\pi\, r}\,\frac{d^2}{dr^2}\int_r^\infty dp\ \delta\left(\frac{r}{p} - (\mathbf{k}\cdot\mathbf{n})\right)\ \widehat{f}(p,\Omega_{\mathbf{n}})\right\}. \tag{4.7}$$

This result has thus the nice form of a summation image in the sense of Barrett [42].

4.2 Radon transforms on $(\Sigma_{(j,\epsilon)})$-spherical caps

The equation of the $(\Sigma_{(j,\epsilon)})$-spherical caps in spherical coordinates is the same as given by eqs. (3.1,3.2), with $t = (\mathbf{n}\cdot\mathbf{k}) = \cos\gamma$. The delta function kernel concentrated on the $(\Sigma_{(j,\epsilon)})$-spherical caps is readily expressed by eqs. (3.4,3.5).

The equation linking the spherical components of $\widehat{f}(\tau,\Omega_{\mathbf{n}})$ to those of $f(r,\Omega_{\mathbf{k}})$is now (see eq. (3.1))

$$\frac{\tau\,\widehat{f}^{(j,\epsilon)}_{lm}(\tau)}{\sqrt{\tau^2 + (-1)^{j-1}}} =$$

$$2\pi\int_q^{q(\epsilon^{j-1}\sqrt{\tau^2+(-1)^{j-1}}+\epsilon^j\tau)} \epsilon\, dr_{(j,\epsilon)}\ P_l\left(\frac{\epsilon^j}{2\tau}\left(\frac{r_{(j,\epsilon)}}{q} + (-1)^j\frac{q}{r_{(j,\epsilon)}}\right)\right)\ r_{(j,\epsilon)}\ f_{lm}(r_{(j,\epsilon)}). \tag{4.8}$$

The inverse formula reads

$$f_{lm}(r_{(j,\epsilon)}) = \frac{\epsilon^j}{\pi\, r_{(j,\epsilon)}}\,\frac{d}{dr_{(j,\epsilon)}}\,\frac{q r^2_{(j,\epsilon)}}{r^2_{(j,\epsilon)} + (-1)^{j-1}q^2}\,\frac{d}{dr_{(j,\epsilon)}}$$

$$\int_{(j-1)}^{\frac{\epsilon^j}{2}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)} d\tau\, P_l\left(\frac{\epsilon^j}{2\tau}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)\right)\frac{\tau\,\widehat{f}^{(j,\epsilon)}_{lm}(\tau)}{\sqrt{\tau^2+(-1)^{j-1}}}. \qquad (4.9)$$

A closed form for the inversion formula can be given as

$$f(r_{(j,\epsilon)}) = -\frac{1}{\pi}\frac{1}{r_{(j,\epsilon)}}\frac{d}{dr_{(j,\epsilon)}}\frac{q r^2_{(j,\epsilon)}}{r^2_{(j,\epsilon)}+(-1)^{j-1}q^2}\frac{d}{dr_{(j,\epsilon)}}\int_{\mathbb{S}^2} d\Omega_{\mathbf{n}}\int_{\frac{\epsilon^j}{2}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)}^{\infty} d\tau$$

$$\delta\left(\frac{\epsilon^j}{2}\left(\frac{r_{(j,\epsilon)}}{q}+(-1)^j\frac{q}{r_{(j,\epsilon)}}\right)-(\mathbf{k}\cdot\mathbf{n})\right)\frac{\tau\,\widehat{f}^{(j,\epsilon)}(\tau,\Omega_{\mathbf{n}})}{\sqrt{\tau^2+(-1)^{j-1}}}. \qquad (4.10)$$

Here also it is not needed to have the Radon data for the full range of τ.

4.3 Possible applications to scattered radiation three-dimensional imaging

With the inversion formulas of the Radon transforms on various spherical caps in $\mathbb{R}^3$, three-dimensional imaging processes using scattered gamma rays can be proposed. They are elaborations of the new Compton Scatter Tomography modalities in [47].

The question is how to generate spherical caps Radon data from the circular arcs Radon data obtained before in two dimensions. One way to achieve this is to rotate the pair source-detector around the reflection symmetry axis. This operation amounts to an azimuthal angular summation of the circular arc Radon data around its its reflection symmetry axis. Technically one may implement discretely this rotation by disposing a very large number of alternate pairs of collimated source-detector all around the circular rim (C_q) for each spatial angular orientation of the rotational symmetry axis of the spherical cap. This is schematically shown in Fig. 1 below. An emission (resp. detection) site S (resp. D) is represented by a blackened (resp. white) circular spot on the spherical cap rim. A photon emitted at $\mathbf{S}$ will be scattered in the bulk of an object at M under a scattering angle ω before being absorbed at $\mathbf{D}$, see *e.g.* [47]. Of course proper mechanical collimation is to be designed and build at S and D in order for the photon path SMD to be in a plane containing the line SD and the rotational symmetry axis of the spherical cap.

But evidently this operation does not produce the required spherical cap Radon data. So there is a need to connect the measured integral data to the needed spherical cap Radon data.

An analogous situation has occurred in cone-beam geometry computed tomography (CT), which generates only ray data. One possible idea to invert the cone-beam CT transform is to convert the ray data into three-dimensional Radon data and use the related inverse Radon formula. However generating planar data from ray data does not yield directly the planar Radon data. A "trick" found by Grangeat [43] shows a way to make this transition. Thus in the present situation, a relation must be established between the angular summation of circular arc Radon data and the spherical cap Radon data.

To show how this conversion can be done, we choose a special spherical coordinate system and compute the integral of a given function $f(r,\theta,\phi)$ first along a meridian arc

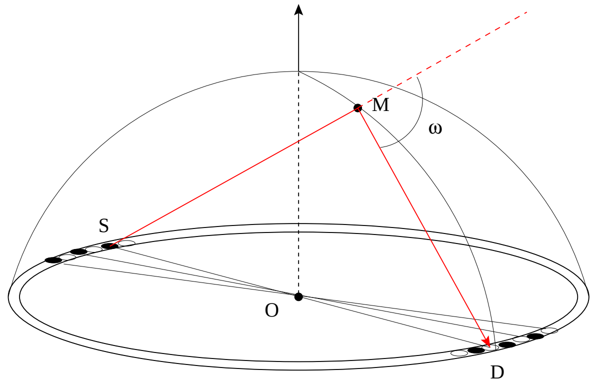

Figure 1: Arrangement of source-detector pairs on the circular rim of a spherical cap

of radius R in its arc measure $R\,d\theta$, *i.e.* ϕ fixed and $0 < \theta < \theta_0 < \pi/2$, then followed by an ϕ-integration from 0 to π. The result is

$$\int_0^{\pi} d\phi \int_0^{\theta_0} d\theta\, R\, f(R,\theta,\phi). \tag{4.11}$$

This integral represents the summation of the Radon data on a circular arc over an angle $\phi \in [0,\pi]$ in this particular spherical coordinate system.

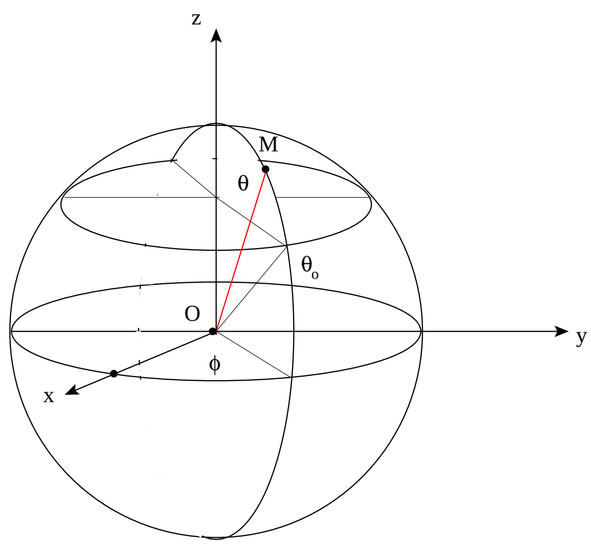

Figure 2: A $\mathbb{R}^3-$spherical cap in a special coordinate system

By rearranging the integrand, we see that this integral can be viewed as a spherical-cap Radon data of the function $g(R,\theta,\phi)$

$$\int_0^{\pi} d\phi \int_0^{\theta_0} d\theta\, R\, f(R,\theta,\phi) = R^2 \int_0^{\pi} d\phi \int_0^{\theta_0} \sin\theta\, d\theta\, g(R,\theta,\phi), \tag{4.12}$$

where

$$g(R,\theta,\phi) = \frac{f(R,\theta,\phi)}{R\,\sin\theta}. \tag{4.13}$$

Note that $R\sin\theta_0$ is the radius of the circular rim of the spherical cap in Fig. 2.

Thus by rotating the pair of points S and D around their reflection symmetry axis by an angle π, we collect the spherical cap Radon data of the function $g(r,\theta,\phi)$ and not of $f(r,\theta,\phi)$. Thus the inversion formulas of the five modalities quoted before give the reconstruction of $g(r,\theta,\phi)$ first. The sought function $f(R,\theta,\phi)$ can be deduced from eq. (4.13) in the special spherical coordinate system given above. In practice the recovery of f from g is more complicated in an arbitrary coordinate system.

4.4 Inclusion of Compton kinematic factor, attenuation and photometric effects

In real physical processes, a Compton kinematic factor, which includes the Compton differential cross section, should be taken into account in the measurement data. It is solely a function of the Compton scattering angle ω, which is related to the τ parameter by

$$\tau = \cot\omega \quad \text{for} \quad j=1, \quad \text{and} \quad \tau = \csc\omega \quad \text{for} \quad j=2.$$

Therefore its inclusion as a factor $P(\tau)$, would not alter the invertibility of the Radon transform on $(\Sigma_{(j,\epsilon)})$ in $\mathbb{R}^{2,3}$.

It is known that radiation flux density, after traveling a distance d is weakened by a factor $1/d^2$ in $\mathbb{R}^3$ due to dispersion in space. So for a radiation pencil emitted at site S, scattered at site M and registered at site D, accounting for this effect means inserting a factor $(SM \times MD)^{-2}$ in the definition of the Radon transform (3.3), see *e.g.* Fig. 1. Fortunately this product can be exactly evaluated and reexpressed as a product of a function of $r_{(j,\epsilon)}$ and a function of τ. The distances SM and MD can be evaluated by applying the cosine theorem for triangles as

$$\begin{aligned} SM^2_{(j,\epsilon)} &= q^2 + r^2_{(j,\epsilon)} - 2q\, r_{(j,\epsilon)} \cos(\psi_{(j,\epsilon)} - \gamma) \\ MD^2_{(j,\epsilon)} &= q^2 + r^2_{(j,\epsilon)} - 2q\, r_{(j,\epsilon)} \cos(\psi_{(j,\epsilon)} + \gamma), \end{aligned}$$

where $\psi_{(j,\epsilon)} = (\pi/2\,\delta_{j1} + \gamma_0\,\delta_{j2})$. Their product can be re-expressed as a product of a function of $r_{(j,\epsilon)}$ and a function of τ as

$$SM^2_{(j,\epsilon)}\, MD^2_{(j,\epsilon)} = (q^2 - r^2_{(j,\epsilon)})^2\, (1 + (-1)^{j-1}\tau^{-2})$$

A quick check shows that the invertibility of the transform via circular harmonic components remains valid, since the factor in τ can be absorbed in the Radon data and the factor in $r_{(j,\epsilon)}$ may be used to redefine a new input function. Moreover in practice, since real objects to be investigated are represented by non-negative integrable functions with compact support, the parameter q may be adjusted so that no divergence arises in the integrals defining the Radon transforms.

However the effect of radiation attenuation in matter, even if it is assumed to be uniform (*e.g.*with a constant linear attenuation coefficient), cannot fit into this inversion scheme, as already pointed out in [28, 41]. Then approximate compensation for non-uniform attenuation is to be worked out.

Conclusion

In this article, explicit inversion formulas for Radon transforms on spherical caps are derived. They generalize and complement inversion formulas already known for Radon transforms on spheres centered on planes or spheres. The main idea is to use the spherical harmonic components and the Funck-Hecke theorem which allows to get a generalized Gegenbauer transform. A further change of variables and functions brings turns this Gegenbauer transform back to a standard form known for the Radon transform on spheres passing a fixed point for which an inverse formula can be written down. Five different cases are studied. In $\mathbb{R}^3$, they suggest potential new imaging processes based on Compton scattering of ionizing radiation. Of course it would be necessary to investigate in depth further mathematical properties of these Radon transforms in order to secure their working in future scanning devices.

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T.T. Truong
Laboratoire de Physique Théorique et Modélisation
(CNRS UMR 8089/Université de Cergy-Pontoise)
2 av. Adolphe Chauvin,
95302 Cergy-Pontoise, France
E-mail: `truong@u-cergy.fr`
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