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AN ASYMPTOTIC EXPANSION FOR A SOLUTION TO VISCOELASTICITY EQUATIONS

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Abstract A problem of waves excited by an arbitrary oriented impulsive point force is investigated for a linear system of viscoelasticity equations. It is assumed that the medium is heterogeneous, isotropic and its properties depend on the prehistory of a wavy process. We suppose that the modulus of elasticity is expressed as the sum of two items. The first one is a function of space variables and the second item presents an integral operator of convolution type with respect to time. The structure of the solution to the Cauchy problem for a system of viscoelasticity equations is examined.

Key words: viscoelasticity equations, point source, asymptotic expansion.

AMS Mathematics Subject Classification: 35C20, 35L15, 35R10.

1 Introduction, main result

At the beginning of the last century J.Hadamard offered a method to solve a Cauchy problem for second order general hyperbolic equations. (see [4]). The solution is based on the series expansion using some infinite system of functions. These functions depend on the square of a distance in pseudo-Riemannian metrics. This metric is naturally associated with a differential equation. The main point of the method and its modern formulation can be found in Babich's paper [1]. In Sobolev's works [8, 9] another method of construction the solution to the Cauchy problem is developed. It is based on an integral equation that is equivalent to the initial problem. The attempt to apply the J. Hadamard method to systems of hyperbolic equations with different speeds of waves leads Babich to creation of the ray method. This method is also based on the expansion of a solution on some unlimited system of functions, but in contrast to Hadamard's series, here some more general function $\gamma(x,t)$ participates as an argument. This function satisfies characteristic equation. So as this equation has, as a rule, some real roots, each root has its own expansion that is similar to J. Hadamard's expansion. Based on plane-wave expansions and their subsequent summation, V.M.Babich builds fundamental solutions for Petrovskiy systems of hyperbolic equations [2] and for equations of elasticity [3]. A singular part of a fundamental solution is written out explicitly for equations of elasticity. This part is a main component that represents the sum of the delta-functions multiplied by some factors. These functions are located on characteristic cones.

In the paper [6] the ray expansion is taken out for a system of elasticity equations. The ray expansion is based, in essence, on Babich's method. Here the function $\gamma(x,t)$ has a form of a conical wave. In this case derivatives of the function $\gamma(x,t)$are not continuous and the higher a derivatives' order has the more a singularity order in a vicinity of cone's top. The last one induces difficulties when receiving explicit formulas of an asymptotic expansion. The situation becomes simpler by assumption that the medium is homogeneous in some vicinity of a source. Thereafter the solution is written out explicitly for sufficiently small spacy-temporal vicinity of the points where a point impulsive force is applied. In fact, the problem reduces to the extension of the solution outside this vicinity. Thus it is possible to describe in details the structure of not only singular part of the solution, but also its regular part. Particularly, it is possible to calculate a jump of the solution and all its derivatives, when passing through characteristic surfaces, which are responsible for compressive and transverse waves. The last one is actual in the inverse problems research area ([7]).

Below, the construction of asymptotic formulas to the solution of linear viscoelasticity equations is given for an arbitrary directed concentrated impulsive force. More precisely, the following Cauchy problem is considered:.

$$Mu \equiv \rho(x)\, u_{tt} - Lu = f^0 \delta(x-y,t), \quad u|_{t<0} \equiv 0. \tag{1.1}$$

Here $u = (u_1, u_2, u_3)$ is a vector of elastic displacement, $f^0 = (f_1^0, f_2^0, f_3^0)$ is a numerical vector characterizing the direction of the force, an operator $L = (L_1, L_2, L_3,)$ is defined by the equations

$$L_i u = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(u)}{\partial x_j}, \quad \sigma_{ij}(u) = \lambda(x)\delta_{ij}\mathrm{div} u(x,t) + \mu(x)\left(\frac{\partial u_i(x,t)}{\partial x_j} + \frac{\partial u_j(x,t)}{\partial x_i}\right)$$

$$+ \int_{-\infty}^{t} \left[p(x,t-s)\delta_{ij}\mathrm{div} u(x,s) + q(x,t-s)\left(\frac{\partial u_i(x,s)}{\partial x_j} + \frac{\partial u_j(x,s)}{\partial x_i}\right)\right] ds, \; i,j = 1,2,3. \tag{1.2}$$

In these equations δ_{ij} is Kronecker's symbol, $\lambda(x)$, $\mu(x)$ are elastic modules, $\rho(x)$ is the density of area, functions $p(x,t)$, $q(x,t)$ characterize the viscoelasticity.

In the sequel, suppose that $\lambda(x) + \mu(x) > 0$, $\mu(x) > 0$, $\rho(x) > 0$.

In the case of homogeneous medium when ρ, λ, μ are constants, $p = q = 0$, problem (1.1) has been solved by Love [5]. The solution is given by the formula ([6] §5)

$$\begin{aligned} u(x,t,y) &= \frac{f^0}{4\pi\rho\, c_s^2 |x-y|}\, \delta\big(t - \tau_s(x,y)\big) \\ &\quad + \frac{1}{4\pi\rho} \nabla \mathrm{div} \Big\{ \frac{f^0}{|x-y|} \left[\theta_1\big(t-\tau_p(x,y)\big) - \theta_1\big(t - \tau_s(x,y)\big)\right]\Big\}, \end{aligned} \tag{1.3}$$

where

$$\tau_p(x,y) = \frac{|x-y|}{c_p}, \quad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \tau_s(x,y) = \frac{|x-y|}{c_s}, \quad c_s = \sqrt{\frac{\mu}{\rho}},$$

c_p, c_s are constant and they define speed of the compressive and transverse waves, $\theta_1(t) = t\theta_0(t)$, $\theta_0(t)$ is Heaviside's function: $\theta_0(t) = 1$ for $t \geq 0$ and $\theta_0(t) = 0$ for $t < 0$.

By calculating operations div and ∇, we see that the formula (1.3) can be expressed as

$$\begin{aligned} u(x,t,y) &= \frac{1}{4\pi\rho}\Big\{\frac{(f^0\cdot\nu)\nu}{c_p^2|x-y|}\delta\big(t-\tau_p(x,y)\big)+\frac{\nu\times(f^0\times\nu)}{c_s^2|x-y|}\,\delta\big(t-\tau_s(x,y)\big)\\ &+\frac{2(f^0\cdot\nu)\nu-\nu\times(f^0\times\nu)}{|x-y|^2}\Big[\frac{1}{c_p}\,\theta_0\Big(t-\tau_p(x,y)\Big)-\frac{1}{c_s}\,\theta_0\big(t-\tau_s(x,y)\big)\Big]\\ &+\frac{2(f^0\cdot\nu)\nu-\nu\times(f^0\times\nu)}{|x-y|^3}\Big[\theta_1\big(t-\tau_p(x,y)\big)-\theta_1\big(t-\tau_s(x,y)\big)\Big]\Big\}, \end{aligned} \tag{1.4}$$

where $\nu=(x-y)/|x-y|$.

Let us consider an infinite system of functions resulting from Heaviside's function by integration and differentiation:

$$\theta_k(t)=\frac{t^k}{k!}\theta_0(t),\quad k=1,2,\dots,\quad \theta_{-k}(t)=\frac{d^k}{dt^k}\theta_0(t)=\delta^{(k-1)}(t),\quad k=1,2,\dots.$$

We notice that the functions of this system at any $k=0,\pm1,\pm2,\dots$, satisfy equality $\theta_k'(t)=\theta_{k-1}(t)$. Let

$$c_p(x)=\sqrt{\frac{\lambda(x)+2\mu(x)}{\rho(x)}},\quad c_s(x)=\sqrt{\frac{\mu(x)}{\rho(x)}},$$

be speeds of compressive and transverse waves in a inhomogeneous medium. Further, we suppose that $\lambda(x)$, $\mu(x)$, $\rho(x)$, $p(x,t)$, $q(x,t)$ are infinitely differentiable functions of their arguments and $c_p(x)>c_s(x)>0$, $\rho(x)>0$ for all $x\in\mathbb{R}^3$. Then suppose that the parameters $\lambda(x)$, $\mu(x)$, $\rho(x)$ are constant in some ε vicinity of the point y, and $p(x,t)=q(x,t)=0$ for $|x-y|<\varepsilon, t\geq 0$.

Define two Riemannian's metric using the element's length $d\tau_p=|dx|/c_p(x)$ and $d\tau_s=|dx|/c_s(x)$, in this case $|dx|$ is an element's length in Euclidean metric. Let $\Gamma_p(x,y)$, $\Gamma_s(x,y)$ be geodesic lines linking points x and y; $\tau_p(x,y)$ and $\tau_s(x,y)$ be Riemannian's lengths. Suppose that these two metrics are simple, i.e. functions $\tau_p(x,y)$ and $\tau_s(x,y)$ are uniquely defined. Formula (1.4) for a homogeneous medium, in which $\rho=\rho(y)$, $c_p=c_p(y)$, $c_s=c_s(y)$, can be represented as a finite ray expansion

$$u(x,t,y) = \sum_{k=-1}^{1}\Big[\alpha^{(k,p)}(x,y)\,\theta_k(t-\tau_p(x,y))+\alpha^{(k,s)}(x,y)\,\theta_k(t-\tau_s(x,y))\Big], \tag{1.5}$$

where the coefficients $\alpha^{(k,p)}(x,y)$ are given by

$$\begin{aligned} \alpha^{(-1,p)}(x,y) &= -\frac{(f^0\cdot\nabla_y\tau_p(x,y))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p(x,y)},\\ \alpha^{(0,p)}(x,y) &= \frac{\nabla\tau_p(x,y)\times(f^0\times\nabla_y\tau_p(x,y))-2(f^0\cdot\nabla_y\tau_p(x,y))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p^2(x,y)},\\ \alpha^{(1,p)}(x,y) &= \frac{\nabla\tau_p(x,y)\times(f^0\times\nabla_y\tau_p(x,y))-2(f^0\cdot\nabla_y\tau_p(x,y)))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p^3(x,y)}, \end{aligned} \tag{1.6}$$

,

and the coefficients $\alpha^{(k,s)}(x,y)$ are given by

$$\begin{aligned}
\alpha^{(-1,s)}(x,y) &= -\frac{\nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s(x,y)},\\
\alpha^{(0,s)}(x,y) &= \frac{2(f^0\cdot\nabla_y\tau_s(x,y))\nabla\tau_s(x,y)-\nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s^2(x,y)},\\
\alpha^{(1,s)}(x,y) &= \frac{2(f^0\cdot\nabla_y\tau_s(x,y))\nabla\tau_s(x,y)-\nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s^3(x,y)}.
\end{aligned} \tag{1.7}$$

By the assumption that the medium is homogeneous in the vicinity of a source, the solution of (1.1) coincides with the solution for a homogeneous area in a sufficiently small vicinity of the point $(y,0)$.

Enter some additional notations. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ be a vector, which depends on the spatial variables x and y, and τ be a scalar function of the same variables. Note via $\kappa_{ij}^m(\alpha,\tau)$ the superposition of functions of x and y, and define them for $i,j=1,2,3$ and integer values of m, by the equalities

$$\begin{aligned}
\kappa_{ij}^0(\alpha,\tau) &= -\lambda(x)\delta_{ij}(\alpha\cdot\nabla\tau)-\mu(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^1(\alpha,\tau) &= \lambda(x)\delta_{ij}\mathrm{div}\alpha+\mu(x)\Big(\frac{\partial\alpha_i}{\partial x_j}+\frac{\partial\alpha_j}{\partial x_i}\Big)\\
&\quad -p_0(x)\delta_{ij}(\alpha\cdot\nabla\tau)-q_0(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^m(\alpha,\tau) &= p_{(m-2)}(x)\delta_{ij}\mathrm{div}\alpha+q_{(m-2)}(x)\Big(\frac{\partial\alpha_i}{\partial x_j}+\frac{\partial\alpha_j}{\partial x_i}\Big)\\
&\quad -p_{(m-1)}(x)\delta_{ij}(\alpha\cdot\nabla\tau)-q_{(m-1)}(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\ m\ge 2,
\end{aligned} \tag{1.8}$$

where

$$p_m(x)=\frac{\partial^m p(x,t)}{\partial t^m}\Big|_{t=0},\quad q_m(x)=\frac{\partial^m q(x,t)}{\partial t^m}\Big|_{t=0}.$$

Further, let $Q^{(n,p)}$, $Q^{(n,s)}$ be vector-functions, which components $Q_i^{(n,p)}$, $Q_i^{(n,s)}$ for $n=1,2,\dots$ are calculated by following formulas:

$$\begin{aligned}
Q_i^{(n,p)}(\alpha,\tau) &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}\Big[\kappa_{ij}^m(\alpha^{(n-m,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j}-\frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,p)},\tau_p)\Big],\\
Q_i^{(n,s)}(\alpha,\tau) &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}\Big[\kappa_{ij}^m(\alpha^{(n-m,s)},\tau_s)\frac{\partial\tau_s}{\partial x_j}-\frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,s)},\tau_s)\Big],
\end{aligned} \tag{1.9}$$

$$i=1,2,3.$$

Assume that the $Q^{(0,p)}=0$, $Q^{(0,s)}=0$.

Let

$$\zeta^p=(\zeta_1^p,\zeta_2^p,\zeta_3^p)=-c_p^2(y)\tau_p(x,y)\nabla_y\tau_p(x,y)$$

and

$$\zeta^s = (\zeta_1^s, \zeta_2^s, \zeta_3^s) = -c_s^2(y)\tau_s(x,y)\nabla_y \tau_s(x,y)$$

be Riemannian's coordinates of x with respect to y in the metrics $d\tau_p = |dx|/c_p(x)$, $d\tau_s = |dx|/c_s(x)$ respectively, and $J_p(x,y) = \det(\frac{\partial \zeta^p}{\partial x})$, $J_s(x,y) = \det(\frac{\partial \zeta^s}{\partial x})$ be Jacobians of transformations of Riemannian's coordinates to cartesian ones. Define a scalar function $A^{(p)}(x,y)$ and a matrix $T^{(s)}(x,y)$ with equalities

$$A^{(p)}(x,y) = \frac{\sqrt{J_p(x,y)}}{4\pi\tau_p(x,y)c_p^2(y)\sqrt{\rho(x)\rho(y)}} \exp\Big(\frac{1}{2}\int\limits_{\Gamma_p(x,y)} \frac{p_0(\xi)+2q_0(\xi)}{\lambda(\xi)+2\mu(\xi)}\,d\tau_p'\Big), \qquad (1.10)$$

$$T^{(s)}(x,y) = \frac{\mathcal{S}(x,y)\sqrt{J_s(x,y)}}{4\pi\tau_s(x,y)c_s^2(y)\sqrt{\rho(x)\rho(y)}} \exp\Big(\frac{1}{2}\int\limits_{\Gamma_s(x,y)} \frac{q_0(\xi)}{\mu(\xi)}\,d\tau_s'\Big). \qquad (1.11)$$

In these equalities ξ is a variable point of geodesics $\Gamma_p(x,y)$, $\Gamma_s(x,y)$, and $\tau_p' = \tau_p(\xi,y)$, $\tau_s' = \tau_s(\xi,y)$, respectively, $\mathcal{S}(x,y)$ is the matrix exponent

$$\mathcal{S}(x,y) = \exp\Big\{\int\limits_{\Gamma_s(x,y)} (\nabla \ln c_s(\xi))^t\,d\xi\Big\}. \qquad (1.12)$$

Here $(\nabla \ln c_s(\xi))^t$ is a column vector, and $d\xi = (d\xi_1, d\xi_2, d\xi_3)$ is a row vector, and multiplication of these vectors is performed according to rules of matrix algebra.

Theorem 1.1. *Let the above assumptions for functions $\lambda(x)$, $\mu(x)$, $\rho(x)$, $p(x,t)$, $q(x,t)$ hold. Then the solution of (1.1) can be represented in the form of asymptotic series:*

$$u(x,t,y) = \sum_{k=-1}^{\infty}\Big[\alpha^{(k,p)}(x,y)\,\theta_k(t-\tau_p(x,y)) + \alpha^{(k,s)}(x,y)\,\theta_k(t-\tau_s(x,y))\Big], \qquad (1.13)$$

where $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$ are functions of the class $\mathbf{C}^\infty(\mathbb{R}^6 \setminus \{(y,y)\})$ defined by (1.6), (1.7) for $|x-y|<\varepsilon$, and given for $|x-y|>\varepsilon$ by formulas

$$\alpha^{(k,p)}(x,y) = c_p(x)[A^{(k,p)}(x,y)\nabla\tau_p(x,y) + \nabla\tau_p(x,y)\times B^{(k,p)}(x,y)],$$
$$\alpha^{(k,s)}(x,y) = c_s(x)[A^{(k,s)}(x,y)\nabla\tau_s(x,y) + \nabla\tau_s(x,y)\times B^{(k,s)}(x,y)],$$

where

$$A^{(-1,p)}(x,y) = -(f^0\cdot\nabla_y\tau_p(x,y))A^{(p)}(x,y), \qquad B^{(-1,p)}(x,y) = 0,$$
$$B^{(-1,s)}(x,y) = -(f^0\times\nabla_y\tau_s(x,y))T^{(s)}(x,y), \quad A^{(-1,s)}(x,y) = 0,$$

and subsequent coefficients are calculated by using following recursion formulas:

$$A^{(n-1,p)}(x,y) = \Big[\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} + \int\limits_{\Gamma_p(x,\xi_p(x,y))} \frac{R^{(n,p)}(\xi,y)}{2A^{(p)}(\xi,y)}\,d\tau_p\Big]A^{(p)}(x,y),\ n\ge 1,$$

$$B^{(n,p)}(x,y) = \frac{\lambda+2\mu}{\rho(\lambda+\mu)}\Big(c_pQ^{(n,p)}\times\nabla\tau_p - [\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0c_p^{-2}]B^{(n-1,p)}$$
$$-c_p\Big[2\mu(\nabla\tau_p\cdot\nabla)\alpha^{(n-1,p)} + \nabla\big((\lambda+\mu)c_p^{-1}A^{(n-1,p)}\big) - c_p^{-1}A^{(n-1,p)}\nabla\mu\Big]\times\nabla\tau_p\Big),\ n\ge 0,$$

$$B^{(n-1,s)}(x,y) = \Big[B^{(n-1,s)}(\xi_s(x,y),y)T^{(s)}(\xi_s(x,y),y) + \frac{1}{2}\int\limits_{\Gamma_s(x,\xi_s(x,y))} R^{(n,s)}(\xi,y)T^{(s)}(\xi,y)\,d\tau_p\Big](T^{(s)})^{-1}(x,y),\ n\ge 1,$$

$$A^{(n,s)}(x,y) = \frac{c_s^2}{\lambda+\mu}\{[(\lambda+\mu)\mathrm{div}\alpha^{(n-1,s)} + \nabla\mu\cdot\alpha^{(n-1,s)} - (p_0+q_0)c_s^{-1}A^{(n-1,s)}]c_s^{-1} + [\mu\Delta\tau_s + \nabla\mu\cdot\nabla\tau_s - q_0c_s^{-2}]A^{(n-1,s)} + [2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)} + c_s\nabla\big((\lambda+\mu)c_s^{-1}A^{(n-1,s)}\big) - A^{(n-1,s)}\nabla\mu - c_sQ^{(n,s)}]\cdot\nabla\tau_s\},\ n\ge 0.$$

In these formulas $\xi_p(x,y)$, $\xi_s(x,y)$ *are the intersection points of geodesics* $\Gamma_p(x,y)$ *and* $\Gamma_s(x,y)$ *with the sphere* $|x-y|=\varepsilon$, *respectively, scalar functions* $R^{(n,p)}$ *and* $R^{(n,s)}$ *are defined by the equalities*

$$\begin{aligned} R^{(n,p)} &= \frac{1}{\rho}\Big(c_pQ^{(n,p)}\cdot\nabla\tau_p - [(\lambda+\mu)c_p^{-1}\mathrm{div}(c_p\nabla\tau_p\times B^{(n-1,p)}) \\ &\quad +\nabla\mu\cdot(\nabla\tau_p\times B^{(n-1,p)}) + 2\mu(\nabla\tau_p\times B^{(n-1,p)})\cdot\nabla\ln c_p]\Big), \\ R^{(n,s)} &= \frac{1}{\rho}\{c_sQ^{(n,s)} + A^{(n-1,s)}[(\lambda+2\mu)\nabla\ln c_s - \nabla\lambda] - (\lambda+\mu)\nabla A^{(n-1,s)}\}\times\nabla\tau_s, \end{aligned}$$

moreover,

$$\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} = -\frac{2(f^0\cdot\nabla_y\tau_p(x,y))[c_p(y)]^n}{|\xi_p(x,y)-y|^n}\begin{cases}1, & n=1,2,\\ 0, & n>2,\end{cases}$$

$$B^{(n-1,s)}(\xi_s(x,y),y)T^{(s)}(\xi_s(x,y),y) = -\frac{(f^0\times\nabla_y\tau_s(x,y))[c_s(y)]^n}{|\xi_s(x,y)-y|^n}\begin{cases}1, & n=1,2,\\ 0, & n>2.\end{cases}$$

Remark 1.1. *Asymptotic series (1.13) is an expansion "by the smoothness", (the term belongs V. M. Babich, see his work [1]) in the vicinity of the characteristic cones.*

This series allows us to calculate the singular part of the solution and jumps of derivatives of any order on the characteristic cones $t=\tau_p(x,y)$ *and* $t=\tau_s(x,y)$. *The jumps of of derivatives of order* $m\ge 0$ *are expressed in terms of coefficients* $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$ *for* $k\le m$. *In particular,*

$$\left[\frac{\partial^m u(x,t,y)}{\partial t^m}\right]_{t=\tau_p(x,y)} = \alpha^{(m,p)}(x,y), \qquad \left[\frac{\partial^m u(x,t,y)}{\partial t^m}\right]_{t=\tau_s(x,y)} = \alpha^{(m,s)}(x,y).$$

Remark 1.2. *In the case where the coefficients of elasticity equations have a finite, but sufficiently high smoothness, we can write only the finite number of terms in the asymptotic (1.13), and the residue can be estimated by using the standard method of energy estimates.*

A structure of the article is as following. In Section 2, for the reader convenience, there are some basic information about the formulas and concepts of Riemannian's geometry. In section 3, the basic relations of the ray method are derived. In sections 4 and 5, formulas calculating coefficients $\alpha^{(k,p)}$ and $\alpha^{(k,s)}$ of the series (1.13) are established.

2 Some facts from Riemannian's geometry

We have already had two Riemannian's metrics, linked with speeds of compressive $c_p(x)$ and transverse $c_s(x)$ waves. It is convenient to set out the series of the facts and suitable notations for some arbitrary isotropic Riemannian's metrics. The element of length is defined by the formula $d\tau = |dx|/c(x)$. Here $c(x)$ is a smooth and positive function defined on any compact domain of the space $\mathbb{R}^3$. Further we will use these facts for $c = c_p$ and $c = c_s$, providing relevant indices to p or s.

Associate the Riemannian's distance $\tau(x, y)$ between arbitrary two points x and y. Assume that the function $\tau(x, y)$ is defined uniquely. In this case, each pair of points x, y corresponds to a unique geodesic $\Gamma(x, y)$ that joins them. As a positive direction on $\Gamma(x, y)$ we take direction from y to x.

The function $\tau(x, y)$ satisfies the first order differential equations

$$c(x)|\nabla_x \tau(x, y)| = 1, \quad c(y)|\nabla_y \tau(x, y)| = 1, \tag{2.1}$$

and to the additional condition

$$\tau(x, y) \sim \frac{|x - y|}{c(y)}, \quad x \to y. \tag{2.2}$$

In sequel, assume that $\nabla_x = \nabla$. For a known function $\tau(x, y)$ defined along a geodesic $\Gamma(x, y)$ we have

$$\frac{dx}{d\tau} = c^2(x)\nabla\tau(x, y), \tag{2.3}$$

where the parameter τ is numerically equal to Riemannian's length $\tau(x, y)$. It follows from (2.3) that the vector $\nabla\tau(x, y)$ is directed at x tangentially to $\Gamma(x, y)$. The derivative of the function $\varphi(x)$ with respect to τ along to $\Gamma(x, y)$ is given by

$$\frac{d\varphi(x)}{d\tau} = \nabla\varphi(x) \cdot \frac{dx}{d\tau} = c^2(x)\nabla\tau(x, y) \cdot \nabla\varphi(x). \tag{2.4}$$

Derivatives of the function $\tau(x, y)$ are smooth everywhere, except the point $x = y$. Then, first order derivatives are bounded but not continuous at the point $x = y$. Derivatives of order $k > 1$ grow indefinitely in the vicinity of this point, with the rate of growth $O(|x - y|^{1-k})$. In contrast, the function $\tau^2(x, y)$ is smooth everywhere with respect to variables x and y and belongs to class $\mathbf{C}^k(\mathbb{R}^6)$ if $c(x) \in \mathbf{C}^k(\mathbb{R}^3)$, $k \geq 2$. Further, we suppose that $c(x) \in \mathbf{C}^\infty(\mathbb{R}^3)$.

Consider the Riemannian's coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ of point x for a fixed point y, determining with equality $\zeta = -c^2(y)\tau(x, y)\nabla_y\tau(x, y)$. From equalities (2.1), (2.3) it is clear that $\zeta = c(y)\tau(x, y)\nu^0$ where ν^0 is the unit tangent vector to $\Gamma(x, y)$ at the point y. The direction of this vector corresponds to the positive direction of $\Gamma(x, y)$, i.e from y to x. From the above, it follows that $\zeta(x, y) \in \mathbf{C}^\infty(\mathbb{R}^6)$ as $x \to y$.

Fix x and y. Let ξ be an arbitrary point of $\Gamma(x, y)$. Equation of the geodesic $\Gamma(x, y)$ can be written in the form $\xi = f(z\zeta, y)$, where $f(\zeta, y)$ is some function of class $\mathbf{C}^\infty(\mathbb{R}^6)$ and z is a dimensionless parameter, and $z = \tau(\xi, y)/\tau(x, y)$. In particular, when $z = 1$, we get that $x = f(\zeta, y)$. Note that $f(\zeta, y) = y + \zeta + O(|\zeta|^2)$ as $\zeta \to 0$.

For the future, note one important formula (the deduction can be found in [7]):

$$c^2\Delta\tau^2 = 6 - c^2\nabla\tau^2\cdot\nabla\ln(c^2J), \quad x\in\mathbb{R}^3. \tag{2.5}$$

Here $J = J(x,y)$ is the Jacobian of transition from Riemannian's coordinates of x to cartesian ones:

$$J(x,y) = \det\Big(\frac{\partial\zeta}{\partial x}\Big) \in \mathbf{C}^\infty(\mathbb{R}^6), \quad J(y,y) = 1. \tag{2.6}$$

From (2.5) follows the formula

$$c^2\Delta\tau = \frac{2}{\tau} - c^2\nabla\tau\cdot\nabla\ln(c^2J), \quad x\in\mathbb{R}^3. \tag{2.7}$$

3 The deduction of main relations

The solution to (1.1) is represented in the form of asymptotic series

$$u(x,t,y) = \sum_{k=-1}^{\infty}\Big[\alpha^{(k,p)}(x,y)\theta_k(t-\tau_p(x,y)) + \alpha^{(k,s)}(x,y)\theta_k(t-\tau_s(x,y))\Big]. \tag{3.1}$$

For the convenience of further calculations, let us set $\alpha^{(k,p)} = \alpha^{(k,s)} = 0$ for $k < -1$.

We should count the value of differential operator M (see (1.1)) at the function $u(x,t,y)$. To avoid the same type of calculations, for the moment, we will determine the value of this operator at the some simple function $v(x,t,y)$, which is defined below. For this aim, let us regard Riemannian's metric that is defined in the Section 2 with an element of length $d\tau = |dx|/c(x)$, $c(x) > 0$, and corresponding function $\tau(x,y)$ as Riemannian's distance between x and y points. Define the function $v(x,t,y)$ by the formula, which is similar to (3.1),

$$v(x,t,y) = \sum_{n=-1}^{\infty}\alpha^n(x,y)\theta_n(\gamma), \quad \gamma = t-\tau(x,y).$$

The following equalities hold:

$$\sigma_{ij}(\alpha^n\theta_n(\gamma)) = \sum_{m=0}^{\infty}\kappa_{ij}^m(\alpha^n,\tau)\theta_{n+m-1}(\gamma),$$

Where $\kappa_{ij}^m(\alpha,\tau)$, $\alpha = (\alpha_1,\alpha_1,\alpha_1)$ are defined by

$$\begin{aligned}
\kappa_{ij}^0(\alpha,\tau) &= -\lambda(x)\delta_{ij}(\alpha\cdot\nabla\tau) - \mu(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j} + \alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^1(\alpha,\tau) &= \lambda(x)\delta_{ij}\mathrm{div}\alpha + \mu(x)\Big(\frac{\partial\alpha_i}{\partial x_j} + \frac{\partial\alpha_j}{\partial x_i}\Big)\\
&\quad -p_0(x)\delta_{ij}(\alpha\cdot\nabla\tau) - q_0(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j} + \alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^m(\alpha,\tau) &= -p_{(m-1)}(x)\delta_{ij}(\alpha\cdot\nabla\tau) - q_{(m-1)}(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j} + \alpha_j\frac{\partial\tau}{\partial x_i}\Big)\\
&\quad +p_{(m-2)}(x)\delta_{ij}\mathrm{div}\alpha + q_{(m-2)}(x)\Big(\frac{\partial\alpha_i}{\partial x_j} + \frac{\partial\alpha_j}{\partial x_i}\Big),\ m\ge 2,
\end{aligned} \tag{3.2}$$

where

$$p_m(x) = \frac{\partial^m p(x,t)}{\partial t^m}\Big|_{t=0}, \quad q_m(x) = \frac{\partial^m q(x,t)}{\partial t^m}\Big|_{t=0}.$$

Then

$$L_i(\alpha^n\theta_n(\gamma)) = \sum_{j=1}^{3}\sum_{m=0}^{\infty}\Big(-\theta_{n+m-2}(\gamma)\kappa_{ij}^m(\alpha^n,\tau)\frac{\partial\tau}{\partial x_j} + \theta_{n+m-1}(\gamma)\frac{\partial}{\partial x_j}\kappa_{ij}^m(\alpha^n,\tau)\Big).$$

On the other hand,

$$\frac{\partial^2 v_i}{\partial t^2} = \sum_{n=-1}^{\infty}\alpha_i^n\theta_{n-2}(\gamma), \quad i = 1,2,3$$

Using these formulas, let find

$$\rho\frac{\partial^2 v_i}{\partial t^2} - L_i v = \sum_{j=1}^{3}\sum_{n=-1}^{\infty}\sum_{m=0}^{\infty} q_{ij}^m(\alpha^n,\tau)\theta_{n+m-2}(\gamma), \quad i = 1,2,3. \tag{3.3}$$

In this formula

$$\begin{aligned} q_{ij}^0(\alpha,\tau) &= \rho\alpha_i\delta_{ij} + \kappa_{ij}^0(\alpha,\tau)\frac{\partial\tau}{\partial x_j}, \\ q_{ij}^m(\alpha,\tau) &= \kappa_{ij}^m(\alpha,\tau)\frac{\partial\tau}{\partial x_j} - \frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha,\tau), \quad m \geq 1. \end{aligned} \tag{3.4}$$

We can rewrite the equalities (3.3) as

$$\rho\frac{\partial^2 v_i}{\partial t^2} - L_i v = \sum_{k=0}^{\infty} r_i^k\theta_{k-3}(\gamma), \quad i = 1,2,3, \tag{3.5}$$

where

$$r_i^k = \sum_{j=1}^{3}\sum_{m=0}^{k} q_{ij}^m(\alpha^{k-m-1},\tau), \quad i = 1,2,3, \quad k \geq 0. \tag{3.6}$$

According to formulas (3.3)-(3.6) we have:

$$\rho\frac{\partial^2 u_i}{\partial t^2} - L_i u = \sum_{k=0}^{\infty}\big[r_i^{(k,p)}\theta_{k-3}(\gamma_p) + r_i^{(k,s)}\theta_{k-3}(\gamma_s)\big], \quad i = 1,2,3, \tag{3.7}$$

where $r_i^{(k,s)}$, $r_i^{(k,s)}$, γ_p, γ_s are defined by equalities

$$r_i^{(k,p)} = \sum_{j=1}^{3}\sum_{m=0}^{k} q_{ij}^m(\alpha^{(k-m-1,p)},\tau_p), \quad \gamma_p = t - \tau_p(x,y), \tag{3.8}$$

$$r_i^{(k,s)} = \sum_{j=1}^{3}\sum_{m=0}^{k} q_{ij}^m(\alpha^{(k-m-1,s)},\tau_s), \quad \gamma_s = t - \tau_s(x,y), \tag{3.9}$$

and $q_{ij}^m(\alpha,\tau)$ are given by formulas (3.4).

Having $\rho\frac{\partial^2 u_i}{\partial t^2} - L_i u = 0$ for $t > 0$, the following equalities should satisfy

$$r_i^{(k,p)} = 0, \quad i = 1,2,3, \quad k = 0,1,2,\ldots, \tag{3.10}$$
$$r_i^{(k,s)} = 0, \quad i = 1,2,3, \quad k = 0,1,2,\ldots. \tag{3.11}$$

These equalities are used to calculate $\alpha^{(n,p)}$, $\alpha^{(n,s)}$, $n \geq -1$.

4 Coefficients $\alpha^{(k,p)}$ calculation

First, let us consider the group of equalities (3.10). Suppose $k = 0$. Then $r_i^{(0,p)} = 0$, $i = 1,2,3$ and to find $\alpha^{(-1,p)}$ we get the system of homogeneous equations

$$\rho\alpha_i^{(-1,p)} + \sum_{j=1}^{3} \kappa_{ij}^0(\alpha^{(-1,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j} = 0, \quad i = 1,2,3, \tag{4.1}$$

where

$$\kappa_{ij}^0(\alpha^{(-1,p)},\tau_p) = -\lambda(x)\delta_{ij}(\alpha^{(-1,p)}\cdot\nabla\tau_p) - \mu(x)\Big(\alpha_i^{(-1,p)}\frac{\partial\tau_p}{\partial x_j} + \alpha_j^{(-1,p)}\frac{\partial\tau_p}{\partial x_i}\Big).$$

It is easier to work with equalities (4.1), via representing them as one vector equality. Notice that,

$$\sum_{j=1}^{3} \kappa_{ij}^0(\alpha^{(-1,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j} = -(\lambda+\mu)(\alpha^{(-1,p)},\nabla\tau_p)\frac{\partial\tau_p}{\partial x_i} - \mu c_p^{-2}\alpha_i^{(-1,p)}. \tag{4.2}$$

When deriving this formula the equality $|\nabla\tau_p|^2 = c_p^{-2}$ is used. Taking into account(4.2), let us write equalities (4.1) as

$$(\rho - \mu c_p^{-2})\alpha^{(-1,p)} - (\lambda+\mu)(\alpha^{(-1,p)}\cdot\nabla\tau_p)\nabla\tau_p = 0. \tag{4.3}$$

It is easy to check that the equations (4.1) are linearly dependent ones. Indeed, taking scalar product both sides of equation (4.3) and $\nabla\tau_p$ and using the equality $c_p^{-2} = \rho/(\lambda+2\mu)$, we obtain zero. It also means that projection of vector $\alpha^{(-1,p)}$ to the direction $\nabla\tau_p$ can not be found from the system (4.1).

Let us show that the projection of this vector to a plane that is orthogonal to the vector $\nabla\tau_p$ is defined by the equality (4.3) uniquely and it is equal to zero. Indeed, any vector $\alpha^{(n,p)}$ can be represent in the form

$$\alpha^{(n,p)} = c_p(x)[A^{(n,p)}\nabla\tau_p + \nabla\tau_p \times B^{(n,p)}], \quad B^{(n,p)}\cdot\nabla\tau_p = 0, \tag{4.4}$$

where the scalar $A^{(n,p)}$ and the vector $B^{(n,p)}$ are calculated as

$$A^{(n,p)} = c_p(x)(\alpha^{(n,p)}\cdot\nabla\tau_p), \quad B^{(n,p)} = c_p(x)(\alpha^{(n,p)}\times\nabla\tau_p). \tag{4.5}$$

Multiplying the equality (4.3) by $c_p\nabla\tau_p$, we get following expression:

$$(\rho - \mu c_p^{-2})B^{(-1,p)} = 0. \tag{4.6}$$

So as $\rho-\mu c_p^{-2} = \rho(\lambda+\mu)/(\lambda+2\mu) > 0$, then from this equality follows, that $B^{(-1,p)} = 0$. The quantity $A^{(-1,p)}$ remains indefinite. In order to calculate it, we use equality (3.10) with $k = 1$. In expanded form it can be written as

$$\sum_{j=1}^{3}\Big[\kappa_{ij}^1(\alpha^{(-1,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j} - \frac{\partial}{\partial x_j}\kappa_{ij}^0(\alpha^{(-1,p)},\tau_p) + \rho\alpha_i^{(0,p)}\delta_{ij} + \kappa_{ij}^0(\alpha^{(0,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j}\Big] = 0, \tag{4.7}$$

$$i = 1,2,3.$$

There according formulas (3.2),

$$\kappa_{ij}^0(\alpha^{(-1,p)},\tau_p) = -\lambda\delta_{ij}(\alpha^{(-1,p)}\cdot\nabla\tau_p) - \mu\Big(\alpha_i^{(-1,p)}\frac{\partial\tau_p}{\partial x_j} + \alpha_j^{(-1,p)}\frac{\partial\tau_p}{\partial x_i}\Big),$$

$$\kappa_{ij}^0(\alpha^{(0,p)},\tau_p) = -\lambda\delta_{ij}(\alpha^{(0,p)}\cdot\nabla\tau_p) - \mu\Big(\alpha_i^{(0,p)}\frac{\partial\tau_p}{\partial x_j} + \alpha_j^{(0,p)}\frac{\partial\tau_p}{\partial x_i}\Big),$$

$$\kappa_{ij}^1(\alpha^{(-1,p)},\tau_p) = \lambda\delta_{ij}\mathrm{div}\alpha^{(-1,p)} + \mu\Big(\frac{\partial\alpha_i^{(-1,p)}}{\partial x_j} + \frac{\partial\alpha_j^{(-1,p)}}{\partial x_i}\Big)$$
$$-p_0\delta_{ij}(\alpha^{(-1,p)}\cdot\nabla\tau_p) - q_0\Big(\alpha_i^{(-1,p)}\frac{\partial\tau_p}{\partial x_j} + \alpha_j^{(-1,p)}\frac{\partial\tau_p}{\partial x_i}\Big).$$

Let us write the system of equality (4.7) in a vector form. Notice, that first summand on the left-hand side of the formula (4.7) can be written in the form

$$\sum_{j=1}^{3}\kappa_{ij}^1(\alpha^{(-1,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j} = \lambda\mathrm{div}\alpha^{(-1,p)}\frac{\partial\tau_p}{\partial x_i} + \mu(\nabla\tau_p\cdot\nabla)\alpha_i^{(-1,p)} + \frac{\partial}{\partial x_i}(\mu\alpha^{(-1,p)}\cdot\nabla\tau_p)$$
$$-\mu(\alpha^{(-1,p)}\cdot\nabla)\frac{\partial\tau_p}{\partial x_i} - (\alpha^{(-1,p)}\cdot\nabla\tau_p)\frac{\partial\mu}{\partial x_i} - (p_0+q_0)(\alpha^{(-1,p)}\cdot\nabla\tau_p)\frac{\partial\tau_p}{\partial x_i} - q_0c_p^{-2}\alpha_i^{(-1,p)}. \tag{4.8}$$

Let us transform second summand of the formula (4.7) as follows

$$-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\kappa_{ij}^0(\alpha^{(-1,p)},\tau_p) = \frac{\partial}{\partial x_i}\big(\lambda\alpha^{(-1,p)}\cdot\nabla\tau_p\big) + \mu(\nabla\tau_p\cdot\nabla)\alpha_i^{(-1,p)}$$
$$+\mu\Big(\alpha_i^{(-1,p)}\Delta\tau_p + \mathrm{div}\alpha^{(-1,p)}\frac{\partial\tau_p}{\partial x_i} + (\alpha^{(-1,p)}\cdot\nabla)\frac{\partial\tau_p}{\partial x_i}\Big)$$
$$+(\nabla\mu\cdot\nabla\tau_p)\alpha_i^{(-1,p)} + \nabla\mu\cdot\alpha^{(-1,p)}\frac{\partial\tau_p}{\partial x_i}. \tag{4.9}$$

Using equalities (4.5), (4.8), (4.9), we can write quantity (4.7) in vector form:

$$[(\lambda+\mu)\mathrm{div}\alpha^{(-1,p)} + \nabla\mu\cdot\alpha^{(-1,p)} - (p_0+q_0)c_p^{-1}A^{(-1,p)}]\nabla\tau_p$$
$$+[\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0c_p^{-2}]\alpha^{(-1,p)} + 2\mu(\nabla\tau_p\cdot\nabla)\alpha^{(-1,p)}$$
$$+\nabla\big((\lambda+\mu)c_p^{-1}A^{(-1,p)}\big) - c_p^{-1}A^{(-1,p)}\nabla\mu$$
$$+(\rho-\mu c_p^{-2})\alpha^{(0,p)} - (\lambda+\mu)c_p^{-1}A^{(0,p)}\nabla\tau_p = 0. \tag{4.10}$$

Replace the equality (4.10) by its projections to vector $\nabla\tau_p$ and to a plane that is orthogonal to this vector.

Multiplying equality (4.10) by $c_p\nabla\tau_p$ we get:

$$\begin{aligned}[(\lambda+\mu)\mathrm{div}\alpha^{(-1,p)} + \nabla\mu\cdot\alpha^{(-1,p)}]c_p^{-1} + [\mu\Delta\tau_p - c_p^{-2}(p_0+2q_0)]A^{(-1,p)} \\ +2\mu\big[\nabla\tau_p\cdot\nabla A^{(-1,p)} - \alpha^{(-1,p)}\cdot(\nabla\tau_p\cdot\nabla)(c_p\nabla\tau_p)\big] \\ +c_p\nabla\big((\lambda+\mu)c_p^{-1}A^{(-1,p)}\big)\cdot\nabla\tau_p = 0. \end{aligned} \tag{4.11}$$

Because the following equalities

$$\begin{aligned}\alpha^{(-1,p)}\cdot(\nabla\tau_p\cdot\nabla)(c_p\nabla\tau_p) &= (\alpha^{(-1,p)}\cdot\nabla\tau_p)(\nabla\tau_p\cdot\nabla c_p) + c_p\alpha^{(-1,p)}\cdot(\nabla\tau_p\cdot\nabla)\nabla\tau_p \\ &= A^{(-1,p)}(\nabla\tau_p\cdot\nabla\ln c_p) - c_p^{-1}(\alpha^{(-1,p)}\cdot\nabla\ln c_p), \\ c_p\nabla\big((\lambda+\mu)c_p^{-1}A^{(-1,p)}\big)\cdot\nabla\tau_p &= (\lambda+\mu)\nabla A^{(-1,p)}\cdot\nabla\tau_p + A^{(-1,p)}\nabla(\lambda+\mu)\cdot\nabla\tau_p \\ &\quad -A^{(-1,p)}(\lambda+\mu)\nabla\ln c_p\cdot\nabla\tau_p\end{aligned}$$

hold, then equality (4.11)is equivalent to the following

$$\begin{aligned}(\lambda+3\mu)\nabla A^{(-1,p)}\cdot\nabla\tau_p + [(\lambda+\mu)\mathrm{div}\alpha^{(-1,p)} + \nabla\mu\cdot\alpha^{(-1,p)} + 2\mu\alpha^{(-1,p)}\cdot\nabla\ln c_p]c_p^{-1} \\ +[\mu\Delta\tau_p + \nabla(\lambda+\mu)\cdot\nabla\tau_p - (\lambda+3\mu)\nabla\tau_p\cdot\nabla\ln c_p - (p_0+2q_0)c_p^{-2}]A^{(-1,p)} = 0\end{aligned} \tag{4.12}$$

Substitute $\alpha^{(-1,p)} = A^{(-1,p)}c_p\nabla\tau_p$ into this formula. Then the equality (4.12) takes the form:

$$\begin{aligned}2(\lambda+2\mu)(\nabla A^{(-1,p)}\cdot\nabla\tau_p) + A^{(-1,p)}[(\lambda+2\mu)\Delta\tau_p \\ +\nabla(\lambda+2\mu)\cdot\nabla\tau_p - (c_p)^{-2}(p_0+2q_0)] = 0.\end{aligned} \tag{4.13}$$

Dividing this equality by ρ, we get its another form

$$2c_p^2(\nabla A^{(-1,p)}\cdot\nabla\tau_p) + A^{(-1,p)}[c_p^2\Delta\tau_p + c_p^2\nabla\ln(c_p^2\rho)\cdot\nabla\tau_p - b_1] = 0. \tag{4.14}$$

Here $b_1 = (c_p)^{-2}(p_0+2q_0)/\rho = (p_0+2q_0)/(\lambda+2\mu)$.

Notice, that due to the formula (2.4) a differentiation of some function $\varphi(x)$ with respect to τ_p along $\Gamma_p(x,y)$ is prescribed by the formula

$$\frac{d\varphi(x)}{d\tau_p} = \nabla\varphi(x)\cdot\frac{dx}{d\tau_p} = c_p^2(x)\nabla\tau_p(x,y)\cdot\nabla\varphi(x). \tag{4.15}$$

Note one more formula

$$c_p^2\Delta\tau_p = \frac{2}{\tau_p} - c_p^2\nabla\tau_p\cdot\nabla\ln(c_p^2J_p), \quad x\in\mathbb{R}^3, \tag{4.16}$$

which corresponds to the equality (2.7).

Due to formulas (4.15), (4.16) we get

$$2\frac{\partial A^{(-1,p)}}{\partial\tau_p} + A^{(-1,p)}\Big[\frac{2}{\tau_p} + \frac{\partial}{\partial\tau_p}\Big(\ln\frac{\rho}{J_p}\Big) - b_1\Big] = 0. \tag{4.17}$$

The latter equation can be written in the following way

$$\frac{\partial}{\partial \tau_p}\Big[A^{(-1,p)}\frac{\tau_p\sqrt{\rho}}{\sqrt{J_p}}\exp\Big(-\frac{1}{2}\int\limits_{\Gamma_p(x,y)} b_1(\xi)\,d\tau_p'\Big)\Big] = 0. \tag{4.18}$$

This equality means that expression in squared brackets in the formula (4.18) must be constant along a geodesic line $\Gamma_p(x,y)$, i.e.

$$A^{(-1,p)}(x,y)\frac{\tau_p(x,y)\sqrt{\rho(x)}}{\sqrt{J_p(x,y)}}\exp\Big(-\frac{1}{2}\int\limits_{\Gamma_p(x,y)} b_1(\xi)\,d\tau_p'\Big) = C_1. \tag{4.19}$$

Let us choose a constant C_1 such that the main components of decomposition (1.5) for homogeneous medium with $c_p = c_p(y)$, $\rho = \rho(y)$ and decomposition (3.1) for inhomogeneous medium coincide at the neighborhood of the point y. It follows from (4.19) that

$$C_1 = \sqrt{\rho(y)}\lim_{x\to y}[\tau_p(x,y)A^{(-1,p)}(x,y)], \tag{4.20}$$

where the limit is calculated along the geodesic $\Gamma_p(x,y)$. On the other hand, this limit can be calculated via formulas (1.5), (1.6). For this aim one should compare formulas (4.4) and (4.5) As a result, we found

$$\lim_{x\to y}[\tau_p(x,y)A^{(-1,p)}(x,y)] = -\frac{1}{4\pi\rho(y)}\lim_{x\to y}\frac{(f^0\cdot\nabla_y\tau_p)}{c_p(x)c_p(y)} = \frac{(f^0\cdot\nu_p^0)}{4\pi c_p^3(y)\rho(y)}. \tag{4.21}$$

Here $\nu_p^0 = -c_p(y)\nabla_y\tau_p$ is a unit vector tangent to $\Gamma_p(x,y)$ at y, which is positively directed. Then,

$$C_1 = \frac{(f^0\cdot\nu_p^0)}{4\pi c_p^3(y)\sqrt{\rho(y)}} = -\frac{(f^0\cdot\nabla_y\tau_p)}{4\pi c_p^2(y)\sqrt{\rho(y)}}. \tag{4.22}$$

The formula (4.19) leads to the equality

$$A^{(-1,p)}(x,y) = -(f^0\cdot\nabla_y\tau_p(x,y))A^{(p)}(x,y), \tag{4.23}$$

where

$$A^{(p)}(x,y) = \frac{\sqrt{J_p(x,y)}}{4\pi\tau_p(x,y)c_p^2(y)\sqrt{\rho(x)\rho(y)}}\exp\Big(\frac{1}{2}\int\limits_{\Gamma_p(x,y)}\frac{p_0(\xi)+2q_0(\xi)}{\lambda(\xi)+2\mu(\xi)}\,d\tau_p'\Big), \tag{4.24}$$

So,

$$\alpha^{(-1,p)}(x,y) = -c_p(x)(f^0\cdot\nabla_y\tau_p(x,y))A^{(p)}(x,y)\,\nabla_x\tau_p(x,y). \tag{4.25}$$

Let us take vector product equality (4.10) and vector $c_p\nabla\tau_p$. As a result, we get

$$B^{(0,p)} = -\frac{\lambda+2\mu}{\rho(\lambda+\mu)}\Big((\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0c_p^{-2})B^{(-1,p)} + \Big[2\mu(\nabla\tau_p\cdot\nabla)\alpha^{(-1,p)} + \nabla\big((\lambda+\mu)c_p^{-1}A^{(-1,p)}\big) - c_p^{-1}A^{(-1,p)}\nabla\mu\Big]\times\nabla\tau_p\Big). \tag{4.26}$$

Notice that in this formula $B^{(-1,p)} = 0$ holds.

So, using equality (3.10) where $k = 1$, we calculate $A^{(-1,p)}$ and $B^{(0,p)}$. Let us show that evaluations are able to be calculated with recurrent formulas. Let $n \geq 1$ be an integer number. Assume that vectors $\alpha^{(k-1,p)}$, $B^{(k,p)}$ are known for all nonnegative $k < n$. Let us show, that equalities (3.10) allow us to find formulas for calculation $A^{(n-1,p)}$ and $B^{(n,p)}$ for $k = n + 1$.

Indeed, write equalities $r_i^{(n+1,p)} = 0$, $i = 1, 2, 3$ in the following way:

$$\sum_{j=1}^{3} \left[q_{ij}^0(\alpha^{(n,p)}, \tau_p) + q_{ij}^1(\alpha^{(n-1,p)}, \tau_p)\right] = Q_i^{(n,p)}, \quad i = 1, 2, 3, \tag{4.27}$$

where $Q_i^{(n,p)}$ are components of the vector $Q^{(n,p)}$ defined by

$$\begin{aligned} Q_i^{(n,p)} &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1} q_{ij}^m(\alpha^{(n-m,p)}, \tau_p), \\ &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1} \Big[\kappa_{ij}^m(\alpha^{(n-m,p)}, \tau_p)\frac{\partial \tau_p}{\partial x_j} - \frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,p)}, \tau_p)\Big], \ i = 1, 2, 3. \end{aligned}$$

According the above assumption, vector $Q^{(n,p)}$ is known. The equality (4.27) can be written in vector form that is quite similar to the equality (4.10), namely,

$$\begin{aligned} &[(\lambda + \mu)\mathrm{div}\alpha^{(n-1,p)} + \nabla\mu \cdot \alpha^{(n-1,p)} - (p_0 + q_0)c_p^{-1}A^{(n-1,p)}]\nabla\tau_p \\ &+[\mu\Delta\tau_p + \nabla\mu \cdot \nabla\tau_p - q_0 c_p^{-2}]\alpha^{(n-1,p)} + 2\mu(\nabla\tau_p \cdot \nabla)\alpha^{(n-1,p)} \\ &+\nabla\big((\lambda + \mu)c_p^{-1}A^{(n-1,p)}\big) - c_p^{-1}A^{(n-1,p)}\nabla\mu \\ &+(\rho - \mu c_p^{-2})\alpha^{(n,p)} - (\lambda + \mu)c_p^{-1}A^{(n,p)}\nabla\tau_p = Q^{(n,p)}. \end{aligned} \tag{4.28}$$

Multiplying it scalarly by $c_p\nabla\tau_p$ we get an equality

$$\begin{aligned} &(\lambda + 3\mu)\nabla A^{(n-1,p)} \cdot \nabla\tau_p + [(\lambda + \mu)\mathrm{div}\alpha^{(n-1,p)} + \nabla\mu \cdot \alpha^{(n-1,p)} \\ &+2\mu\alpha^{(n-1,p)} \cdot \nabla \ln c_p]c_p^{-1} + [\mu\Delta\tau_p + \nabla(\lambda + \mu) \cdot \nabla\tau_p \\ &-(\lambda + 3\mu)\nabla\tau_p \cdot \nabla \ln c_p - (p_0 + 2q_0)c_p^{-2}]A^{(n-1,p)} = c_p Q^{(n,p)} \cdot \nabla\tau_p, \end{aligned} \tag{4.29}$$

which is an analog of (4.12). Let us substitute in this formula the expression

$$\alpha^{(n-1,p)} = A^{(n-1,p)}c_p\nabla\tau_p + c_p\nabla\tau_p \times B^{(n-1,p)}$$

and take into account, that vector $B^{(n-1,p)}$ is known, by the assumption. Transfer known terms to the right-hand part of the equality and divide the result by ρ. As a result we will get the equation to define $A^{(n-1,p)}$:

$$2c_p^2(\nabla A^{(n-1,p)} \cdot \nabla\tau_p) + A^{(n-1,p)}[c_p^2\Delta\tau_p + c_p^2\nabla \ln(c_p^2\rho) \cdot \nabla\tau_p - b_1] = R^{(n,p)}. \tag{4.30}$$

Here a vector $R^{(n,p)}$ is defined by

$$\begin{aligned} R^{(n,p)} &= \frac{1}{\rho}\Big(c_p Q^{(n,p)} \cdot \nabla\tau_p - [(\lambda + \mu)c_p^{-1}\mathrm{div}(c_p\nabla\tau_p \times B^{(n-1,p)}) \\ &+\nabla\mu \cdot (\nabla\tau_p \times B^{(n-1,p)}) + 2\mu(\nabla\tau_p \times B^{(n-1,p)}) \cdot \nabla \ln c_p]\Big). \end{aligned} \tag{4.31}$$

Along geodesic $\Gamma_p(x,y)$ this equality can be written in the following form

$$\frac{d}{d\tau_p}\Big(\frac{A^{(n-1,p)}(x,y)}{A^{(p)}(x,y)}\Big) = \frac{R^{(n,p)}(x,y)}{2A^{(p)}(x,y)}, \tag{4.32}$$

where the function $A^{(p)}(x,y)$ is determined by the formula (4.24).

Let $\xi_p(x,y)$ be an intersection point of geodesic $\Gamma_p(x,y)$ with sphere $S_\varepsilon = \{x \in \mathbb{R}^3 | \, |x| = \varepsilon\}$. Integrating the equality (4.32) form the point $\xi_p(x,y)$ to the point $\xi_p(x,y)$ we obtain the equality:

$$A^{(n-1,p)}(x,y) = \Big[\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} + \int\limits_{\Gamma_p(x,\xi_p(x,y))} \frac{R^{(n,p)}(\xi,y)}{2A^{(p)}(\xi,y)}\, d\tau_p'\Big] A^{(p)}(x,y), \; n \geq 1. \tag{4.33}$$

Let us notice, that the first summand of the expression in square brackets is easy to calculate by using formulas (1.6). As a result we find that

$$\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} = -\frac{2(f^0 \cdot \nabla_y \tau_p(x,y))[c_p(y)]^n}{|\xi_p(x,y)-y|^n} \begin{cases} 1, & n=1,2, \\ 0, & n>2. \end{cases} \tag{4.34}$$

Let us find a formula to calculate $B^{(n,p)}(x,y)$. Multiplying the equality (4.28) vectorially by $c_p \nabla \tau_p$, we get the expression

$$\begin{aligned}[]
[\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0 c_p^{-2}]B^{(n-1,p)} + 2\mu c_p(\nabla\tau_p\cdot\nabla)\alpha^{(n-1,p)} \times \nabla\tau_p \\
+ c_p\nabla\big((\lambda+\mu)c_p^{-1}A^{(n-1,p)}\big) \times \nabla\tau_p - A^{(n-1,p)}\nabla\mu \times \nabla\tau_p \\
+ (\rho - \mu c_p^{-2})B^{(n,p)} = c_p Q^{(n,p)} \times \nabla\tau_p,
\end{aligned}$$

that allows us to find recurrent formula for $B^{(n,p)}(x,y)$ when $n \geq 1$ in the form

$$\begin{aligned}
B^{(n,p)} = \frac{\lambda+2\mu}{\rho(\lambda+\mu)}\Big(c_p Q^{(n,p)} \times \nabla\tau_p - [\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0 c_p^{-2}]B^{(n-1,p)} \\
-c_p\Big[2\mu(\nabla\tau_p\cdot\nabla)\alpha^{(n-1,p)} + \nabla\big((\lambda+\mu)c_p^{-1}A^{(n-1,p)}\big) - c_p^{-1}A^{(n-1,p)}\nabla\mu\Big] \times \nabla\tau_p\Big).
\end{aligned} \tag{4.35}$$

Comparing this formulas with the formula (4.26), we notice that (4.35) is still true for $n=0$ if we assume that $Q^{(0,p)}=0$.

5 The calculation of coefficients $\alpha^{(k,s)}$

Now let us consider the group of equalities (3.11). Let $k=0$. Then $r_i^{(0,s)}=0$, $i=1,2,3$ and we obtain the system of homogeneous equations for $\alpha^{(-1,s)}$

$$\rho\alpha_i^{(-1,s)} + \sum_{j=1}^{3} \kappa_{ij}^0(\alpha^{(-1,s)},\tau_s)\frac{\partial\tau_s}{\partial x_j} = 0, \quad i=1,2,3, \tag{5.1}$$

in which

$$\kappa_{ij}^0(\alpha^{(-1,s)},\tau_s) = -\lambda(x)\delta_{ij}(\alpha^{(-1,s)}\cdot\nabla\tau_s) - \mu(x)\Big(\alpha_i^{(-1,s)}\frac{\partial\tau_s}{\partial x_j} + \alpha_j^{(-1,s)}\frac{\partial\tau_s}{\partial x_i}\Big).$$

Using the equality $\rho(x)-\mu(x)c_s^2(x)=0$, it is not difficult to check that the equations (5.1) can be written in the following vector form:

$$-(\lambda+\mu)(\alpha^{(-1,s)}\cdot\nabla\tau_s)\nabla\tau_s=0. \tag{5.2}$$

Represent the vector $\alpha^{(n,s)}$ in the form likewise (4.4):

$$\alpha^{(n,s)}=c_s(x)[A^{(n,s)}\nabla\tau_s+\nabla\tau_s\times B^{(n,s)}],\quad B^{(n,s)}\cdot\nabla\tau_s=0, \tag{5.3}$$

in which $A^{(n,s)}$ пїS $B^{(n,s)}$ are calculated by formulas

$$A^{(n,s)}=c_s(x)(\alpha^{(n,s)}\cdot\nabla\tau_s),\quad B^{(n,s)}=c_s(x)(\alpha^{(n,s)}\times\nabla\tau_s). \tag{5.4}$$

It follows from the equality (5.2) that $A^{(-1,s)}=0$. The vector $B^{(-1,s)}$ remains indefinite. In order to find it, let us consider equalities $r_i^{(1,s)}=0$, $i=1,2,3$. In the vector form, they are completely analogous to the equality (4.10):

$$\begin{aligned}&[(\lambda+\mu)\mathrm{div}\alpha^{(-1,s)}+\nabla\mu\cdot\alpha^{(-1,s)}-(p_0+q_0)c_s^{-1}A^{(-1,s)}]\nabla\tau_s\\&+[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]\alpha^{(-1,s)}+2\mu(\nabla\tau_s\cdot\nabla)\alpha^{(-1,s)}\\&+\nabla\big((\lambda+\mu)c_s^{-1}A^{(-1,s)}\big)-c_s^{-1}A^{(-1,s)}\nabla\mu\\&+(\rho-\mu c_s^{-2})\alpha^{(0,s)}-(\lambda+\mu)c_s^{-1}A^{(0,s)}\nabla\tau_s=0.\end{aligned} \tag{5.5}$$

In this equality $\rho-\mu c_s^{-2}=0$. Multiplying (5.5) vectorially by $c_s\nabla\tau_s$ and using the equality $A^{(-1,s)}=0$ we get the expression below:

$$[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]B^{(-1,s)}+2c_s\mu(\nabla\tau_s\cdot\nabla)\alpha^{(-1,s)}\times\nabla\tau_s=0. \tag{5.6}$$

Let us transform last term of this expression in the following form

$$\begin{aligned}2c_s\mu(\nabla\tau_s\cdot\nabla)\alpha^{(-1,s)}\times\nabla\tau_s&=2\mu[(\nabla\tau_s\cdot\nabla)B^{(-1,s)}-c_s(\nabla\tau_s\cdot\nabla\ln c_s)\alpha^{(-1,s)}\times\nabla\tau_s\\&\quad-c_s\alpha^{(-1,s)}\times(\nabla\tau_s\cdot\nabla)\nabla\tau_s]\\&=2\mu[(\nabla\tau_s\cdot\nabla)B^{(-1,s)}-(\nabla\tau_s\cdot\nabla\ln c_s)B^{(-1,s)}\\&\quad+(\nabla\tau_s\times B^{(-1,s)})\times\nabla\ln c_s]\\&=2\mu[(\nabla\tau_s\cdot\nabla)B^{(-1,s)}-(B^{(-1,s)}\cdot\nabla\ln c_s)\nabla\tau_s].\end{aligned} \tag{5.7}$$

Here the vector equality $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ is used. Divide both parts of the equality (5.6) by ρ and use the formula (5.7). Then we get the relation

$$\begin{aligned}&2c_s^2(\nabla\tau_s\cdot\nabla)B^{(-1,s)}-(B^{(-1,s)}\cdot\nabla c_s^2)\nabla\tau_s\\&\qquad+[c_s^2\Delta\tau_s+c_s^2\nabla\ln(\rho c_s^2)\cdot\nabla\tau_s-b_2]B^{(-1,s)}=0,\end{aligned} \tag{5.8}$$

where $b_2=b_2(x)=c_s^{-2}(x)q_0(x)/\rho(x)=q_0(x)/\mu(x)$. Along the geodesic $\Gamma_s(x,y)$ the following equality holds:

$$\begin{aligned}&2\Big\{c_s^2(\nabla\tau_s\cdot\nabla)B^{(-1,s)}+\Big[\frac{1}{\tau_s}+c_s^2\nabla\tau_s\cdot\nabla\ln\Big(\frac{\sqrt{\rho}}{\sqrt{J_s}}\Big)-\frac{b_2}{2}\Big]B^{(-1,s)}\\&\qquad\qquad-c_s(B^{(-1,s)}\cdot\nabla c_s)\nabla\tau_s\Big\}\\&=\frac{2c_s\sqrt{J_s}}{\tau_s\sqrt{\rho}}\Big[\frac{d}{d\tau_s}\Big(\frac{\tau_s\sqrt{\rho}\,B^{(-1,s)}\mathcal{S}^{-1}}{\sqrt{J_s}}\exp\Big(-\frac{1}{2}\int\limits_{\Gamma_s(x,y)}b_2(\xi)\,d\tau_s'\Big)\Big)\Big]\\&\qquad\qquad\times\mathcal{S}\exp\Big(\frac{1}{2}\int\limits_{\Gamma_s(x,y)}b_2(\xi)\,d\tau_s'\Big).\end{aligned} \tag{5.9}$$

Here $J_s = J_s(x,y) = \det(\frac{\partial \zeta^s}{\partial x})$, $\mathcal{S}$ is some matrix exponential and $\mathcal{S}^{-1}$ is its inverse matrix. Let us explain an origin of the matrix exponential. Use the notation for the term $c_s(B^{(-1,s)} \cdot \nabla c_s)\nabla \tau_s$ in matrix form:

$$c_s(B^{(-1,s)} \cdot \nabla c_s)\nabla \tau_s = B^{(-1,s)} S, \quad S = c_s (\nabla c_s)^t \nabla \tau_s = (\nabla \ln c_s)^t \frac{dx}{d\tau_s},$$

where $(\nabla c_s)^t$ means transposed vector ∇c_s, i.e. represents a column vector. Integration of the matrix $S(x,y)$ along the geodesic $\Gamma_s(x,y)$ gives a matrix exponential that is defined by the equalities:

$$\mathcal{S}(x,y) = \exp\Big\{ \int\limits_{\Gamma_s(x,y)} S(\xi,y)\, d\tau_s' \Big\} = \exp\Big\{ \int\limits_{\Gamma_s(x,y)} (\nabla \ln c_s(\xi))^t\, d\xi \Big\}.$$

In these equalities ξ is a variable point of the geodesic $\Gamma_s(x,y)$, $d\xi = (d\xi_1, d\xi_2, d\xi_3)$, and $\tau_s' = \tau_s(\xi, y)$.

Now let us turn to the calculation of $B^{(-1,s)}$. Using the formula (5.8) we conclude that the expression in the formula (5.9) under the derivative operation must be constant along the geodesic $\Gamma_s(x,y)$, i.e.

$$\frac{\tau_s(x,y)\sqrt{\rho(x)}\, B^{(-1,s)}(x,y)\mathcal{S}^{-1}(x,y)}{\sqrt{J_s(x,y)}} \exp\Big(-\frac{1}{2} \int\limits_{\Gamma_s(x,y)} b_2(\xi)\, d\tau_s' \Big) = C_2. \qquad (5.10)$$

In order to choose a constant C_2 we use the calculations that are analogous to have been used above, when one chose C_1. On the one hand, this constant is defined by a limiting equality

$$\begin{aligned} C_2 &= \lim_{x\to y} \frac{\sqrt{\rho(x)}\, \tau_s(x,y) B^{(-1,s)}(x,y)\mathcal{S}^{-1}(x,y)}{\sqrt{J_s(x,y)}} \exp\Big(-\frac{1}{2} \int\limits_{\Gamma_s(x,y)} b_2(\xi)\, d\tau_s' \Big) \\ &= \sqrt{\rho(y)} \lim_{x\to y} [\tau_s(x,y) B^{(-1,s)}(x,y)], \end{aligned}$$

where the limit is calculated along the geodisic $\Gamma_s(x,y)$. On the other hand, this limit can be calculated by using the formulas(5.4), (1.7). Namely,

$$\lim_{x\to y} [\tau_s(x,y) B^{(-1,s)}(x,y)] = -\lim_{x\to y} \frac{(f^0 \times \nabla_y \tau_s)}{4\pi\rho(y)c_s(x)c_s(y)} = \frac{(f^0 \times \nu_s^0)}{4\pi\rho(y)c_s^3(y)},$$

where ν_s^0 is a unit vector tangent to $\Gamma_s(x,y)$ at the point y.

Then, we find the expression for the constant

$$C_2 = \frac{(f^0 \times \nu_s^0)}{4\pi c_s^3(y)\sqrt{\rho(y)}} = -\frac{(f^0 \times \nabla_y \tau_s(x,y))}{4\pi c_s^2(y)\sqrt{\rho(y)}}.$$

The formula (5.10) leads to the equality

$$B^{(-1,s)}(x,y) = -(f^0 \times \nabla_y \tau_s(x,y)) T^{(s)}(x,y), \qquad (5.11)$$

where

$$T^{(s)}(x,y) = \frac{\mathcal{S}(x,y)\sqrt{J_s(x,y)}}{4\pi\tau_s(x,y)c_s^2(y)\sqrt{\rho(x)\rho(y)}} \exp\Big(\frac{1}{2}\int\limits_{\Gamma_s(x,y)} \frac{q_0(\xi)}{\mu(\xi)}\,d\tau_s'\Big). \tag{5.12}$$

Finally, we obtain the formula for the coefficient $\alpha^{(-1,s)}$:

$$\alpha^{(-1,s)}(x,y) = -c_s(x)\nabla_x\tau_s(x,y)\times\Big((f^0\times\nabla_y\tau_s(x,y))T^{(s)}(x,y)\Big). \tag{5.13}$$

Calculating inner product of equality (5.5) and vector $c_s\nabla\tau_s$ and taking into account that $A^{(-1,s)}=0$ we get the relation

$$\begin{aligned}&[(\lambda+\mu)\mathrm{div}\alpha^{(-1,s)}+\nabla\mu\cdot\alpha^{(-1,s)}]c_s^{-1}\\ &\qquad +2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(-1,s)}\cdot\nabla\tau_s-(\lambda+\mu)c_s^{-2}A^{(0,s)}=0,\end{aligned}$$

that yields to

$$A^{(0,s)} = \frac{c_s^2}{\lambda+\mu}\Big((\lambda+\mu)\mathrm{div}\alpha^{(-1,s)}+\nabla\mu\cdot\alpha^{(-1,s)}+2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(-1,s)}\cdot\nabla\tau_s\Big) \tag{5.14}$$

Thereby, using the equalities (3.11) for $k=1$, we have calculated $B^{(-1,s)}$ and $A^{(0,s)}$. Further calculations can be done by recurrent formulas. Assume that vectors $\alpha^{(k-1,s)}$, $A^{(k,s)}$ are given for all $k<n$. Let us show that the equalities (3.11) for $k=n+1$ allow us to find the formulas for calculations of $B^{(n-1,s)}$ and $A^{(n,s)}$. Indeed, write equalities $r_i^{(n+1,s)}=0$, $i=1,2,3$ in the form

$$\sum_{j=1}^{3}\left[q_{ij}^0(\alpha^{(n,s)},\tau_p)+q_{ij}^1(\alpha^{(n-1,s)},\tau_p)\right]=Q_i^{(n,s)},\quad i=1,2,3, \tag{5.15}$$

here $Q_i^{(n,s)}$ are the components of the vector $Q^{(n,s)}$ defined by

$$\begin{aligned}Q_i^{(n,s)} &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}q_{ij}^m(\alpha^{(n-m,s)},\tau_s),\\ &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}\Big[\kappa_{ij}^m(\alpha^{(n-m,s)},\tau_s)\frac{\partial\tau_s}{\partial x_j}-\frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,s)},\tau_s)\Big],\ i=1,2,3.\end{aligned}$$

According to the assumption the vector $Q^{(n,s)}$ is known. The equality (5.15) can be written in vector form, quite similar to the equality (5.5), namely

$$\begin{aligned}&[(\lambda+\mu)\mathrm{div}\alpha^{(n-1,s)}+\nabla\mu\cdot\alpha^{(n-1,s)}-(p_0+q_0)c_s^{-1}A^{(n-1,s)}]\nabla\tau_s\\ &\quad+[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]\alpha^{(n-1,s)}+2\mu(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}\\ &\qquad+\nabla\big((\lambda+\mu)c_s^{-1}A^{(n-1,s)}\big)-c_s^{-1}A^{(n-1,s)}\nabla\mu\\ &\qquad\quad-(\lambda+\mu)c_s^{-1}A^{(n,s)}\nabla\tau_s=Q^{(n,s)}.\end{aligned} \tag{5.16}$$

Multiplying the equality (5.16) vectorially by $c_s\nabla\tau_s$, we get the ratio

$$\begin{aligned}&[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]B^{(n-1,s)}+2c_s\mu(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}\times\nabla\tau_s\\ &=\{c_sQ^{(n,s)}+A^{(n-1,s)}[(\lambda+\mu)\nabla\ln c_s-\nabla\lambda]-(\lambda+\mu)\nabla A^{(n-1,s)}\}\times\nabla\tau_s.\end{aligned}\tag{5.17}$$

So as the equality holds

$$\begin{aligned}2c_s\mu(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}\times\nabla\tau_s &= 2\mu[(\nabla\tau_s\cdot\nabla)B^{(n-1,s)}-c_s(\nabla\tau_s\cdot\nabla\ln c_s)\alpha^{(n-1,s)}\times\nabla\tau_s\\ &\quad -c_s\alpha^{(n-1,s)}\times(\nabla\tau_s\cdot\nabla)\nabla\tau_s]\\ &= 2\mu[(\nabla\tau_s\cdot\nabla)B^{(n-1,s)}-(\nabla\tau_s\cdot\nabla\ln c_s)B^{(n-1,s)}\\ &\quad +[A^{(n-1,s)}\nabla\tau_s+(\nabla\tau_s\times B^{(n-1,s)})]\times\nabla\ln c_s]\\ &= 2\mu[(\nabla\tau_s\cdot\nabla)B^{(n-1,s)}-(B^{(n-1,s)}\cdot\nabla\ln c_s)\nabla\tau_s\\ &\quad +A^{(n-1,s)}\nabla\tau_s\times\nabla\ln c_s].\end{aligned}$$

Then the equality (5.17) divided by $\rho(x)$, can be written in the form

$$\begin{aligned}&2c_s^2[(\nabla\tau_s\cdot\nabla)B^{(n-1,s)}-(B^{(n-1,s)}\cdot\nabla\ln c_s)\nabla\tau_s\\ &+[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]B^{(n-1,s)}=R^{(n,s)},\end{aligned}$$

in which

$$R^{(n,s)}=\frac{1}{\rho}\{c_sQ^{(n,s)}+A^{(n-1,s)}[(\lambda+2\mu)\nabla\ln c_s-\nabla\lambda]-(\lambda+\mu)\nabla A^{(n-1,s)}\}\times\nabla\tau_s.$$

Along geodesic $\Gamma_s(x,y)$ this equality can be written in the form

$$\frac{d}{d\tau_s}\Big(B^{(n-1,s)}(x,y)(T^{(s)}(x,y))^{-1}\Big)=\frac{1}{2}R^{(n,s)}(x,y)(T^{(s)}(x,y))^{-1},\tag{5.18}$$

where $(T^{(s)}(x,y))^{-1}$ is a matrix inverse with respect to $T^{(s)}(x,y)$ defined by (5.12).

Let $\xi_s(x,y)$ be an intersection of the geodesic $\Gamma_s(x,y)$ with the sphere $S_\varepsilon=\{x\in\mathbb{R}^3|\,|x|=\varepsilon\}$. Integrating the equality (5.18) between the points $\xi_s(x,y)$ and x we get the equality

$$\begin{aligned}B^{(n-1,s)}(x,y)&=\Big[B^{(n-1,s)}(\xi_s(x,y),y)(T^{(s)}(\xi_s(x,y),y))^{-1}\\ &+\frac{1}{2}\int\limits_{\Gamma_s(x,\xi_p(x,y))}R^{(n,s)}(\xi,y)(T^{(s)}(\xi,y))^{-1}\,d\tau_p'\Big]T^{(s)}(x,y),\ n\geq 1.\end{aligned}$$

The first term of the expression in square brackets is easy calculated by using formula (1.7). As a result we find, that

$$B^{(n-1,s)}(\xi,y)(T^{(s)}(\xi,y))^{-1}|_{\xi=\xi_s(x,y)}=-\frac{(f^0\times\nabla_y\tau_s(x,y))[c_s(y)]^n}{|\xi_s(x,y)-y|^n}\begin{cases}1, & n=1,2,\\ 0, & n>2.\end{cases}$$

Let us find a formula to calculate $A^{(n,s)}(x,y)$. Scalar multiplying the equality (5.16) by $c_s\nabla\tau_s$, we get the relation

$$\begin{aligned}&[(\lambda+\mu)\mathrm{div}\alpha^{(n-1,s)}+\nabla\mu\cdot\alpha^{(n-1,s)}-(p_0+q_0)c_s^{-1}A^{(n-1,s)}]c_s^{-1}\\ &+[\mu\Delta\tau_s+\nabla\mu\cdot\nabla\tau_s-q_0c_s^{-2}]A^{(n-1,s)}+2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}\cdot\nabla\tau_s\\ &+c_s[\nabla\big((\lambda+\mu)c_s^{-1}A^{(n-1,s)}\big)-c_s^{-1}A^{(n-1,s)}\nabla\mu]\cdot\nabla\tau_s\\ &-(\lambda+\mu)c_s^{-2}A^{(n,s)}=c_sQ^{(n,s)}\cdot\nabla\tau_s,\end{aligned}$$

from which it is possible to find recurrent formula to find $A^{(n,s)}(x,y)$ for $n \geq 1$ in the form

$$A^{(n,s)} = \frac{c_s^2}{\lambda+\mu}\{[(\lambda+\mu)\mathrm{div}\alpha^{(n-1,s)} + \nabla\mu\cdot\alpha^{(n-1,s)} - (p_0+q_0)c_s^{-1}A^{(n-1,s)}]c_s^{-1}$$
$$+[\mu\Delta\tau_s + \nabla\mu\cdot\nabla\tau_s - q_0c_s^{-2}]A^{(n-1,s)} + [2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}$$
$$+c_s\nabla\big((\lambda+\mu)c_s^{-1}A^{(n-1,s)}\big) - A^{(n-1,s)}\nabla\mu - c_sQ^{(n,s)}]\cdot\nabla\tau_s\}.$$

Thus, all the formulas participating in the theorem 1.1, are derived.

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